Ex 1

a) c_0 fermé dans ℓ^∞:

$x_n = (\xi_j^{(n)})$, $x_n \to x = (\xi_j) \in \ell^\infty$

$\sup_j |\xi_j^{(n)} - \xi_j| \to 0 \Rightarrow$

$\varepsilon > 0$: $\exists n_0$: $|\xi_j^{(n)} - \xi_j| \le \varepsilon/2 \;\forall n \ge n_0, \forall j$

$\Rightarrow |\xi_j| \le |\xi_j - \xi_j^{(n_0)}| + |\xi_j^{(n_0)}| \le \varepsilon/2 + \varepsilon/2 \;\forall j \ge \dots$

$\Rightarrow \lim \xi_j = 0 \Rightarrow x \in c_0$.

$F \subseteq X$ Banach, F fermé $\Rightarrow F$ Banach: $f_n \in F$

$\|f_n - f_m\| \to 0 \Rightarrow \exists x \in X : \|f_n - x\| \to 0$

F fermé, $\Rightarrow x \in F$.

b) $\mathrm{Vect}\langle e_1, e_2, \dots \rangle$ dense dans c_0:

$(\xi_j) = x \in c_0$, $\varepsilon > 0$, $\xi_j \to 0 \Rightarrow |\xi_j| < \varepsilon \;\forall j \ge j_0$

$x_n := \sum_1^n \xi_j e_j = (\xi_1, \dots, \xi_n, 0, 0, \dots) \in c_0$

$\|x_n - x\|_\infty = \sup_{j \ge n+1} |\xi_j| \left[\to 0 \quad n \to \infty \right]$

$\le \varepsilon \quad \forall n \ge j_0$.

c_0 séparable car: $\{ \sum_1^n q_i e_i : n \in \mathbb{N}, q_i \in \mathbb{Q} \}$ dense

dans c_0.

$x \in c_0$, $\varepsilon > 0$, $\exists n_0$: $\|x - x_n\|_\infty < \varepsilon/2 \;\forall n \ge n_0$; $\exists q_i \in$

$|q_i - \xi_i| < \varepsilon/2n_0 \quad i = 1, \dots, n_0$

$\Rightarrow |\sum_1^{n_0} q_i e_i - x| \le \sum_1^{n_0} |q_i - \xi_i| + \varepsilon/2 \le \varepsilon$.

(c) $(c_0)^* \underset{\text{isom}}{\cong} \ell^1$

$$\phi : \begin{cases} \ell^1 \longrightarrow (c_0)^* \\ y=(y_k) \longmapsto f : \begin{cases} c_0 \longrightarrow \mathbb{C} \\ x=(x_k) \longmapsto \sum_{k=1}^{\infty} y_k x_k \end{cases} \end{cases}$$

• f bien défini : $|f(x)| \leq \sum_1^{\infty} |y_k|\,|x_k| \leq \|x\|_\infty \|y\|_1 < \infty$

• f lin : $f(\lambda x + \beta x') = \sum y_k(\lambda x_k + \beta x_k')$
$= \lambda \underbrace{\sum y_k x_k}_{\text{conv}} + \beta \underbrace{\sum y_k x_k'}_{\text{conv}}$ conv.
$= \lambda f(x) + \beta f(x')$

• (1) $\Rightarrow$ $\|f\|_{op} \leq \|y\|_1$ (2)
$\Rightarrow$ f continue. $\Rightarrow f \in (c_0)^*$
$\Rightarrow \phi$ bien déf.

• ϕ lin : $\left[\phi(\lambda y + \beta y')\right](x) = \sum_{k=1}^{\infty} (\lambda y_k + \beta y_k') x_k$
$= \lambda \underbrace{\sum_{k=1}^{\infty} y_k x_k}_{\text{conv}} + \beta \underbrace{\sum^{\infty} y_k' x_k}_{\text{conv}}$ conv.
$= \lambda\, \phi(y)(x) + \beta\, \phi(y')(x)$
$= (\lambda\, \phi(y) + \beta\, \phi(y'))(x)$ $\forall x \in c_0$
$\Rightarrow \phi(\lambda y + \beta y') = \lambda\, \phi(y) + \beta\, \phi(y')$

• $\|\phi(y)\|_{op} = \|f\|_{op} \leq \|y\|_1$ $\underset{0}{\Rightarrow}$ ϕ borné $\Rightarrow \phi$ continue
et $\|\phi\|_{op} \leq 1$.

ϕ inj : $y \in \ker \phi \Rightarrow f := \phi(y) \equiv 0$

$\Rightarrow \phi(y)(x) = 0 \ \forall x$; $x = e_n$:

$\Rightarrow 0 = \phi(y)(e_n) = f(e_n) = y_n \quad \forall n$

$\Rightarrow y \equiv 0. \Rightarrow \ker \phi = \{0\}.$

ϕ surj : $f \in (c_0)^*$, $y_k := f(e_k)$

$\Rightarrow y = (y_k) \in \ell^1$ car :

$|y_k| = |f(e_k)| \leq \|f\|_{op} \cdot \|e_k\|_\infty = \|f\|_{op} \quad \forall k$

$\sum_1^n |y_k| = \sum_1^n |f(e_k)| = \sum_1^n \varepsilon_k f(e_k) \overset{f \text{ lin}}{=} f(\sum_1^n \varepsilon_k e_k)$

$= |f(\sum_1^n \varepsilon_k e_k)| \leq \|f\|_{op} \|\sum_1^n \varepsilon_k e_k\|_\infty =$

$\|f\|_{op} \underbrace{\sup_k |\varepsilon_k|}_{=1} = \|f\|_{op}. \quad \forall n$

$\Rightarrow (y_k) \in \ell^1$ et $\underline{\|y\|_{\ell^1} \leq \|f\|_{op}}$. 13

On a : $\phi(y) = f$ car :

$\phi(y)(e_k) = y_k = f(e_k) \quad \forall k$

$\phi(y)$ lin, f lin $\Rightarrow \phi(y)(\sum_1^n x_i e_i)$

$= \sum_1^n x_i \phi(y)(e_i) = \sum_1^n x_i f(e_i)$

$= f(\sum_1^n x_i e_i) \quad \forall n.$

(f lin.)

$\mathrm{Vect}(e_1, e_2, \dots)$ dense dans c_0

$\phi(y), f$ cont $\Rightarrow \forall x \in c_0$:

$\phi(y)(x) = \phi(y)(\lim \sum_1^n x_i e_i) = \lim \phi(y)(\sum_1^n x_i e_i)$

$= \lim f(\sum_1^n x_i e_i) = f(\lim \sum x_i e_i) = f(x).$

• Φ isométrie :

$\Phi(y) = f$

$\|f\|_{op} \underset{(2)}{\leq} \|y\|_1 \underset{(3)}{\leq} \|f\|_{op}$

$\Rightarrow \|\Phi(y)\|_{op} = \|y\|_{op}$.

Ex 2 : a) $Tf(x) = \int_0^1 \underbrace{\min(y,x)}_{=:w(x,y)} f(y)\,dy$

Tf f et continue $\Rightarrow$

T op. de Fredholm $\underset{TD}{\Rightarrow}$ T compact

$|Tf(x)| \leq \int_0^1 |f(y)|\,dy \leq \|f\|_\infty$

$\Rightarrow \|T\|_{op} \leq 1$.

b) $\boxed{Tf(x) = \int_0^x y f(y)\,dx + x\int_x^1 f(y)\,dy = \lambda f(x)}$

• $\ker(T)$: $\int_0^1 \min(y,x) f(y)\,dy \equiv 0 \quad \forall x$

$\underset{\text{diff}}{\Rightarrow}$ $x f(x) + \int_x^1 f(y)\,dy - x f(x) \equiv 0$

$\Rightarrow \int_x^1 f(y)\,dy = 0 \quad \forall x \underset{\text{diff}}{\Rightarrow} f(x) = 0 \ \forall x$

$\Rightarrow f \equiv 0$.

$\lambda \neq 0$ diff $\Rightarrow$ $\lambda f'(x) = \int_x^1 f(y)\,dy$ (2)

$\Rightarrow f \in C^2$ et $\lambda f''(x) = -f(x)$

$\Rightarrow \boxed{\lambda f'' + f \equiv 0}$

$x=0 \Rightarrow 0 = \lambda f(0) \underset{\lambda \neq 0}{\Rightarrow} f(0) = 0$

$x=1 \Rightarrow \lambda f'(1) = 0 \underset{\lambda \neq 0}{\Rightarrow} f'(1) = 0$

$$\begin{cases} \lambda f'' + f = 0 \\ f(0) = f'(1) = 0 \end{cases}$$

syst fond. $\left\{ \cos \frac{1}{\sqrt{\lambda}} x, \ \sin \frac{1}{\sqrt{\lambda}} x \right\}$

si $\lambda > 0$

si $\lambda < 0$: $\left\{ e^{\frac{1}{\sqrt{|\lambda|}} x}, \ e^{-\frac{1}{\sqrt{|\lambda|}} x} \right\}$

forme complexe $\{ e^{\mu x}, e^{-\mu x} \}$ $\mu \in \mathbb{C} \setminus \{0\}$

$\lambda \mu^2 e^{\mu x} + e^{\mu x} \equiv 0$

$\lambda \mu^2 + 1 = 0 \iff \mu^2 = -\frac{1}{\lambda} \iff \lambda = -\frac{1}{\mu^2}$

$$\begin{cases} f(x) = a e^{\mu x} + b e^{-\mu x} \\ f'(x) = \mu a e^{\mu x} - b \mu e^{-\mu x} \end{cases} \qquad \begin{matrix} x = 0 \\ x = 1 \end{matrix} \begin{cases} a + b = 0 \\ \mu a e^{\mu} - b \mu e^{-\mu} = 0 \end{cases}$$

$$\begin{cases} a = -b \\ a[e^{\mu} + e^{-\mu}] = 0 \end{cases} \quad e^{2\mu} + 1 = 0 \Rightarrow 2\mu = i(\pi + 2k\pi)$$

$a \neq 0$ (car sinon $a = b = 0$)

$\Rightarrow \lambda = -\frac{1}{\mu^2} = \frac{1}{(\frac{\pi}{2} + k\pi)^2}$ $k \in \mathbb{Z}$.

$\Rightarrow \lambda > 0$.

$\sigma_p(T) \subseteq \left\{ \frac{1}{(\frac{\pi}{2} + k\pi)^2}, k \in \mathbb{Z} \right\}$.

$\tilde{f}_k(x) = e^{\mu x} - e^{-\mu x} = e^{i(\frac{\pi}{2} + k\pi)x} - e^{-i(\frac{\pi}{2} + k\pi)x}$

$= 2i \sin(\frac{\pi}{2} + k\pi)$

choisir $f_k(x) = \sin(\frac{\pi}{2} + k\pi)x$; $f(0) = 0$

$T f_k = \lambda_k f_k$ car :

$\int_x^1 x \sin \pi(\frac{1}{2} + k) \cancel{y} \, dy + \int_0^x \cancel{y} \sin \pi (\frac{1}{2} + k) y \, dy$

$= \frac{1}{(\pi(\frac{1}{2} + k))^2} \sin \pi(\frac{1}{2} + k) x.$