My solutions to Problems in

Amer. Math. Monthly
Math. Magazine
Elemente der Math
CRUX

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Raymond Mortini

1. American Math. Monthly

12470. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh(2^{n-1})}\right).$$

$\rightarrow 1$

12422. *Proposed by Mohammed Aassila, Strasbourg, France.* Let a, b, c be integers such that $a \neq 0$ and $an^2 + bn + c \neq 0$ for all positive integers n.
(a) Prove that if there is a positive integer k such that $b^2 - 4ac = k^2a^2$, then

$$\sum_{n=1}^{\infty} \frac{1}{an^2 + bn + c}$$

is rational.
(b)* Is the converse of **(a)** true?

12460. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Prove that the following are equivalent:

(1) Whenever $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences of rationals such that $\langle a_n + b_n \rangle$ converges, the sequence $\langle f(a_n) + b_n g(b_n) \rangle$ also converges.

(2) There are constants m and c such that $f(x) = mx + c$ and $g(x) = m$.

$a + q, a$ $-q, 0$

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12459. *Proposed by Hervé Grandmontagne, Paris, France.* Let α be a real number greater than 1. Evaluate

$$\int_0^\infty \frac{\mathrm{Li}_2(-x^\alpha) + \mathrm{Li}_2\left(-x^{-\alpha}\right)}{1+x^\alpha}\,dx,$$

where Li_2 is the dilogarithm function, defined by $\mathrm{Li}_2(x) = \sum_{k=1}^\infty x^k/k^2$ when $|x| < 1$ and extended by analytic continuation.

$$\frac{e^{-s_N}}{e^{-s_N}}$$

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12451. *Proposed by Adam L. Bruce, Dexter, MI.* Let A and B be complex n-by-n and n-by-m matrices, respectively, let $0_{m,n}$ denote the m-by-n zero matrix, let I_m denote the m-by-m identity matrix, and let exp be the matrix exponential function. Prove

$$\exp\begin{bmatrix} A & B \\ 0_{m,n} & 0_{m,m} \end{bmatrix} = \begin{bmatrix} \exp(A) & \left(\int_0^1 \exp(tA)dt\right) \cdot B \\ 0_{m,n} & I_m \end{bmatrix}.$$

12436. *Proposed by Lorenzo Sauras-Altuzarra, Vienna University of Technology, Vienna, Austria.* For a positive integer n, evaluate

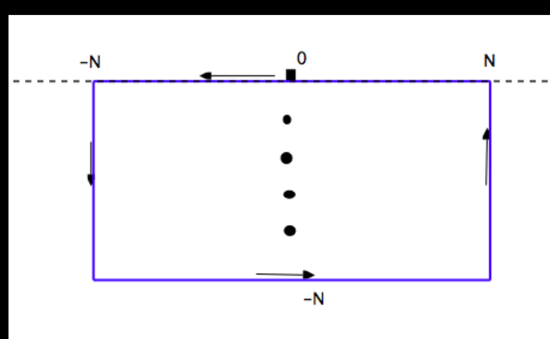
$$\prod_{k=1}^{n}\left(x+\sin^2\left(\frac{k\pi}{2n}\right)\right).$$

$$\sin^2\left(\frac{k\pi}{2n}\right) \qquad \frac{1}{2}\left(1-\cos\left(\frac{k\pi}{n}\right)\right)$$

12433. *Proposed by Etan Ossip, student, Queen's University, Kingston, ON, Canada.* For $x > 1$, prove

$$\frac{i}{2}\int_{-\infty}^{\infty} \frac{\tanh(\pi t)}{\left(\frac{1}{2}+it\right)^{x}}\, dt = \zeta(x),$$

where ζ is the Riemann zeta function.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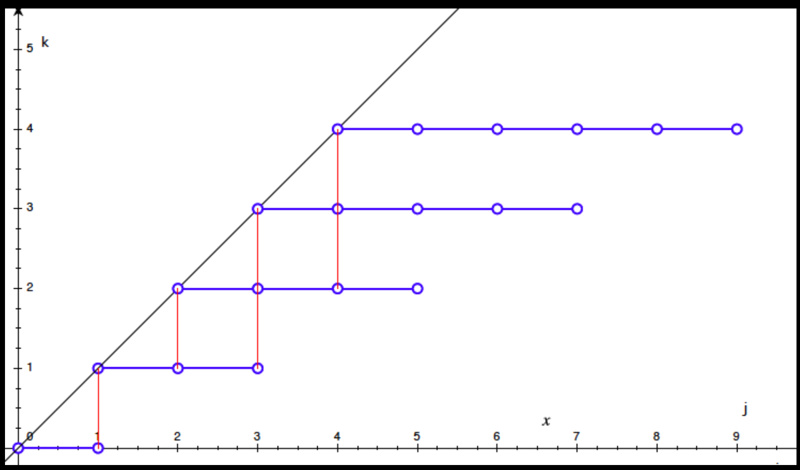
12415. *Proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.* For a nonnegative integer n, evaluate

$$\sum_{j=0}^{2n} \sum_{k=\lfloor j/2 \rfloor}^{j} \binom{2n+2}{2k+1}\binom{n+1}{2k-j}.$$



$k \le j \le 2k+1,\ k = 0, 1, 2, 3, 4$ $\quad$ $\lfloor j/2 \rfloor \le k \le j$ for $j = 0, 1, 2, 3, 4$

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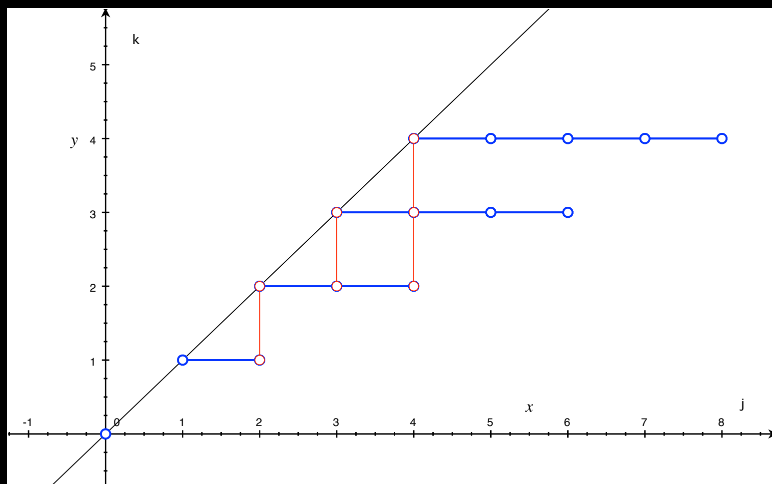
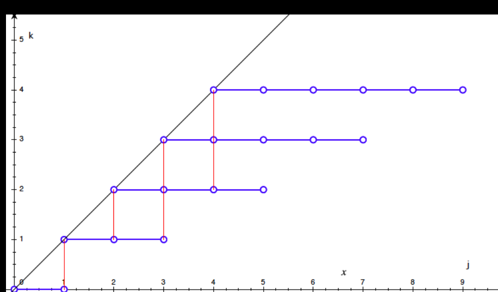
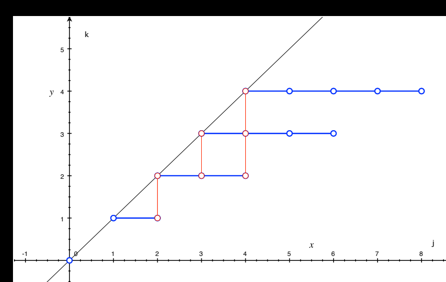


$k \leq j \leq 2k,\ k = 0, 1, 2, 3, 4$ $\lceil j/2 \rceil \leq k \leq j$ for $j = 0, 1, 2, 3, 4$





12406. *Proposed by Raymond Mortini, University of Luxembourg, Esch-sur-Alzette, Luxembourg, and Rudolf Rupp, Nuremberg Institute of Technology, Nuremberg, Germany.* For fixed $p \in \mathbb{R}$, find all functions $f : [0, 1] \to \mathbb{R}$ that are continuous at 0 and 1 and satisfy $f\left(x^2\right) + 2pf(x) = (x + p)^2$ for all $x \in [0, 1]$.

[redacted]

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12407. *Proposed by an anonymous contributor, New Delhi, India.* Let r be a positive real number. Evaluate

$$\int_0^\infty \frac{x^{r-1}}{\left(1+x^2\right)\left(1+x^{2r}\right)}\,dx.$$

12398. *Proposed by Lawrence Glasser, Clarkson University, Potsdam, NY.* Evaluate

$$\sum_{n=0}^{\infty} \operatorname{csch}\left(2^n\right).$$

$+1-1$

12389. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f(x) = \sum_{n=1}^{\infty} |\sin(nx)|/n^2$. Prove $\lim_{x \to 0+} f(x)/(x \ln x) = -1$.

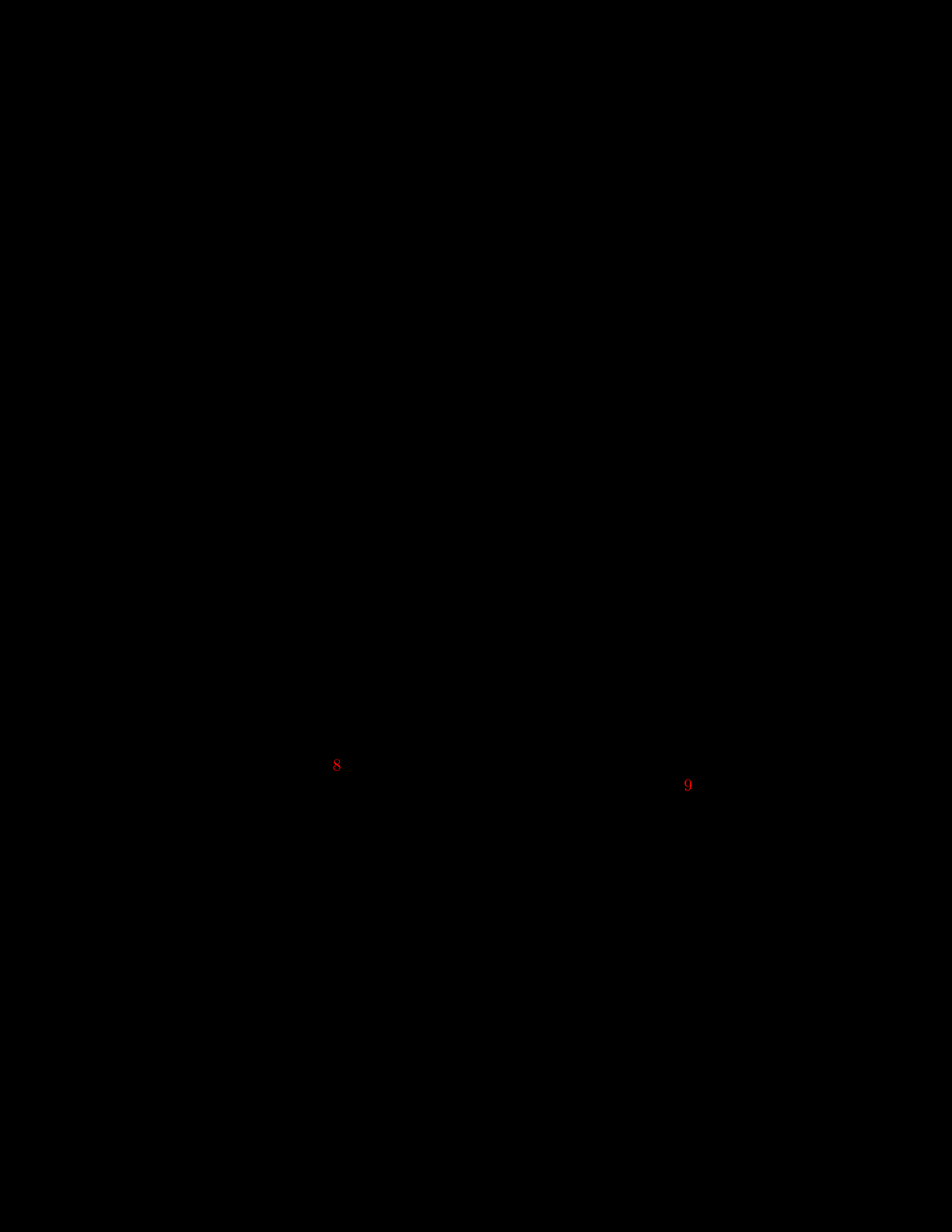
12388. *Proposed by Antonio Garcia, Strasbourg, France.* Let α be a real number. Evaluate

$$\int_0^\infty \frac{(\ln x)^2 \arctan(x)}{1 - 2(\cos\alpha)x + x^2}\,dx.$$

12380. *Proposed by Dorin Mărghidanu, Alexandru Ioan Cuza National College, Corabia, Romania.* Let m, n, and p be positive integers, and let a, b, and c be nonnegative real numbers with $a + b + c = 3$. Prove

$$\sqrt[m]{a + \sqrt[n]{b + \sqrt[p]{c}}} + \sqrt[m]{b + \sqrt[n]{c + \sqrt[p]{a}}} + \sqrt[m]{c + \sqrt[n]{a + \sqrt[p]{b}}} \le 3\sqrt[m]{1 + \sqrt[n]{2}},$$

and determine when equality occurs.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12372. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* For $\alpha > 0$, evaluate

$$\int_0^1 \frac{\ln |x^\alpha - (1-x)^\alpha|}{x}\, dx.$$

97

12375. *Proposed by Hongwei Chen, Christopher Newport University, Newport News, VA.* Let

$$I_n = \int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^n \, dx.$$

Problem 12288 [2021, 946] in this MONTHLY asked for a proof that $I_2 = \pi/5$. Prove that I_n is a rational multiple of π whenever n is a positive integer.

12362. *Proposed by Antonio Garcia, Strasbourg, France.* Evaluate

$$\lim_{n\to\infty}\int_0^{\pi/2}\frac{n}{\left(\sqrt{2}\cos x\right)^n+\left(\sqrt{2}\sin x\right)^n}\,dx.$$

12347. *Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bîrlad, Romania.* Let a and b be real numbers with $0 < a < 1 < b$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$.

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12340. *Proposed by Antonio Garcia, Strasbourg, France.* Let $g : [0, 1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = Cg(1/2)$$

for some constant C (independent of g) and determine the value of C.

12338. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^\infty \frac{\cos(x) - 1}{x\ (e^x - 1)}\, dx = \frac{1}{2} \ln\bigl(\pi \operatorname{csch}(\pi)\bigr).$$

12338. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^\infty \frac{\cos(x) - 1}{x\ (e^x - 1)}\, dx = \frac{1}{2} \ln\bigl(\pi \operatorname{csch}(\pi)\bigr).$$

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https://www.youtube.com/watch?v=p8ok5QNNlsc

12312. *Proposed by Martin Tchernookov, University of Wisconsin, Whitewater, WI.* Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ such that, for all positive x,

$$f(x)\left(f(x) - \frac{1}{x}\int_0^x f(t)\,dt\right) \geq (f(x)-1)^2.$$

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Solution to problem 12312, AMM 129 (3) (2022), p. 286 , by
Gerd Herzog, Raymond Mortini

We show that the constant function 1 is the only solution

Let $y = y(x) := \int_0^x f(t)dt$ and suppose that the continuous function $f : [0, \infty[\to \mathbb{R}$ satisfies on $]0, \infty[$

$$f(x)\left(f(x) - \frac{1}{x}\int_0^x f(t)dt\right) \geq (f(x)-1)^2.$$

Then

$$y'\left(2 - \frac{y}{x}\right) \geq 1 \text{ for } x > 0 \text{ and } y(0) = 0. \tag{25}$$

Note that this implies that $y'(0) = 1$, because, by letting $x \to 0$,

$$y'(0)(2 - y'(0)) \geq 1 \iff (y'(0)-1)^2 \leq 0$$

Let the function $w : [0, \infty[\to \mathbb{R}$ be given by

$$w(x) := \begin{cases} \frac{y(x)}{x} & \text{if } x > 0 \\ y'(0) & \text{if } x = 0. \end{cases}$$

Then $w \in C([0, \infty[) \cap C^1(\,]0, \infty[)$. We claim that

$$w(x) = 1 \text{ for every } x \geq 0, \tag{26}$$

from which we conclude that $y(x) = x$ and so $f(x) = y'(x) = 1$ for $x \geq 0$.

To see this, note that by (25), $w(x) \neq 2$. Since w is continuous on $[0, \infty[$, $w(0) = 1$, and w does not take the value 2, we have that $w(x) < 2$ for each $x > 0$. Hence, for $x > 0$,

$$\begin{aligned} w'(x) &= \frac{xy'(x) - y(x)}{x^2} \geq \frac{1}{x}\left(\frac{1}{2 - w(x)} - w(x)\right) \\ &= \frac{1}{x} \cdot \frac{(1 - w(x))^2}{2 - w(x)} \end{aligned} \tag{27}$$

Thus we may deduce from (27) that $w' \geq 0$; that is w is increasing [2].

Now suppose that (26) is not true.

Case 1 There is $x_0 > 0$ with $w(x_0) < 1$. This is not possible, though, as w is increasing, but $w(0) = 1$.

Case 2 There is $x_0 > 0$ with $w(x_0) > 1$. As w is increasing, $w > 1$ for $x \geq x_0$. Note that we already know that $w < 2$. Since the map $t \mapsto \frac{(1-t)^2}{2-t}$ is increasing on $[1, 2[$, we deduce from (27) that for $x \geq x_0$

$$w'(x) \geq \frac{1}{x} \cdot \frac{(1 - w(x_0))^2}{2 - w(x_0)} =: c\frac{1}{x}.$$

[2] in the weak sense; or funnily called nondecreasing, a very ambiguous word.

Hence, by integration, for $x \geq x_0$,

$$w(x) \geq w(x_0) + c \log(x/x_0) \to \infty \ (x \to \infty).$$

An obvious contradiction. □

12308. *Proposed by Cezar Lupu, Yanqi Lake BIMSA and Tsinghua University, Beijing, China.* What is the minimum value of $\int_0^1 \left(f'(x)\right)^2 dx$ over all continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1$?

Solution to problem 12308, AMM 129 (3) (2022), p. 285 , by
Raymond Mortini

We show that the minimal value is given by $105/2$ and is obtained by the polynomial $f(x) = -105/16x^4 + 105/8x^2 - 33/16$

Let p be any polynomial. Then, by Cauchy-Schwarz,

$$\left(\int_0^1 f'p\,dx\right)^2 \leq \left(\int_0^1 f'^2 dx\right)\left(\int_0^1 p^2 dx\right).$$

A primitives of $f'p$ is given by $fp - \int fp'dx$. Now choose p so that $p(0) = p(1) = 0$ and $p'(x) = \alpha x^2 + \beta$. To this end, put

$$p(x) = ax(x^2 - 1).$$

Then

$$I := \int_0^1 f'p\,dx \quad = \quad fp\Big|_0^1 - \int_0^1 f(3ax^2 - a)dx = -3a + a = -2a$$

Moreover,

$$\int_0^1 p^2 dx = a^2 \int_0^1 (x^6 + x^2 - 2x^4)dx = a^2\left(\frac{1}{7} + \frac{1}{3} - \frac{2}{5}\right).$$

Hence

$$\int f'^2 dx \geq \frac{4a^2}{a^2\left(\frac{1}{7} + \frac{1}{3} - \frac{2}{5}\right)} = \frac{105}{2}.$$

Equality in the Cauchy-Schwarz inequality is given whenever $f' = p$. Thus

$$f(x) = \frac{a}{4}x^4 - \frac{a}{2}x^2 + c.$$

Now a and c have to be chosen so that $\int f = \int x^2 f = 1$. This yields the linear system

$$-7a + 60c = 60$$
$$-27a + 140c = 420$$

whose solution is $a = -105/4$ and $c = -33/16$. Consequently

$$f(x) = -105/16x^4 + 105/8x^2 - 33/16.$$

Note that

$$f'(x)^2 = \big(-\frac{105}{4}x(x^2 - 1)\big)^2.$$

12326. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every fixed $y \in \mathbb{R}$, $f(x+y) - f(x)$ is a polynomial in x. Prove that f is a polynomial function.

Solution to problem 12326, AMM 129 (5) (2022), p. 487 , by
Raymond Mortini, Peter Pflug, Amol Sasane

By considering the symmetric function $p(x,y) := f(x+y) - f(x) - f(y)$ we get from the assumption that as well $p(\cdot, y)$ and $p(x, \cdot)$ are polynomials in their variables separately. Hence, by [1], $p(x,y)$ is a polynomial.

Case 1 $f \in C^1(\mathbb{R})$. Write $p(x,y) = \sum a_{i,j} x^i y^j$ with symmetrical coefficients and $a_{0,0} = -f(0)$ (the sum being finite of course) If we take $y = 0$, then for all x

$$-f(0) = f(x+0) - f(x) - f(0) = a_{0,0} + \sum a_{i,0} x^i.$$

Hence $a_{i,0} = 0$ for all $i \geq 1$. Due to symmetry, we also have $a_{0,j} = 0$ for all $j \geq 1$. Thus we have only coefficients $a_{i,j}$ for $i, j \geq 1$. Consequenlty

$$\frac{f(x+y) - f(x) - (f(y) - f(0)))}{y} = \sum_{i,j \geq 1} a_{i,j} x^i y^{j-1}.$$

As f is assumed to be differentiable, we may take $y \to 0$ and get

$$f'(x) - f'(0) = \sum_{i \geq 1} a_{i,1} x^i.$$

Integration yields

$$f(x) - f(0) - x f'(0) = \sum_{i \geq 1} a_{i,1} \frac{x^{i+1}}{i+1}.$$

Thus f is a polynomial.

Case 2 $f \in C(\mathbb{R})$. Let $F(x) := \int_0^x f(t)dt$ be a primitive of f. Then with

$$G(x,y) := F(x+y) - F(x) - F(y)$$

$$\begin{aligned}
G(x,y) &= \int_0^{x+y} f(t)dt - \int_0^x f(t)dt - \int_0^y f(t)dt \\
&\underset{t=y+s}{=} \int_{-y}^{x} f(y+s)ds - \int_0^x f(t)dt - \int_0^y f(t)dt \\
&= \int_{-y}^{0} f(y+s)ds + \int_0^x \big(f(y+s) - f(s)\big)ds - \int_0^y f(t)dt \\
&\underset{t=y+s}{=} \int_0^y f(t)dt + \int_0^x \big(f(y+s) - f(s)\big)ds - \int_0^y f(t)dt \\
&= \int_0^x p(y,s)ds + f(y)x
\end{aligned}$$

which is a polynomial in x. Again, by symmetry, and the Carroll argument, G is a polynomial. Hence, by Case 1, F is a polynomial and so does $f = F'$.

References

[1] F.W. Carroll A polynomial in each variable separately is a polynomial, Amer. Math. Soc. 68 (1961), 42 44

12288. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx = \frac{\pi}{5}.$$

Solution to problem 12288 in Amer. Math. Monthly 128 (2021), 946, by
Raymond Mortini and Rudolf Rupp

A change of the variable $x \to 1/x$ yields that

$$J := \int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx = \int_0^\infty \frac{(x^2 - \sin^2 x)^2}{x^6}\, dx.$$

Note that

$$(x^2 - \sin^2 x)^2 = x^4 - 2x^2 \sin^2 x + \sin^4 x.$$

Now we "linearize" the trigonometric powers: $\sin^2 x = (1/2)(1 - \cos 2x)$ and $\sin^4 x = (3/8) - (1/2)\cos 2x + (1/8)\cos 4x$. Thus $J = I/2$, where

$$I := \int_{\mathbb{R}} \frac{\frac{3}{8} + x^4 - x^2 + (x^2 - \frac{1}{2})\cos(2x) + \frac{1}{8}\cos(4x)}{x^6}\, dx.$$

Next we consider the meromorphic function

$$f(z) := \frac{\frac{3}{8} + z^4 - z^2 + (z^2 - \frac{1}{2})e^{2iz} + \frac{1}{8}e^{4iz}}{z^6}.$$

Then we add in the numerator the polynomial

$$p(z) := i\left(\frac{1}{2}z - \frac{4}{3}z^3 + \frac{2}{5}z^5\right),$$

that is we consider the function

$$F(z) := f(z) + \frac{p(z)}{z^6}.$$

Note that this polynomial is chosen so that F has a removable singularity at $z = 0$ (in other words, $-\frac{p(z)}{z^6}$ is the principal part in the Laurent expansion of f around the origin). Hence $\int_\Gamma F(z)dz = 0$, where Γ is the boundary of the half-disk $|z| \leq R$, $\operatorname{Im} z \geq 0$, consisting of the half circle Γ_R and the interval $[-R, R]$. Hence, by letting $R \to \infty$ and taking real parts,

$$0 = \operatorname{Re} \lim_{R\to\infty} \int_{\Gamma_R} F(z)dz + I.$$

By Jordan's Lemma, $\limsup_{R\to\infty} \int_{\Gamma_R} |e^{inz}||dz| < \infty$. Hence,

$$\lim_{R\to\infty} \int_{\Gamma_R} F(z)dz = 0 + 0 + i \lim_{R\to\infty} \int_{\Gamma_R} \frac{\frac{2}{5}z^5}{z^6} dz = -\frac{2\pi}{5}.$$

We conclude that the value of the original integral J is $\pi/5$.

12290. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Find all analytic functions $f:\mathbb{C}\to\mathbb{C}$ that satisfy

$$|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$$

for all real numbers x and y.

Solution to problem 12290 in Amer. Math. Monthly 128 (2021), 946, by
Raymond Mortini and Rudolf Rupp

We show that all solutions are given by az, $b\sin(kz)$ and $c\sinh(kz)$ where $a,b,c\in\mathbb{C}$ and $k\in\mathbb{R}$.

First we note that any solution f necessarily satisfies $f(0)=0$. Now let $h(z) := |f(z)|^2 = (f\overline{f})(z)$. Since $f_x = f'$ and $f_y = if_x = if'$, we see that $f_{xy} = (f')_y = i(f')_x = if''$. Moreover $(\overline{f})_x = \overline{f_x}$. Hence

$$\begin{aligned} h_{xy} &= (f_x\overline{f} + f\overline{f}_x)_y = f_{xy}\overline{f} + f_x\overline{f}_y + f_y\overline{f}_x + f\overline{f}_{xy} \\ &= 2\mathrm{Re}(f_{xy}\overline{f}) + 0 = 2\mathrm{Re}(if''\overline{f}) = -2\mathrm{Im}(f''\overline{f}). \end{aligned}$$

Now $|f(z)|^2 = |f(x)|^2 + |f(iy)|^2$ implies that the mixed derivative of the right hand side is 0. We conclude that $\mathrm{Im}(f''\overline{f}) = 0$ in $\mathbb{C}$. Let $U = \mathbb{C}\setminus Z(f)$, where $Z(f) = \{z\in\mathbb{C} : f(z) = 0\}$. Then on U, this is equivalent to

$$0 = \mathrm{Im}\left(\frac{f''}{f}|f|^2\right) = \mathrm{Im}\left(\frac{f''}{f}\right).$$

Thus, a necessary condition for $f\not\equiv 0$ being a solution is that f''/f is a real constant λ. The differential equation $f'' = \lambda f$ in $\mathbb{C}$ has the solutions $az+d$ if $\lambda = 0$, or $\alpha e^{\sqrt{\lambda}z} + \beta e^{-\sqrt{\lambda}z}$ if $\lambda > 0$, and $\alpha e^{i\sqrt{|\lambda|}z} + \beta e^{-i\sqrt{|\lambda|}z}$ if $\lambda < 0$. Since $f(0) = 0$, we have $d = 0$ and $\beta = -\alpha$. So, with $k := \sqrt{|\lambda|}$,

$$f(z) = az,\ c\sinh kz \text{ if } \lambda > 0 \text{ and } c\sin kz \text{ if } \lambda < 0.$$

It is now easy to check that these are solutions indeed (wlog for $k=1$):

$$\begin{aligned} \sin(x+iy) &= \cos(iy)\sin x + \cos x\sin(iy) \\ &= \frac{e^{-y}+e^{y}}{2}\sin x - i\cos x\frac{e^{-y}-e^{y}}{2} \\ &= \cosh y\sin x + i\cos x\sinh y \end{aligned}$$

$$\begin{aligned} |\sin z|^2 &= \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y \\ &= \sin^2 x\cosh^2 y + (1-\sin^2 x)\sinh^2 y \\ &= \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y \\ &= \sin^2 x + \sinh^2 y \\ &= \sin^2 x + |\sin^2(iy)|. \end{aligned}$$

as $\sin(iy) = i\sinh y$

12256. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Prove

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx = -\frac{5}{8}\zeta(3),$$

where $\zeta(3)$ is Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$.

Solution to problem 12256 in Amer. Math. Monthly 128 (2021), 478, by
Raymond Mortini and Rudolf Rupp

Using that $4ab = (a+b)^2 - (a-b)^2$, we obtain

$$4\int_0^1 \frac{\log(1+x)\log(1-x)}{x}dx = \int_0^1 \frac{\log^2(1-x^2)}{x}dx - \int_0^1 \frac{\log^2\frac{1+x}{1-x}}{x}dx =: I_1 - I_2.$$

For I_1, we make the substitution $1-x^2=t^2$. Hence, due to $-xdx = tdt$,

$$I_1 = \int_0^1 \frac{\log^2 t^2}{1-t^2} t\,dt$$

Using that $\int\sum = \sum\int$ (Lebesgue), and twice intregation by parts,

$$I_1 = 4\sum_{n=0}^{\infty}\int_0^1 t^{2n+1}\log^2 t\,dt = 8\sum_{n=0}^{\infty}\frac{1}{(2n+2)^3} = \xi(3).$$

For the second one, I_2, we make the substitution $t=\frac{1+x}{1-x}$. Then $x = \frac{t-1}{t+1}$ and $dx = \frac{2}{(t+1)^2}dt$. Hence

$$I_2 = 2\int_1^{\infty}\frac{\log^2 t}{1-t^2}\,dt \overset{t=1/s}{=} 2\int_0^1 \frac{\log^2 s}{1-s^2}\,ds = 2\sum_{n=0}^{\infty}\int_0^1 s^{2n}\log^2 s\,ds = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = 4\frac{7}{8}\xi(3).$$

Consequently, $4I = (1-\frac{7}{2})\xi(3) = -\frac{5}{2}\xi(3)$ and so

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}dx = -\frac{5}{8}\xi(3).$$

11684. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France, and Rudolf Rupp, Georg-Simon-Ohm Hochschule Nürnberg, Nuremberg, Germany.* For complex a and z, let $\phi_a(z) = (a-z)/(1-\bar{a}z)$ and $\rho(a,z) = |a-z|/|1-\bar{a}z|$.

(a) Show that whenever $-1 < a, b < 1$,

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\rho(a,b), \text{ and}$$

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2.$$

(b) For complex α, β with $|\alpha| = |\beta| = 1$, let

$$m(z) = m_{a,b,\alpha,\beta}(z) = |\alpha\phi_a(z) - \beta\phi_b(z)|.$$

Determine the maximum and minimum, taken over z with $|z| = 1$, of $m(z)$.

original statement

Given $a, b, \alpha, \beta \in \mathbb{C}$ with $|a| < 1$, $|b| < 1$ and $|\alpha| = |\beta| = 1$, let $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ and $\rho(a,b) = |a-b|/|1-\bar{a}b|$ the pseudohyperbolic distance between a and b.

i) Show that whenever $a, b \in\,]-1,1[$,

$$M^- := \max_{|z|\le 1} |\varphi_a(z) - \varphi_b(z)| = 2\rho(a,b)$$

and

$$M^+ := \max_{|z|\le 1} |\varphi_a(z) + \varphi_b(z)| = 2.$$

ii) Determine

$$M := \max_{|z|=1} |\alpha\varphi_a(z) - \beta\varphi_b(z)|$$

and

$$m := \min_{|z|=1} |\alpha\varphi_a(z) - \beta\varphi_b(z)|.$$

Solution to problem 11684 AMM 120 (2013), 76 by
Raymond Mortini, Rudolf Rupp

i) That $M^+ = 2$ is easy: just take $z = 1$ and evaluate:

$$|\varphi_a(1) + \varphi_b(1)| = |-1-1| = 2.$$

Since $M^+ \le 2$, we are done.

ii) We first observe that ϕ_b is its own inverse. Let $c = (b-a)/(1-a\bar{b})$ and $\lambda = -(1-a\bar{b})/(1-\bar{a}b)$. Since ϕ_b is a bijection of the unit circle onto itself,

$$\max_{|z|=1} |\alpha\varphi_a(z) - \beta\varphi_b(z)| = \max_{|z|=1} |\alpha\overline{\beta}\varphi_a(\varphi_b(z)) - z| = \max_{|z|=1} |\alpha\overline{\beta}\lambda\varphi_c(z) - z|.$$

The same identities hold when replacing the maximum with the minimum.

Put $\gamma := \alpha\overline{\beta}\lambda$ and let $-\pi < \arg\gamma \le \pi$. For $|z| = 1$ we obtain

$$H(z) := |\gamma\phi_c(z) - z| = \left|\gamma\frac{z(c\overline{z}-1)}{1-\bar{c}z} - z\right|$$

$$= \left|\gamma\frac{1-c\overline{z}}{1-\bar{c}z} + 1\right| = \left|\gamma\,\frac{w}{\overline{w}} + 1\right|,$$

where
$$w = 1 - c\bar{z} = 1 - c\frac{1}{z}.$$
If z moves on the unit circle, then w moves on the circle $|w-1| = |c|$. Let $w = |w|e^{i\theta}$. Then (see figure 3) the domain of variation of θ is the interval $[-\theta_m, \theta_m]$ with $|\theta_m| < \pi/2$ and $\sin\theta_m = |c| = \rho(a,b)$. Now

$$H(z) = |\gamma e^{2i\theta} + 1| = 2|\cos(\tfrac{\arg\gamma}{2} + \theta)|.$$

Hence,
$$M = \max_{|z|=1} H(z) = 2\max\{|\cos(\tfrac{\arg\gamma}{2} + \theta)| : |\theta| \leq \arcsin(\rho(a,b))\}$$
and
$$m = \min_{|z|=1} H(z) = 2\min\{|\cos(\tfrac{\arg\gamma}{2} + \theta)| : |\theta| \leq \arcsin(\rho(a,b))\}.$$
In particular, if $a, b \in]-1,1[$ and $\alpha = \beta = 1$, then $\gamma = -1$, and so (using the maximum principle at $*$)
$$M^- \stackrel{*}{=} \max_{|z|=1} H(z) = 2\max\{|\sin\theta| : |\theta| \leq \arcsin(\rho(a,b))\} = 2\rho(a,b).$$
If $a, b \in]-1,1[$ and $\alpha = 1, \beta = -1$, then $\gamma = 1$, and so
$$M^+ \stackrel{*}{=} \max_{|z|=1} H(z) = 2\max\{|\cos\theta| : |\theta| \leq \arcsin(\rho(a,b))\} = 2.$$

We note that $m = 0$, that is $H(z_0) = 0$ for some z_0 with $|z_0| = 1$, if and only if $\gamma\phi_c$ has a fixed point on the unit circle (namely z_0). This is equivalent to the condition $|\cos(\frac{\arg\gamma}{2})| \leq |c|$. Moreover, $M = 2$ if and only if $|\sin(\frac{\arg\gamma}{2})| \leq |c|$.

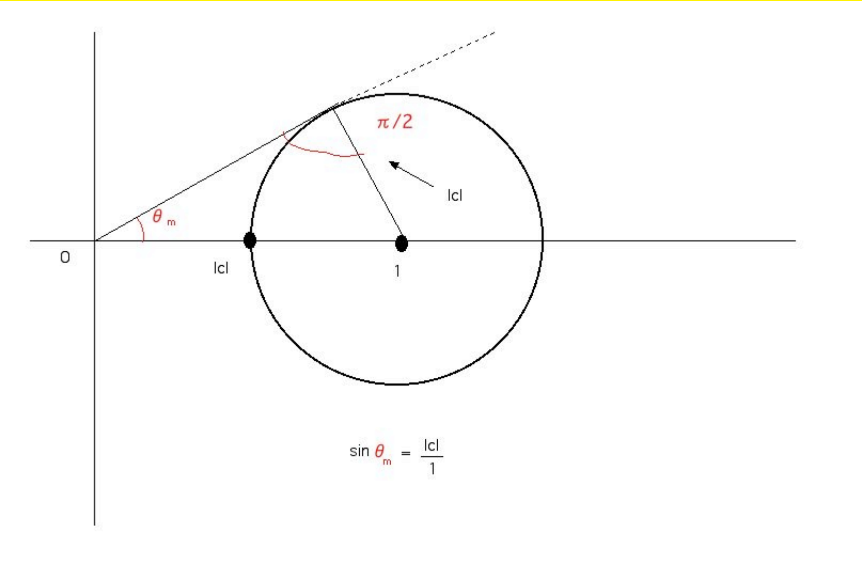
!


FIGURE 3. The domain of variation of $\arg w$

Solution by the proposers.
(**b**) Observe that ϕ_a is its own inverse. Let $c = (b-a)/(1-a\overline{b})$ and let

$$\lambda = -\frac{1-a\overline{b}}{1-\overline{a}b}.$$

Since ϕ_b is a bijection of the unit circle onto itself,

$$\max_{|z|=1} \left|\alpha\phi_a(z) - \beta\phi_b(z)\right| = \max_{|z|=1} \left|\alpha\overline{\beta}\phi_a\big(\phi_b(z)\big) - z\right| = \max_{|z|=1} \left|\alpha\overline{\beta}\lambda\phi_c(z) - z\right|.$$

The same identities hold when the maximum is replaced by the minimum. Put $\gamma = \alpha\overline{\beta}\lambda$, and let $-\pi < \arg\gamma \le \pi$. For $|z| = 1$, let $H(z) = |\gamma\phi_c(z) - z|$. We have

$$H(z) = \left|\gamma\,\frac{z(c\overline{z}-1)}{1-\overline{c}z} - z\right| = \left|\gamma\,\frac{1-c\overline{z}}{1-\overline{c}z} - 1\right| = \left|\gamma\,\frac{w}{\overline{w}} + 1\right|,$$

where $w = 1 - c\overline{z} = 1 - c/z$. As z moves around the unit circle, w moves around the circle $|w-1| = |c|$. Write $w = |w|e^{i\theta}$. Note that θ varies on the interval $[-\theta_m, \theta_m]$, where $|\theta_m| < \pi/2$ and $\sin\theta_m = |c| = \rho(a,b)$. Now

$$H(z) = \left|\gamma e^{2i\theta} + 1\right| = 2\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right|.$$

Hence

$$\max_{|z|=1} H(z) = 2\max\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a,b)\right\} \qquad (*)$$

and

$$\min_{|z|=1} H(z) = 2\min\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a,b)\right\}.$$

(**a**) Specialize $(*)$ by taking $a, b \in (-1,1)$ and $\alpha = \beta = 1$, so that $\gamma = -1$. By the maximum principle, the maximum on the disk is achieved on the boundary, so

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\max\left\{|\sin\theta| : |\theta| \le \arcsin\rho(a,b)\right\} = 2\rho(a,b).$$

For the other part of (**a**), instead specialize $(*)$ by taking $a, b \in (-1,1)$ and $\alpha = 1$, $\beta = -1$, so that $\gamma = 1$. This gives

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2\max\left\{|\cos\theta| : |\theta| \le \arcsin\rho(a,b)\right\} = 2.$$

Also solved by P. P. Dályay (Hungary) and R. Stong. Part (a) only by A. Alt, D. Beckwith, D. Fleischman, O. P. Lossers (Netherlands), and T. Smotzer.

Solution to problem 11584 AMM 118 (2011), 558 by
Raymond Mortini, Jérôme Noël

By the Schwarz-Pick inequality, $\frac{(1-|z|^2)|B'(z)|}{1-|B(z)|^2} \le 1$ for any holomorphic self-map of the unit disk. Then, if we let B be the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n}\,\frac{a_n - z}{1 - \overline{a}_n z}$$

associated with the zeros (a_n), we get:

11584. *Proposed by Raymond Mortini and Jérôme Noël, Université Paul Verlaine, Metz, France.* Let $\langle a_j \rangle$ be a sequence of nonzero complex numbers inside the unit circle such that $\prod_{k=1}^{\infty} |a_k|$ converges. Prove that

$$\left| \sum_{j=1}^{\infty} \frac{1-|a_j|^2}{a_j} \right| \le \frac{1-\prod_{j=1}^{\infty} |a_j|^2}{\prod_{j=1}^{\infty} |a_j|}.$$

$$\frac{|B'(0)|}{1-|B(0)|^2} \le 1.$$

But

$$\frac{B'(z)}{B(z)} = -\sum_{n=1}^{\infty} \frac{1-|a_j|^2}{(1-\overline{a}_j z)(a_j - z)}.$$

Hence

$$\left| \sum_{j=1}^{\infty} \frac{1-|a_j|^2}{a_j} \right| = \frac{|B'(0)|}{|B(0)|} \le \frac{1-|B(0)|^2}{|B(0)|} = \frac{1-\prod_{j=1}^{\infty} |a_j|^2}{\prod_{j=1}^{\infty} |a_j|}.$$

Motivation for posing this as a problem to AMM: We are interested in a direct elementary proof.

11578. ***Proposed by Roger Cuculière, Clichy la Garenne, France.*** Let E be a real normed vector space of dimension at least 2. Let f be a mapping from E to E, bounded on the unit sphere $\{x \in E\colon \|x\| = 1\}$, such that whenever x and y are in E, $f(x + f(y)) = f(x) + y$. Prove that f is a continuous, linear involution on E.

Solution to problem 11578 in Amer. Math. Monthly 118 (2011), 464

Lemma 1. *Let $0 < ||x|| < 1$ and $s \in S$. Let s' be the (second) uniquely determined intersection point of the half-line starting at s and passing through x with S. Then the map $Q : S \to [0, \infty[,\ s \mapsto ||x - s||/||x - s'||$ is a nonconstant continuous map.*

Proof. Q obviously is continuous. If we suppose that Q is constant κ, then this constant is necessarily 1 (just interchange s with s'). Now $x = (1-t)s+ts'$. Thus $x-s = t(s'-s)$ and $x-s' = (1-t)(s-s')$ and so $Q(s) = t/(1-t)$. Hence $1 = \kappa = \frac{t}{1-t}$. So $t = 1/2$. Now $x/||x||$ and $-x/||x||$ belong to S and with $t = (1-||x||)/2$ we have $x = (1-t)\frac{x}{||x||} + t\frac{-x}{||x||}$. So $t = 1/2$ implies that $x = 0$. □

Lemma 2. *The unit sphere S is connected whenever* $\dim E \geq 2$.

Proof. Let $x, y \in S$, $x \neq y$. If x is linear independent of y, then the segment $\{tx + (1-t)y : 0 \leq t \leq 1\}$ does not pass through the origin; hence

$$t \mapsto \frac{tx + (1-t)y}{||tx + (1-t)y||}$$

is a path joining y with x on S.

If $y = \lambda x$ for some $\lambda \in \mathbb{R}$, then we use the hypothesis that $\dim E \geq 2$ to guarantee the existence of a vector u linear independent of x. Thus $v := u/||u|| \in S$. By the first case, we may join x with v and then v with y by a path in S. □

Solution to Problem 11578

The first step is to show that $f(0) = 0$.

(1) Let $x = 0$, $y = -f(0)$. Then $f(f(-f(0))) = f(0) - f(0) = 0$;

(2) Let $x = y = 0$. Then $f(f(0)) = f(0)$;

(3) Let $x = -f(y)$. Then $f(0) = f(-f(y)) + y$. With $y = 0$ this gives $f(0) = f(-f(0))$.

(4) Applying f yields $f(f(0)) = f(f(-f(0))) \underset{(1)}{=} 0$. Thus, by (2), $f(0) = 0$.

(5) Let $x = 0$. Then $f(f(y)) = f(0) + y = y$. Hence f is an involution.

(6) f is additive since

$$f(x + y) \underset{(5)}{=} f(x + f(f(y))) = f(x) + f(y).$$

(7) Next we show that f is $\mathbb{Q}$-homogeneous by induction. Indeed, by (5),

$$f((n + 1)x) = f(nx + x) = f(\underbrace{nx}_{X} + f(\underbrace{f(x)}_{Y})) = f(nx) + f(x).$$

Thus $f(mx) = mf(x)$ for every $m \in \mathbb{N}$.

Now

$$0 = f(0) = f(-x + x) \underset{(5)}{=} f(-x + f(f(x))) = f(-x) + f(x).$$

Thus $f(-x) = -f(x)$. Hence, for $p \in \mathbb{Z}$, we have $f(px) = pf(x)$.

Next, if $n \in \mathbb{N}$, then

$$nf(\frac{x}{n}) = f(\frac{x}{n}) + (n-1)f(\frac{x}{n}) = f(\frac{x}{n}) + f(\frac{n-1}{n}x)$$
$$= f\left(\underbrace{\frac{x}{n}}_{X} + f\Big(\underbrace{f(\frac{n-1}{n}x)}_{Y}\Big)\right) \underset{(5)}{=} f\left(\frac{x}{n} + \frac{n-1}{n}x\right) = f(x)$$

Hence $f\left(\frac{x}{n}\right) = \frac{1}{n}f(x)$. Therefore $f\left(\frac{p}{n}\right) = \frac{p}{n}f(x)$ for $p \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(8) By hypothesis, $||f(s)|| \leq C$ for every $s \in S$. Let $0 < ||x|| < 1$. Consider, as in Lemma 1, the map $H : S \to [0, \infty[, s \mapsto ||x - s||/||x - s'||$. H is continuous and non-constant. Since dim $E \geq 2$, S is connected by Lemma 2. Hence $H(S)$ is an interval. In particular, there is $s \in S$ such that $r := ||x - s||/||x - s'||$ is rational. Thus, with $t = r/(1 + r)$,

$$x = (1 - t)s + ts'$$

is a rational convex-combination of two elements in the sphere.

Since f is $\mathbb{Q}$-linear, we conclude that

$$||f(x)|| \leq (1 - t)||f(s)|| + t||f(s')|| \leq (1 - t)C + tC = C.$$

Now let $x \in E$ be arbitrary. Choose a null-sequence ϵ_n of positive numbers so that $q_n := ||x|| + \epsilon_n$ is rational. Then, $||x/q_n|| \leq 1$. Since f is $\mathbb{Q}$-linear, we obtain

$$||f(x)|| = q_n||f(x/q_n)|| \leq q_n C.$$

Letting n tend to infinity, we get

$$||f(x)|| \leq C||x||.$$

Thus f is continuous at the origin. Since f is additive, we deduce that f is continuous everywhere; just use $f(x_0 + x) = f(x_0) + f(x) \to f(x_0)$ if $x \to 0$.

(9) It easily follows now that f is homogeneous: if $\alpha \in \mathbb{R}$, choose a sequence (r_n) of rational numbers converging to α. Then, due to continuity,

$$f(\alpha x) = \lim_n r_n f(x) = \alpha f(x).$$

To sum up, we have shown that f is a continuous linear involution.

Remarks

If $n = 1$, then the unit sphere S is just a two point set, and so every function is automatically bounded on S. There exist, though, non-continuous linear involutions in $\mathbb{R}$. To this end, let $\mathcal{B}$ be a Hamel basis of the $\mathbb{Q}$-vector space $\mathbb{R}$, endowed with the usual Euclidean norm. We may assume that $\mathcal{B}$ is dense in $\mathbb{R}$. Fix two elements b_0 and $b_1 \in \mathcal{B}$. Let f be defined by $f(b_0) = b_1$, $f(b_1) = b_0$ and $f(b) = b$ if $b \in \mathcal{B} \setminus \{b_0, b_1\}$. Linearly extend f (in a unique way). Then, obviously, f is a linear involution. But f is not continuous at b_0. In fact, let $(b_k)_{n\geq 2} \in \mathcal{B}^{\mathbb{N}}$ converge to b_0. Then $f(b_k) = b_k \to b_0 = f(b_1) \neq f(b_0)$.

11548. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f be a twice-differentiable real-valued function with continuous second derivative, and suppose that $f(0) = 0$. Show that

$$\int_{-1}^{1} (f''(x))^2\, dx \geq 10 \left(\int_{-1}^{1} f(x)\, dx \right)^2.$$

Solution to problem 11548 in Amer. Math. Monthly 118 (2011), 85, by
Raymond Mortini and Jérôme Noël

Let $f \in C^2([-1,1])$, $f(0) = 0$. Then

$$\left(\int_{-1}^{1} f(x)dx \right)^2 \leq \frac{1}{10} \int_{-1}^{1} (f''(x))^2 dx.$$

Moreover, the constant $1/10$ is best possible.

Solution We consider the auxiliary integral

$$I = \frac{1}{2} \left[\int_0^1 (t-1)^2 f''(t)dt + \int_{-1}^{0} (1+t)^2 f''(t) \right].$$

We first show that $I = \int_{-1}^{1} f(t)dt$. In fact, twice integration by parts yields:

$$\int_0^1 (t-1)^2 f''(t)dt = -f'(0) - 2\int_0^1 (t-1)f'(t)dt = -f'(0) + 2\int_0^1 f(t)dt,$$

as well as

$$\int_{-1}^{0} (t+1)^2 f''(t)dt = f'(0) - 2\int_{-1}^{0} (t+1)f'(t)dt = f'(0) + 2\int_{-1}^{0} f(t)dt.$$

This proves the first claim. Now we use the Cauchy-Schwarz inequality to estimate I:

$$\left(\int_0^1 (t-1)^2 f''(t)dt \right)^2 \leq \int_0^1 (t-1)^4 dt \int_0^1 (f''(t))^2 dt = \frac{1}{5} \int_0^1 (f''(t))^2 dt,$$

and similarily for the second integral. Hence, by using that $(A+B)^2 \leq 2(A^2+B^2)$, we obtain

$$I^2 \leq 2\frac{1}{4} \left(\frac{1}{5} \int_0^1 (f''(t))^2 dt + \frac{1}{5} \int_{-1}^{0} (f''(t))^2 dt \right) = \frac{1}{10} \int_{-1}^{1} (f''(t))^2 dt.$$

The constant $1/10$ is obtained for the function

$$f(t) = \begin{cases} \frac{1}{12}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 & \text{if } -1 \leq t \leq 0 \\ \frac{1}{12}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 & \text{if } 0 \leq t \leq 1. \end{cases}$$

Indeed, this follows from the fact that in the Cauchy-Schwarz inequality we actually have equality if the functions are colinear: $p''(t) = (1+t)^2$ if $-1 \leq t \leq 0$ and $p''(t) = (1-t)^2$ if $0 \leq t \leq 1$. A computation then shows that

$$\left(\int_{-1}^{1} p(x)dx \right)^2 = \frac{1}{10} \int_{-1}^{1} (p''(x))^2 dx = \frac{1}{25}.$$

Remark If $f \in C^2([-1,1])$ satisfies $f(1) = f(-1) = f'(1) = f'(-1) = 0$, then the inequality above holds, too. In fact.

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} 1 \cdot f(x)dx = xf(x)|_{-1}^{1} - \int_{-1}^{1} xf'(x)dx =$$

$$= -\frac{1}{2}x^2 f'(x)|_{-1}^{1} + \frac{1}{2}\int_{-1}^{1} x^2 f''(x)dx = \frac{1}{2}\int_{-1}^{1} x^2 f''(x)dx$$

Thus, by Cauchy-Schwarz,

$$\left(\int_{-1}^{1} f(x)dx\right)^2 \leq \frac{1}{4}\int_{-1}^{1} x^4 dx \int_{-1}^{1} (f''(x))^2 dx = \frac{1}{4} \cdot \frac{2}{5}\int_{-1}^{1} (f''(x))^2 dx.$$

11456. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France.* Find

$$\lim_{n\to\infty} n \prod_{m=1}^{n}\left(1-\frac{1}{m}+\frac{5}{4m^2}\right).$$

Solution to problem 11456 AMM 116 (2009), 747

$$a_m := 1-\frac{1}{m}+\frac{5}{4}\frac{1}{m^2} = \frac{1+\left(\frac{2m-1}{2}\right)^2}{m^2}$$

$$\prod_{m=1}^{n} a_m = \frac{\prod_{m=1}^{n}\left(1+\left(\frac{2m-1}{2}\right)^2\right)}{\prod_{m=1}^{n} m^2}\ \frac{\prod_{m=1}^{2n} m^2}{\prod_{m=1}^{n}(2m-1)^2\prod_{m=1}^{n}(2m)^2}$$

$$= \frac{\prod_{m=1}^{n}\left(\frac{1}{(2m-1)^2}+\frac{1}{4}\right)\ (2n)!^2}{4^n (n!)^4} = \frac{\prod_{m=1}^{n}\left(\frac{4}{(2m-1)^2}+1\right)\ (2n)!^2}{16^n (n!)^4}.$$

Now, by Stirlings formula,

$$\frac{(2n)!}{4^n n!^2} \sim \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{(n^n e^{-n}\sqrt{2\pi n})^2 2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

Since $\cos(\pi z) = \prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2}\right)$, we have

$$\lim_n n\prod_{m=1}^{n} a_m = \frac{\cos(\pi i)}{\pi} = \frac{\cosh \pi}{\pi}.$$

We note that

$$\sqrt{\prod_{m=1}^{n} a_m} = \frac{1}{n!}\prod_{m=1}^{n}\left|i-\frac{2m-1}{2}\right| = (n+1)\frac{2}{\sqrt{5}}\frac{|f^{(n+1)}(0)|}{(n+1)!},$$

where $f(z) = (1-z)^{i+\frac{1}{2}}$, an interesting function in the Wiener algebra (its Taylor coefficients behave like $n^{-3/2}$ by the above calculations).

11402. *Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.* Let $f\colon [0,1] \to [0,\infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval (0,1) on the x-axis and the other two vertices on the graph of f.

Solution to problem 11402, AMM 115 (10), (2008), p. 949.

The problem obviously is equivalent to show the existence of two points $0 < a < b < 1$ with $f(a) = f(b) = b - a$, or in other words, find $0 < a < b < 1$ with $b - f(b) = a$ and $f(b) = f(a)$.

To this end, consider the function $h(x) := f(x - f(x)) - f(x)$, where we have continuously extended f by the value 0 for $x < 1$ and $x > 1$. Then h is continuous. We have to show that h admits a zero b in $]0,1[$ with $f(b) < b$. Then $a := b - f(b) \in\,]0,1[$ and $b - a = f(b) = f(a)$.

To do this, we prove that h takes positive and negative values on $[0,1]$. Since $h(0) = h(1) = 0$, the continuity of h implies that h has a zero b in $]0,1[$. Our construction will guarantee that $f(b) < b$.

Let ξ_0 be the largest fixed point of f (note hat $0 \leq \xi_0 < 1$). For later purposes, we note that $f(x) \leq x$ whenever $\xi_0 \leq x \leq 1$. If $\xi_0 = 0$, we let x_0 be the be the smallest point for which $f(x_0) = M := \max_{x\in[0,1]} f(x)$. Note that $x_0 \in\,]0,1[$. Finally, let $x_1 \in\, [\xi_0, 1[$ be the largest point with $f(x_1) = M_1 := \max_{x\in[\xi_0,1]} f(x)$. Then $0 < x_0 \leq x_1 < 1$. Since the function $x - f(x)$ is 0 at ξ_0 and 1 at 1, the intermediate value theorem for continuous functions implies that there exists $y_1 \in\,]\xi_0, 1[$ such that $y_1 - f(y_1) = x_1$. Since $f > 0$, $y_1 > x_1$. Thus

$$h(y_1) = f(y_1 - f(y_1)) - f(y_1) = f(x_1) - f(y_1) = M_1 - f(y_1) > 0.$$

On the other hand, $h(\xi_0) = f(\xi_0 - f(\xi_0)) - f(\xi_0) = 0 - f(\xi_0) < 0$ if $\xi_0 > 0$, and if $\xi_0 = 0$, then, $h(x_0) = f(x_0 - f(x_0)) - f(x_0) < 0$ (since $x_0 - f(x_0)$ is left from the smallest maximal point x_0 of f.)

In both cases, there exists b such that $h(b) = 0$. Since $\xi_0 < b < y_1$ if $\xi_0 > 0$ and $0 < x_0 < b < y_1$ if $\xi_0 = 0$, we see that $f(b) < b$.

11333. *Proposed by Pablo Fernández Refolio, Universidad Autónoma de Madrid, Madrid, Spain.* Show that

$$\prod_{n=2}^{\infty}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right)=\pi.$$

Solution to problem 11333, AMM 114 (10), (2007), p. 926.

Let

$$P_N=\prod_{n=2}^{N}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right).$$

a) We have the following equalities:

$$\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^2-1}=\frac{(N+1)^{N^2-1}}{N^{N(N+2)}}\,(N!)^2,$$

b)

$$\prod_{n=2}^{N}\left(\frac{n+1}{n-1}\right)^{n}=\frac{(N+1)^N N^{N+1}}{2(N!)^2}.$$

Hence

$$\sqrt{P_N}=\frac{(N+1)^{N^2-1}}{N^{N(N+2)}}\,(N!)^2\,\frac{(N+1)^{N/2}\,N^{(N+1)/2}}{\sqrt{2}N!}=$$

$$\left(\frac{N+1}{N}\right)^{N^2-1}\frac{N^{N^2-1}}{N^{N(N+2)}}\,N!\,\frac{(N+1)^{N/2}\,N^{(N+1)/2}}{\sqrt{2}}=$$

$$\left(\frac{N+1}{N}\right)^{N^2-1}N!\,\frac{(N+1)^{N/2}}{\mathbf{N^{N/2}}\sqrt{2}}\,\frac{N^{(N+1)/2}\mathbf{N^{N/2}}}{N^{2N+1}}=$$

$$\left(\frac{N+1}{N}\right)^{N^2-1}N!\,\frac{\left(1+\frac{1}{N}\right)^{N/2}}{\sqrt{2}}\,\frac{\sqrt{N}}{N^{N+1}}.$$

We are now using Stirling's formula telling us that $n!\sim e^{-n}n^n\sqrt{2\pi n}$. Hence

$$\sqrt{P_N}\sim\frac{\sqrt{e}}{\sqrt{2}}N^Ne^{-N}\sqrt{2\pi N}\left(\frac{N+1}{N}\right)^{N^2-1}\frac{\sqrt{N}}{N^{N+1}}=$$

$$\sqrt{e}\sqrt{\pi}e^{-N}\left(\frac{N+1}{N}\right)^{N^2-1}.$$

But $a_N:=e^{-N}\left(\frac{N+1}{N}\right)^{N^2-1}\to\frac{1}{\sqrt{e}}$ as $N\to\infty$; in fact, by taking logarithms we obtain

$\log a_n=(N^2-1)\log(1+\frac{1}{N})-N\sim N^2\log(1+\frac{1}{N})-N=N^2(\frac{1}{N}-\frac{1}{2N}\pm\cdots)-N\sim-\frac{1}{2}.$

Hence $\sqrt{P_N}\to\sqrt{\pi}$ and so $P_N\to\pi$.

11226. *Proposed by Franck Beaucoup, Ottawa, Canada, and Tamás Erdélyi, Texas A& M University, College Station, TX.* Let $a_1, \ldots, a_n$ be real numbers, each greater than 1. If $n \geq 2$, show that there is exactly one solution in the interval (0, 1) to

$$\prod_{j=1}^{n} (1 - x^{a_j}) = 1 - x.$$

Solution to problem 11226, AMM 113 (5), (2006), p. 460.

Let $h(x) = \prod_{j=1}^{n}(1-x^{a_j})$. Then $h'(x)/h(x) = -\sum_{j=1}^{n} \frac{a_j x^{a_j-1}}{1-x^{a_j}}$ and hence $h'(x) = -\sum_{j=1}^{n} a_j x^{a_j-1} \prod_{k\neq j}(1-x^{a_k})$. Clearly $h'(0) = h'(1) = 0$. Let

$$f(x) = (1-x)^{-1} \prod_{j=1}^{n} (1-x^{a_j})$$

if $0 \leq x < 1$. Note that $f(0) = 1$ and $\lim_{x\to 1} f(x) = -h'(1) = 0$. Thus, if we show that $f'(0) > 0$ and that the derivative of f has a unique zero in the open interval $]0,1[$, we are done (that is we can then conclude by the intermediate value theorem that there is a unique x_0 with $0 < x_0 < 1$ so that $f(x_0) = 1$, and hence $h(x_0) = 1 - x_0$.)

Now, $f'(x)/f(x) = \frac{1}{1-x} + h'(x)/h(x)$. In particular, $f'(0) = 1$. Thus we have to look for $x \in]0,1[$ so that $g(x) := \sum_{j=1}^{n} a_j x^{a_j-1} \frac{1-x}{1-x^{a_j}} = 1$. But $g(0) = 0$, and, by de l'Hopital's rule, $\lim_{x\to 1} g(x) = n$. The intermediate value theorem yields the existence of x. The uniqueness of such an x follows from the fact that g is strictly increasing. This is due to the fact that the function $\frac{x^{a-1}-x^a}{1-x^a}$ is strictly increasing on $]0,1[$ whenever $a > 1$.

The latter follows from the fact that

$$\frac{d}{dx}\frac{x^{a-1}-x^a}{1-x^a} = \frac{x^{a-2}\big((a-1)+x^a-ax\big)}{(1-x^a)^2}$$

and that $k(x) := a - 1 + x^a - ax \geq 0$ for $0 \leq x \leq 1$, because $k(0) = a - 1 > 0$, $k(1) = 0$ and $k'(x) = a(x^{a-1} - 1) \leq 0$.

11210. *Proposed by Michael S. Becker, University of South Carolina at Sumter, Sumter, SC.* Show that

$$\prod_{n=0}^{\infty} \frac{(2n+1)^4}{(2n+1)^4 - (2/\pi)^4} = \frac{2e\sec(1)}{e^2+1}.$$

Solution to problem 11210, AMM 113 (3), (2006), p. 267.

We note that

$$p_n := \frac{(2n+1)^4 - (2/\pi)^4}{(2n+1)^4} = \left(1 - \left[\frac{2}{\pi(2n+1)}\right]^4\right) =$$

$$\left(1 - \frac{4}{\pi^2(2n+1)^2}\right)\left(1 + \frac{4}{\pi^2(2n+1)^2}\right).$$

Multiplying in the numerator and denominator (which is 1) with the "missing" factors

$$\left(1 - \frac{4}{\pi^2(2n)^2}\right)\left(1 + \frac{4}{\pi^2(2n)^2}\right)$$

we obtain

$$P := \prod_{n=0}^{\infty} p_n = \prod_{k=1}^{\infty} \frac{\left(1 - \frac{4}{\pi^2 k^2}\right)\left(1 + \frac{4}{\pi^2 k^2}\right)}{\left(1 - \frac{1}{\pi^2 k^2}\right)\left(1 + \frac{1}{\pi^2 k^2}\right)}.$$

Using the standard infinite product representation of the sinus

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 k^2}\right),$$

we obtain

$$P = \frac{\frac{\sin 2}{2}\,\frac{\sin(2i)}{2i}}{\frac{\sin 1}{1}\,\frac{\sin i}{i}} = \cos 1 \cosh 1 = (\cos 1)\,\frac{e^2+1}{2}.$$

11202. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Prove that if $\langle a_n \rangle$ is a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n < \infty$, then for all p in $(0, 1)$

$$\lim_{n\to\infty} n^{1-1/p} \left(a_1^p + \cdots + a_n^p\right)^{1/p} = 0.$$

Solution to problem 11202, AMM 113 (2), (2006), p. 179.

The assertion is an immediate consequence of Hölder's inequality: Wlog let $0 \le a_j \le 1$ and let $q \in]0,1[$ be such that $p+q=1$ (note that $p \in]0,1[$.)

$$n^{p-1}\sum_{j=1}^{n} a_j^p = n^{p-1}\left(\sum_{j=1}^{N} a_j^p + \sum_{j=N+1}^{n} a_j^p \cdot 1\right) \le$$

$$\frac{N}{n^{1-p}} + \left(\sum_{j=N+1}^{n} (a_j^p)^{1/p}\right)^p \left(\sum_{j=N+1}^{n} 1^{1/q}\right)^q \frac{1}{n^{1-p}} \le$$

$$\frac{N}{n^{1-p}} + \left(\sum_{j=N+1}^{\infty} a_j\right)^p \frac{n^q}{n^{1-p}} = \frac{N}{n^{1-p}} + \left(\sum_{j=N+1}^{\infty} a_j\right)^p \le \epsilon$$

if N and $n > N$ is sufficiently big.

11185. *Proposed by Rainer Brück, University of Dortmund, Dortmund, Germany, and Raymond Mortini, University of Metz, Metz, France.* Find all natural numbers n and positive real numbers α such that the integral

$$I(\alpha, n) = \int_0^\infty \log\left(1 + \frac{\sin^n x}{x^\alpha}\right) dx$$

converges.

Solution to problem 11185 AMM 112 (2005), 840 by

Rainer Brück, Raymond Mortini

We claim that

$I(\alpha,p)$ converges if and only if $(\alpha,p) \in\,]1,\infty[\times \mathbb{N}$ or $(\alpha,p) \in \left]\frac{1}{2},1\right] \times (2\mathbb{N}+1)$.

First we discuss the behaviour of the integrand at the origin. For $\alpha > 0$ we have $\left|\log\left(1+\frac{\sin^p x}{x^\alpha}\right)\right| \leq \log\left(1+x^{-\alpha}\right)$. Substituting $\frac{1}{x}$ by t, we obtain

$$\int_0^1 \log\left(1+x^{-\alpha}\right) dx = \int_1^\infty \frac{\log\left(1+t^\alpha\right)}{t^2}\, dt\,,$$

and this integral is convergent. Hence, our integral $I(\alpha,p)$ converges at 0 for every $\alpha > 0$ and $p \in \mathbb{N}$.

Now we discuss the behaviour at infinity. Since $\lim\limits_{t\to 0} \frac{\log(1+t)}{t} = 1$, we see that at infinity

$$A(x) := \log\left(1+\frac{\sin^p x}{x^\alpha}\right) \sim \frac{\sin^p x}{x^\alpha} =: B(x)\,.$$

Hence $\int_1^\infty A(x)\,dx$ converges absolutely if and only if $\int_1^\infty B(x)\,dx$ does. Note that by Riemann's convergence test $\int_1^\infty |B(x)|\,dx \leq \int_1^\infty \frac{dx}{x^\alpha} < \infty$ whenever $\alpha > 1$. Hence, $\int_1^\infty A(x)\,dx$ is absolutely convergent for $\alpha > 1$.

Now suppose that $0 < \alpha \leq 1$. On the intervals $J_k := \left[\frac{\pi}{6}+2k\pi, \frac{\pi}{2}+2k\pi\right]$, $k \geq 1$, we have $|\sin x| \geq \frac{1}{2}$ and $x \geq 1$. Hence $\frac{\sin^p x}{x^\alpha} \geq \frac{2^{-p}}{x} \geq \frac{2^{-p}}{2\pi(k+1)}$. Therefore,

$$\int_{J_k} |B(x)|\,dx \geq \frac{1}{3}\cdot\frac{2^{-p-1}}{k+1}\,.$$

Since $\int_1^\infty |B(x)|\,dx \geq \sum_{k=1}^\infty \int_{J_k} |B(x)|\,dx$, we see that $\int_1^\infty |B(x)|\,dx$ and hence $\int_1^\infty |A(x)|\,dx$ diverges (absolutely) for $0 < \alpha \leq 1$. In particular, $\int_1^\infty A(x)\,dx$ diverges whenever p is even, since in that case $|A(x)| = A(x)$.

To continue, we may thus assume that $p = 2n+1$ is odd. We use that for every $\alpha > 0$ and $n \in \mathbb{N}$ the integral $\int_1^\infty \frac{\sin^{2n+1} x}{x^\alpha}\,dx$ converges. Indeed, let $I_m(x) := \int_1^x \frac{\sin^m t}{t^\alpha}\,dt$ and let F_m be a primitive of $\sin^m t$ with $F_m(1) = 0$. For m odd, F_m is periodic, hence bounded. By partial integration we obtain

$$I_{2n+1}(x) = \frac{F_{2n+1}(x)}{x^\alpha} + \alpha\int_1^x \frac{F_{2n+1}(t)}{t^{\alpha+1}}\,dt\,,$$

and we conclude that $I_{2n+1}(x)$ converges as $x \to \infty$.

Now we use the Taylor development

$$\log(1+u) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k}\, u^k + \frac{(-1)^{m-1}}{m}\, u^m\left(1+\varepsilon(u)\right)\,,$$

where ε is a continuous function of u and $\varepsilon(0) = 0$. In particular, $|\varepsilon(u)| < 1$ whenever $|u| \leq \delta$ with $\delta > 0$ sufficiently small. Now, we set $u = u(x) = \frac{\sin^{2n+1} x}{x^\alpha}$, where $x > 0$ is so large that $|u| \leq \delta$. Then for sufficiently large real numbers $M > N$, we have

$$I := \int_N^M \log\left(1 + \frac{\sin^{2n+1} x}{x^\alpha}\right) dx = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k} \int_N^M \left(\frac{\sin^{2n+1} x}{x^\alpha}\right)^k dx$$
$$+ \frac{(-1)^{m-1}}{m} \int_N^M \left(\frac{\sin^{2n+1} x}{x^\alpha}\right)^m (1 + \varepsilon(u(x)))\, dx =: \sum_{k=1}^{m-1} I_k + \widetilde{I}_m\,.$$

Choosing $m \in \mathbb{N}$ such that $m\alpha > 1$ and $(m-1)\alpha \leq 1$, the boundedness of $\varepsilon(u)$ yields the absolute convergence of the last integral $\widetilde{I}_m$. If $\frac{1}{2} < \alpha \leq 1$, then $m = 2$ and hence $I = I_1 + \widetilde{I}_2$. But I_1 and $\widetilde{I}_2$ converge, and hence I converges. If $0 < \alpha \leq \frac{1}{2}$, then $m \geq 3$ and at least a third integral I_2 above appears. That integral is divergent, since the exponent of the sin is an even one (note that by the choice of m, the exponent of x is still at most 1). Since all those divergent integrals I_{2q} come up with the same sign, we finally get the divergence of $I_1 + I_2 + \cdots + I_{m-1}$, and thus I diverges.

Finally, we note that the example $p = 1$ and $\alpha = \frac{1}{2}$ yields examples of functions f and g such that at infinity, $f \sim g$, but for which $\int_0^\infty f(x)\,dx$ diverges and $\int_0^\infty g(x)\,dx$ converges, namely $f(x) = \log\left(1 + \frac{\sin x}{\sqrt{x}}\right)$ and $g(x) = \frac{\sin x}{\sqrt{x}}$.

11147. *Proposed by Pamela Gorkin, Bucknell University, Lewisburg, PA, and Raymond Mortini, Université Paul Verlaine, Metz, France.* For each nonzero integer n let $a_n = i\pi n/(i\pi n - 1)$, and $a_n^* = 1/\overline{a_n}$. Note that a_n^* is the reflection of a_n in the unit circle. Show that the expression

$$\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-a_n} - \frac{1}{z-a_n^*}\right)$$

converges uniformly on compact subsets of $\mathbb{C}\setminus\{1\}$ to a zero-free meromorphic function.

Solution to problem 11147 AMM 112 (2005), 366 by
Pamela Gorkin, Raymond Mortini

Let $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$ be the atomic inner function. Put

$$f = \frac{1/e - S}{1-(1/e)S}.$$

Then f is an inner function (that is it has radial limts of modulus one almost everywhere). Since f does not have radial limit zero, it must be a pure Blaschke product (see Garnett, p.76), that is

$$f(z) = e^{i\theta} z \prod_{n\in\mathbb{Z}\setminus\{0\}} \frac{|a_n|}{a_n}\,\frac{a_n - z}{1-\overline{a}_n z}.$$

Its zeros are exactly the numbers a_n for $n \in \mathbb{Z}\setminus\{0\}$, including the the origin. Since the derivative of S is $S'(z) = -S(z)\frac{2}{(1-z)^2}$, it follows that the derivative of f does not vanish either. But

$$\frac{S'(z)}{S(z)} = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{z-a_n} - \frac{1}{z-a_n^*}\right).$$

11136. *Proposed by Raymond Mortini, Université de Metz, Metz, France*. Prove that there exists a sequence $\langle\lambda_n\rangle$ of distinct complex numbers in the closed unit disk D and a summable sequence $\langle a_n\rangle$ in ℓ^1 such that, for every continuous function u on D that is harmonic on the interior of D and satisfies $u(0) = 0$,

$$\sum_n a_n u(\lambda_n) = 0.$$

Solution to problem 11136 AMM 112 (2005), 181

Let $D_n = D(\lambda_n, r_n)$ be a sequence of pqirwisw disjoint, closed disks contained in the open unit disk $\mathbb{D}$ such that the area measure of $\mathbb{D} \setminus \bigcup D_n$ is zero. Noticing that by the mean-value area theorem for harmonic functions

$$\int\int_{D(\lambda,r)} u(z)dA(z) = \pi r^2 u(\lambda),$$

we obtain the assertion

$$0 = u(0) = \int\int_{\mathbb{D}} u(z)dA(z) = \sum_n \int\int_{D_n} u(z)dA(z) = \pi \sum_n r_n^2 u(\lambda).$$

Remark The problem was motivated by the question, circulating in England, and communicated to me by Joel F. Feinstein, whether the set of exponentials $\{e^{i\lambda z} : \lambda \in \mathbb{C}\}$ is countably linear independent! The method for the proof above presumably appeared for the first time in a paper of J. Wolff [Comptes Rendus Acad. Sci. Paris 173 (1921), 1056-1058].

11070. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata", Rome, Italy.* Let f and g be two commuting analytic maps from a nonempty open connected set $D \subseteq \mathbb{C}$ into D. Suppose that $z_0 \in D$ is a fixed point of both f and g, and that neither $f'(z_0)$ nor $g'(z_0)$ is a root of unity. Suppose also that there exists an integer $N \geq 1$ such that $f^{(k)}(z_0) = g^{(k)}(z_0) = 0$ for $1 \leq k \leq N-1$, while $f^{(N)}(z_0) = g^{(N)}(z_0) \neq 0$. Prove that the restrictions of f and g to D are equal.

solution of problem 11070, AMM 111 (2004), p. 258

Let $\mathbb{N} = \{1, 2, \cdots\}$ and $f, g \in C^n(\Omega)$. Then the result follows from the following formula:

$$(f \circ g)^{(n)}(z) = \sum_{j=1}^{n} f^{(j)}(g(z)) \Bigg(\sum_{\substack{k \in \mathbb{N}^j \\ |k|=n}} C_k^n \, g^{(k)}(z) \Bigg), \tag{Mo$_n$}$$

where $k = (k_1, k_2, \ldots, k_j) \in \mathbb{N}^j$ is an ordered multi-index with $k_1 \leq k_2 \leq \cdots \leq k_j$, $|k| = \sum_{i=1}^{j} k_i$, $g^{(k)} = g^{(k_1)} g^{(k_2)} \ldots g^{(k_j)}$ and $C_k^n = \frac{1}{\prod_i [A_k(i)!]} \binom{n}{k}$. Here $A_k(i)$ denotes the cardinal of how often i appears within the ordered index k and $\binom{n}{k} = \frac{n!}{k_1! k_2! \ldots k_j!}$.

This formula has many advantages vis-å-vis the Faa di Bruno formula

$$(f \circ g)^{(n)} = \sum \binom{n}{p} (f^{(p)} \circ g) \prod_{j=1}^{n} \left(\frac{g^{(j)}}{j!} \right)^{p_j},$$

where $p_j \in \{0, 1, 2, \cdots\}$, $p = p_1 + p_2 + \cdots + p_n$ and $p_1 + 2p_2 + \cdots + np_n = n$, since one immediately can write down all the factors that occur without solving the above equations for p_j.

Case 1: Let $f(z_0) = g(z_0) = z_0$, $A := f'(z_0) = g'(z_0) \neq 0$, $A^p \neq 1$ $\forall p \in \mathbb{N}$ and $f \circ g = g \circ f$.

In order to show that $f \equiv g$ it is enough to prove that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all n. The proof is done inductively:

$n = 2$: Since $(f \circ g)'' = (f'' \circ g) g'^2 + (f' \circ g) g''$ and $f \circ g = g \circ f$ we get: $f''(z_0) A^2 + A g''(z_0) = g''(z_0) A^2 + A f''(z_0)$. Hence $f''(z_0)(A-1) = g''(z_0)(A-1)$. Since $A \neq 1$ we obtain that $f''(z_0) = g''(z_0)$.

$n \to n+1$:

$$(f \circ g)^{(n+1)} = (f' \circ g) g^{(n+1)} + \sum_{j=2}^{n} (f^{(j)} \circ g) \sum_{\substack{k \in \mathbb{N}^j \\ |k|=n+1}} C_k^{n+1} g^{(k)} + (f^{(n+1)} \circ g)(g')^{n+1}$$

Evaluating at z_0 and noticing that, by induction hypotheses, all derivatives appearing in the middle term coincide at z_0 with those when f is replaced by g, we get that

$$A g^{(n+1)}(z_0) + f^{(n+1)}(z_0) A^{n+1} = A f^{(n+1)}(z_0) + g^{(n+1)}(z_0) A^{n+1}.$$

Hence $f^{(n+1)}(z_0)(A^n - 1) = g^{(n+1)}(z_0)(A^n - 1)$, from which we conclude that $f^{(n+1)}(z_0) = f^{(n+1)}(z_0)$, because $A^n \neq 1$.

Case 2: $f(z_0) = g(z_0) = z_0$, $f^{(j)}(z_0) = g^{(j)}(z_0) = 0$ for $1 \leq j < n_0$, but $f^{(n_0)}(z_0) = g^{(n_0)}(z_0) \neq 0$ and $f \circ g = g \circ f$.

Suppose that $f^{(j)}(z_0) = g^{(j)}(z_0)$ has been shown to be true for $j < n$, where $n = pn_0 + q$, with $0 \leq q < n_0$ and $p \geq 1$. We show that this holds then for $j = n$.

Let $N = n_0^2 + (p-1)n_0 + q$ and consider $(f \circ g)^{(N)}(x_0)$. All the terms in $(\mathrm{Mo})_N$ with $j < n_0$ disappear, since $f^{(j)}(g(z_0)) = f^{(j)}(z_0) = 0$. Moreover, as we are going to show, all other terms, excepted the term for $j = n_0$ and the index $k = (n_0, \cdots, n_0, pn_0 + q) \in \mathbb{N}^{n_0}$, coincide for f and g; hence can be thrown off when regarding the equality $(f \circ g)^{(N)} = (g \circ f)^{(N)}$. Thus that equality is equivalent to

$$f^{(n_0)}(g(z_0))(g^{(n_0)})^{n_0-1}(z_0)g^{(pn_0+q)}(z_0) = g^{(n_0)}(f(z_0))(f^{(n_0)})^{n_0-1}(z_0)f^{(pn_0+q)}(z_0)$$

But this implies of course that $f^{(pn_0+q)}(z_0) = g^{(pn_0+q)}(z_0)$, which is what we were after.

That one can restrict to this single index $k = (n_0, \cdots, n_0, pn_0 + q) \in \mathbb{N}^{n_0}$ is seen as follows: Let $k' \in \mathbb{N}^{n_0}$, be an ordered index with $|k'| = |k| = (n_0 - 1)n_0 + pn_0 + q = N$. Suppose that the last coordinate of k' (which is the maximum) is strictly bigger than the last coordinate of k. Then at least one of the previous coordinates of k' must be strictly smaller than n_0. But the associated derivatives of g (resp f) vanish at z_0. Thus this term does not appear in the formula for $(f \circ g)^{(N)}(z_0)$. On the other hand, if the last coordinate of k' is strictly less than $pn_0 + q$ (hence all of the coordinates of k'), then by induction all the associated derivatives of g (in $(f \circ g)^{(N)}$) coincide with those for f (in $(g \circ f)^{(N)}$) at z_0. Thus these terms can be thrown away.

Now let $k' \in \mathbb{N}^j$ with $n_0 < j \leq N$ and $|k'| = N$. Then the maximum of the coordinates of k' is strictly less than $pn_0 + q$, since otherwise $|k'| \geq (j-1)n_0 + pn_0 + q \geq n_0^2 + pn_0 + q > N$, a contradiction. Thus, as above, also these terms can be thrown away.

10991. *Proposed by Raymond Mortini, Départment de Mathematiques, Université de Metz, Ile du Saucy, France.* For complex $a, z \in \mathbb{D} = \{s: \ |s| < 1\}$, let $F(a, z) = (a+z)/(1+\bar{a}z)$ be a map of $\mathbb{D}$ onto $\mathbb{D}$. Let $\rho(a, b) = |(a-b)/(1-\bar{a}b)|$ be the pseudohyperbolic distance.

(**a**) Prove that there exists a function $C\colon\ \mathbb{D} \to \mathbb{R}^+$ so that $\rho(F(a, z), F(b, z)) \leq C(z)\rho(a, b)$ for every $a, b, z \in \mathbb{D}$.

(**b**) Find the minimal value of $C(z)$ for which this bound holds.

No own solution of problem 10991, AMM 110 (2003), p. 155

10890. *Proposed by Raymond Mortini, Université de Metz, Metz, France.* Let d_1 and d_2 be two metrics on a nonempty set X with the property that every ball in (X, d_1) contains a ball in (X, d_2) and vice versa. Must d_1 and d_2 generate the same topology?

solution of problem 10890, AMM 108 (2001), p. 668

Let d denote the Euclidean metric on $\mathbb{R}$ and let f be an injective real-valued function on $\mathbb{R}$. It is easy to see that the function $\rho(x,y) = |f(x) - f(y)|$ defines a second metric on $\mathbb{R}$, i.e. satisfies the axioms

(D1) $\rho(x,y) \geq 0, \rho(x,y) = 0 \iff x = y$,
(D2) $\rho(x,y) = \rho(y,x)$
(D3) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for all $x, y, z \in \mathbb{R}$.

Let $B_d(x_0, \epsilon)$ resp. $B_\rho(x_0, \epsilon)$ denote the open balls of radius ϵ and center x_0 with respect to the distances d and ρ.

Let us now additionally assume that f is increasing, one-sided continuous but not continuous, and has only a finite number of discontinuities. This guarantees that $I := f(\mathbb{R})$ is a union of non-degenerated intervals, with pairwise disjoint closures. The inverse function $f^{-1} : I \to \mathbb{R}$ then is continuous on I. Fix x_0. Hence for every $\epsilon > 0$ there exists $\delta > 0$ such that $B_\rho(x_0, \delta) \subseteq B_d(x_0, \epsilon)$.

Let x_0 be a point at which f is, say, left-continuous. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x < x_0$, $d(x, x_0) < \delta$ implies $\rho(x, x_0) = |f(x) - f(x_0)| < \epsilon$. Let $x_1 = x_0 - \frac{1}{2}\delta$. Then $B_d(x_1, \delta/2) \subseteq B_\rho(x_0, \epsilon)$.

Thus each ball in the d-metric contains a ball in the ρ-metric, and vice-versa.

It is clear that the identity map id: $(\mathbb{R}, \rho) \to (\mathbb{R}, d)$, although being continuous, has no continuous inverse. Note that id : $(\mathbb{R}, d) \to (\mathbb{R}, \rho)$ is continuous at x_0 if and only if f is continuous at x_0. Thus the two topologies are distinct.

Remark If we additionally assume that (X, d_j) are topological vector spaces, then the answer is yes. This is due to the fact that these topologies can be generated by translation invariant metrics d'_1 and d'_2. In fcat, $\forall \varepsilon > 0\ \exists \delta > 0 : B_{d'_1}(x_0, \delta) \subseteq B_{d'_2}(0, \varepsilon/2)$. In particular, x_0 and $-x_0$ are in $B_{d'_2}(0, \varepsilon/2)$. Hence

$$B_{d'_1}(0, \delta) = -x_0 + B_{d'_1}(x_0, \delta) \subseteq B_{d'_2}(0, \varepsilon/2) + B_{d'_2}(0, \varepsilon/2) \subseteq B_{d'_2}(0, \varepsilon).$$

The problem was alspo solved by Matthias Bueger and Dietmar Voigt (Germany).

10857 [2001,172]. *Proposed by Harold Diamond, University of Illinois, Urbana IL.*
(a) Show that

$$\frac{x+\frac{x^3}{3!}+\cdots+\frac{x^{2n-1}}{(2n-1)!}}{1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}} < \tanh x < \frac{x+\frac{x^3}{3!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}}{1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}}.$$

(b) Show that

$$\frac{x+\frac{x^3}{3!}+\cdots+\frac{x^{2n-1}}{(2n-1)!}+\frac{x^{2n+1}}{2(2n+1)!}}{1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}} < \tanh x < \frac{x+\frac{x^3}{3!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}}{1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}+\frac{x^{2n+2}}{2(2n+2)!}},$$

whenever n is a natural number and $0 < x < 2n$.

solution of problem 10857 (a), AMM 108 (2001), p. 172

Let $C_{2n} = \sum_{j=0}^{n} \frac{x^{2j}}{(2j)!}$ and $S_{2n+1} = \sum_{j=0}^{n} \frac{x^{2j+1}}{(2j+1)!}$. We show that, for every $x > 0$, the sequence $(\frac{S_{2n+1}}{C_{2n}})$ is strictly decreasing, whereas $(\frac{S_{2n-1}}{C_{2n}})$ is stricly increasing. Since both sequences converge to $\tanh x$ we get that $\frac{S_{2n-1}}{C_{2n}} < \tanh < \frac{S_{2n+1}}{C_{2n}}$.

i) We have the following equivalences:

$$\left(\frac{S_{2n+1}}{C_{2n}}\right)\searrow \iff \frac{S_{2n+1}}{S_{2n-1}} < \frac{C_{2n}}{C_{2n-2}} \iff \frac{S_{2n-1}+\frac{x^{2n+1}}{(2n+1)!}}{S_{2n-1}} < \frac{C_{2n-2}+\frac{x^{2n}}{(2n)!}}{C_{2n-2}} \iff$$

$$\iff 1+\frac{\frac{x^{2n+1}}{(2n+1)!}}{S_{2n-1}} < 1+\frac{\frac{x^{2n}}{(2n)!}}{C_{2n-2}} \iff xC_{2n-2} < (2n+1)S_{2n-1} \iff$$

$$\sum_{j=0}^{n-1} \frac{x^{2j+1}}{(2j)!} < (2n+1)\sum_{j=0}^{n-1} \frac{x^{2j+1}}{(2j+1)!} \tag{1}$$

But $\frac{1}{(2j)!} < (2n+1)\frac{1}{(2j+1)!} \iff 2j+1 < 2n+1$, which is true. Since $x > 0$ we get (1).

ii) That $(\frac{S_{2n-1}}{C_{2n}})$ is stricly increasing, is shown in exactly the same way.

To sum up, we get

$$\frac{C_{2n+2}}{C_{2n}} < \frac{S_{2n+1}}{S_{2n-1}} < \frac{C_{2n}}{C_{2n-2}}.$$

Continuous Additive Functions.

10854 [2001,171]. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, China.* Find every function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at zero and satisfies

$$f(x+2f(y)) = f(x) + y + f(y)$$

for all real numbers x and y.

solution of problem 10854 AMM 108 (2001), p. 171

Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function, continuous at the origin, and satisfying

$$f(x+2f(y)) = f(x) + f(y) + y \tag{1}$$

for all $x, y \in \mathbf{R}$. First, we shall show that f is continuous everywhere. In fact,

$$f\big(x+2f(x+2f(y))\big) = f(x+2[f(x)+f(y)+y]) = f\big([x+2y+2f(y)]+2f(x)\big) =$$
$$= f\big((x+2y)+2f(y)\big) + f(x) + x = f(x+2y) + f(y) + y + f(x) + x. \tag{2}$$

On the other hand:

$$f\big(x+2f(x+2f(y))\big) = f(x) + f(x+2f(y)) + x + 2f(y) =$$
$$= f(x) + [f(x)+f(y)+y] + x + 2f(y) = 2f(x) + 3f(y) + y + x. \tag{3}$$

By (2) and (3) we get that $f(x+2y) = f(x) + 2f(y)\ \forall (x,y) \in \mathbf{R}^2$.

In particular, by setting $x = y = 0$, we see that $f(0) = 0$.
It easily follows that f is continuous at every point $x \in \mathbf{R}$.

So, in order to continue, we may assume that f is a continuous solution of (1).
Let $x = y$. Then

$$f(y+2f(y)) = y + 2f(y). \tag{4}$$

First we shall determine all continuous solutions of (4). Let $g(y) = y + 2f(y)$. Since g is continuous, $g(\mathbf{R})$ is either a singleton or a nondegenerate interval I. If g is constant, say $g \equiv c$, then $f(y) = \frac{c-y}{2}$ and so $c = f(y+2f(y)) = f(c)$, from which we conclude that $c = 0$. Hence $f(y) = -\frac{y}{2}$. If g is not constant, take $z \in I$; that is $y + 2f(y) = g(y) = z$ for some y. Then $f(z) = z$. Hence f is the identity on I. It follows that $3z = z + 2f(z) = f(z+2f(z)) = g(z)$. Therefore $3z \in I$ and so $I = \langle m, \infty[$ for some $m \in \mathbf{R} \cup \{-\infty\}$. Thus $f(x) = x$ for every $x > m$. Since $g \geq m$, we have that $f \geq \frac{m-y}{2}$ on $]-\infty, m]$.

To prove the converse, choose $m \in \mathbf{R}$. Let f^* be any continuous function on $]-\infty, m]$ such that $f^*(y) \geq \frac{m-y}{2}$ for $y \leq m$ and so that $f^*(m) = m$. Then

$$\widetilde{f}(y) = \begin{cases} f^*(y) & \text{if } y \leq m \\ y & \text{if } y \geq m \end{cases} \tag{5}$$

is a continuous solution of (4).

We deduce that any continuous solution of (1) necessarily has the form (5) or equals $-\frac{1}{2}y$. We shall now show that only for $f^* = id$, we really get a solution of (1).

So let f be a continuous solution of (1). Then $f = \widetilde{f}$ for some f^*. Fix $x < m$. Take $y > m$ so that $x + 2y > m$. Then
$f(x + 2f(y)) = f(x + 2y) = x + 2y$ and $f(x) + y + f(y) = f^*(x) + 2y$. Hence (1) implies that $f^*(x) = x$.

We conclude that f is a continuous solution of (1) if and only if $f(x) = x$ or $f(x) = -\frac{x}{2}$ on $\mathbf{R}$.

10768. ***Proposed by Sung Soo Kim, Hanyang University, Ansan, Kyunggi, Korea.***
(a) Show that there is a continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f + g$ is not increasing for any differentiable function g.
(b) Show that there is a differentiable function $f: \mathbb{R} \to \mathbb{R}$ such that $f + g$ is not increasing for any continuously differentiable function g.
(c) Show that, for any continuously differentiable function $f: \mathbb{R} \to \mathbb{R}$, there is a real analytic function g such that $f + g$ is increasing.

solution of problem 10768 AMM 106 (1999), 963

a) Let $f(x) = \sqrt{|x|}\sin\dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is continuous on $\mathbb{R}$. Let g be a differentiable function on $\mathbb{R}$. Then, in every neigborhood of 0, $h := f + g - g(0)$ takes negative and positive values. In fact, suppose that $h \geq 0$ on $[0, \varepsilon]$. Then $\frac{h(x)}{x} \geq 0$ on $[0, \varepsilon]$. But $\liminf_{x\to 0^+} \frac{h(x)}{x} = g'(0) + \liminf_{x\to 0^+} \frac{1}{\sqrt{x}} \sin\frac{1}{x} = -\infty$, a contradiction. Thus $f + g$ is not monotone on any interval centered at 0.

b) Let $f(x) = x^2 \sin\dfrac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $\mathbb{R}$, $f'(0) = 0$, but $f'(x) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$ takes arbitrarily large negative and positive values in any neighborhood U of 0. Let g be any $C^1(\mathbb{R})$ function. In particular, g' is bounded on every compact interval centered at 0. Hence $f' + g'$ takes arbitrary large negative and positive values in U. Thus $f + g$ is not monotone on any interval centered at 0.

c) We show that for every function $f \in C^1(\mathbb{R})$ there exists an entire function g (that is a function holomorphic on the whole plane), real-valued on $\mathbb{R}$, such that $f + g$ is increasing on $\mathbb{R}$. In fact, $f' + 2|f'| + 2\varepsilon \geq 2\varepsilon > 0$ on $\mathbb{R}$. Let $q = 2|f'| + 2\varepsilon$. Then q is continuous on $\mathbb{R}$. By Carleman's theorem (see [C] and [G], p. 125), there exists an entire function Q such that $\|q - Q\|_\infty \leq \varepsilon$, where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{R}$. Let $G(x) = \operatorname{Re} Q(x)$. Then $\|q - G\| \leq \varepsilon$. Moreover, the function $H(z) = \frac{1}{2}\big(Q(z) + \overline{Q(\overline{z})}\big)$ is analytic in $\mathbb{C}$, and H coincides on $\mathbb{R}$ with G.

Now it is easy to check that $f' + G \geq \varepsilon > 0$. Let g be a primitive of G. Then g is the trace of an entire function and $f + g$ is (strictly) increasing, since its derivative is strictly positive.

References

[G] Dieter Gaier, *Approximation im Komplexen*, Birkhæuser-Verlag, Basel, 1980.

[C] Carleman, T,: *sur un théorème de Weierstrass*, Ark. Mat. Astronom. Fys., 20B (1927), 1-5.

10747. *Proposed by Athanasios Kalakos, Athens, Greece.* **Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that are twice differentiable on an open interval containing 0, have exactly one real root, satisfy $f(1) = 1$, and satisfy $f'(f(t)) = 2f(t)$ for every $t \in \mathbb{R}$.**

solution of problem 10747 AMM 106 (1999), p. 685

We claim that all differentiable solutions f of $f'(f(t)) = 2f(t)$, $t \in \mathbb{R}, f(1) = 1$, and having only one real root, have the form $f(t) = t^2$ for $t \geq 0$ and $f(t) = g(t)$ for $t < 0$, where g is an arbitrary differentiable function, defined on $]-\infty, 0]$ satisfying $g(t) > 0$ for $t < 0$ and $g(0) = g'(0) = 0$. The assumption, that f should be twice differentiable in a neighboorhood of 0, is not important.

Proof Let f be a solution of the problem. Put $h = f \circ f - f^2$. Then $h' = (f' \circ f)f' - 2f'f = f'(f' \circ f - 2f) \equiv 0$. Hence h is a constant, say C. Because $h(1) = 0$, we see that $C = 0$ and so $f \circ f = f^2$. Let $y \in f(\mathbb{R})$. Then $f(x) = y$ for some $x \in \mathbb{R}$. Therefore $f(y) = f(f(x)) = f^2(x) = y^2$. By hypothesis, $\{0, 1\} \subseteq f(\mathbb{R})$. By continuity we conclude that $[0, 1] \subseteq f(\mathbb{R})$. Since the left derivative at $x = 1$ is 2, the differentiability of f now implies that there exists points x_0 greater than 1 for which $f(x_0) > f(1) = 1$. Since $f_{n+1} = f^{2^n}$, we obtain that $f_{n+1}(x_0) = [f(x_0)]^{2^n} \to \infty$. Hence f is unbounded. By the intermediate value theorem, we then get that $[0, \infty] \subseteq f(\mathbb{R})$. Hence $f(x) = x^2$ for $x \geq 0$.

To determine the behaviour of f for negative values, we use the hypothesis that f should have only one zero. Since $f(0) = 0$, by continuity, we conclude that either $f(x) < 0$ for all $x < 0$ or $f(x) > 0$ for all $x < 0$. But $f(x_0) < 0$ for some (all) $x_0 < 0$ implies that $f(f(x_0)) = f^2(x_0) > 0$, a contradiction. Thus $f(x) > 0$ for $x > 0$.

It is easy to check that every function of the form $f(x) = x^2$ for $x \geq 0$ and $f(x) = g(x)$ for $x < 0$, where $g > 0$ is differentiable and satisfies $g(0) = g'(0) = 0$, is a solution of $f \circ f = f^2$. Hence, by differentiating, $f'(f(x))f'(x) = 2f'(x)f(x)$. If $f'(x) \neq 0$, then we are done. If $f'(x_0) = g'(x_0) = 0$ for some $x_0 < 0$, then we use the fact that $y := f(x_0) > 0$ and that for these positive values $f(y) = y^2$. Hence, $f'(f(x_0)) = 2f(x_0)$. So we obtain a solution of our functional equation.

10739. *Proposed by Oscar Ciaurri, Logroño, Spain.* **Suppose that $f\colon [0,1] \to \mathbb{R}$ has a continuous second derivative with $f''(x) > 0$ on $(0,1)$, and suppose that $f(0) = 0$. Choose $a \in (0,1)$ such that $f'(a) < f(1)$. Show that there is a unique $b \in (a,1)$ such that $f'(a) = f(b)/b$.**

solution of problem 10739 AMM 106 (1999), p. 586

Let $H(x) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$. Since $f''(x) > 0$, the function f is strictly convex and both its derivative and the quotient H are strictly increasing (see e.g. W. Walter, Analysis 1, Springer-Verlag, p. 303). Moreover, H is continuous on $]0,1]$. Note that $H(1) = f(1)$ and that $H(0) := \lim_{x\to 0} f'(x)$ exists in $[-\infty, f(1)]$. Hence, by the intermediate value theorem, there exists for every value w with $H(0) < w < H(1)$ a point $b \in]0,1[$ with $H(b) = w$. Now choose $a \in]0,1[$ such that $w := f'(a)$ satisfies $H(0) < w < H(1)$ (such a choice obviously is possible). Thus there exists $b \in]0,1[$ so that $\frac{f(b)}{b} = H(b) = f'(a)$. Choose $x_a \in]0,a[$ so that $H(a) = f'(x_a)$. Due to the monotonicity of f' we obtain: $H(a) = f'(x_a) < f'(a) = H(b)$. Since H is monotone, b is unique and satisfies $a < b < 1$.

10697. ***Proposed by José L. Diaz, Universitat Politècnica de Catalunya, Terrassa, Spain.***
Given n distinct nonzero complex numbers $z_1, z_2, \ldots, z_n$, show that

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n}.$$

solution of problem 10697 AMM 105 (1998), p. 955

This is nothing but a Lagrange interpolatory argument:

In fact let $w_1, \cdots, w_n \in \mathbb{C}$. Then

$$p(z) = \sum_{k=1}^{n} w_k \frac{\prod_{j=1,j\neq k}^{n}(z - z_j)}{\prod_{j=1,j\neq k}^{n}(z_k - z_j)}$$

is the unique polynomial of degree at most $n-1$ satisfying $p(z_k) = w_k,\ k = 1, \cdots, n$. Now choose $w_k = 1$ for every k. Since $q(z) \equiv 1$ satisfies the interpolation $q(z_k) = w_k$, we obtain from uniqueness that $q = p$. Let $z = 0$. Then

$$1 = q(0) = \sum_{k=1}^{n} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{(-z_j)}{z_k - z_j} = (-1)^{n-1} \sum_{k=1}^{n} \frac{\prod_{j=1,j\neq k}^{n} z_j}{\prod_{j=1,j\neq k}^{n}(z_k - z_j)}.$$

Dividing by $\prod_{j=1}^{n} z_j$, yields the assertion

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n-1}}{\prod_{j=1}^{n} z_j}.$$

10651. *Proposed by W. K. Hayman, Imperial College, London, U.K..* If u_1 and u_2 are nonconstant real functions of two variables, and if u_1, u_2, and u_1u_2 are all harmonic in a simply connected plane domain D, prove that $u_2 = av_1 + b$, where v_1 is a harmonic conjugate of u_1 in D, and a and b are real constants.

solution of problem 10651 AMM 105 (1998), p. 271

We prove a stronger version than in the formulated problem.

Proposition 1 *Let u and v be two non constant harmonic functions on a domain $D \subseteq \mathbb{C}$. Suppose that uv is harmonic. Then u has an harmonic conjugate $\tilde{u}$ on D and there are constants $a, b \in \mathbb{R}$ such that*

$$v = a\tilde{u} + b. \tag{1}$$

Remarks. (1) If u is a constant, then (1) is not true (because v may be chosen to be any harmonic function).

(2) If v is a constant then (1) is true for $a = 0$, provided a harmonic conjugate exists. A well known sufficient condition for the existence of a harmonic conjugate being that D is simply connected.

(3) Of course, if v is any harmonic function satisfying (1), then uv is harmonic.

Solution Let Δ be the Laplace operator. Because $\Delta u = \Delta v = 0$ we obtain:

$$0 = \Delta(uv) = \big(u_{xx}v + 2u_xv_x + v_{xx}\big) + \big(u_{yy}v + 2u_yv_y + v_{yy}\big) = 2\big(u_xv_x + u_yv_y\big).$$

Let $f = u_x - iu_y$ and $g = v_x - iv_y$. The harmonicity of u and v imply that f and g satisfy the Cauchy-Riemann differential equations; hence f and g are holomorphic. It is easy to see that $\operatorname{Re} f\overline{g} = u_xv_x + u_yv_y$. Thus $\operatorname{Re} f\overline{g} \equiv 0$ on D.

Let $Z(g) = \{z \in D : g(z) = 0\}$ denote the zero set of g. It is a discrete subset of D provided that $g \not\equiv 0$. Since v is assumed not to be a constant, we see that $g \not\equiv 0$. Then on $D \setminus Z(g)$ we have $\operatorname{Re} \dfrac{f}{g} = \operatorname{Re} \dfrac{f\overline{g}}{|g|^2}$. Thus $\operatorname{Re} \dfrac{f}{g} \equiv 0$ on $D \setminus Z(g)$. This implies, in view of the analyticity, that $\frac{f}{g}$ is a pure imaginary constant, say $\frac{f}{g} \equiv i\lambda$ on $D \setminus Z(g)$. Hence $f = i\lambda g$ on D. The definitions of f and g now yield that $u_x = \lambda v_y$ and $u_y = -\lambda v_x$. Consequently, by the Cauchy-Riemann equations, the function $u + i\lambda v$ is holomorphic on D. In particular, u has an harmonic conjugate on D. (Note that we do not have assumed that D is simply connected.) Thus, for any other harmonic conjugate $\tilde{u}$ of u, we have $\lambda v = \tilde{u} + c$ for some constant $c \in \mathbb{R}$. Note that u not constant implies that $\lambda \neq 0$. Thus v has the desired form (1).

A natural question now is the following. Let u and v be two harmonic functions on a domain $D \subseteq \mathbb{C}$. Then $(u + iv)^2 = u^2 - v^2 + 2iuv$. Assume that $u^2 - v^2$ is harmonic. What can be said for v? We have the following result:

Proposition 2 *Assume that u, v and $u^2 - v^2$ are harmonic in a simply connected domain $D \subseteq C$. Then there exists $a \in \mathbb{R}$ and $\theta \in [0, 2\pi[$ such that*

$$v = \cos\theta\, u - \sin\theta\, \tilde{u} + a. \tag{2}$$

Conversely, every function v satisfying (2) for a harmonic function u has the property that $u^2 - v^2$ is harmonic.

Proof Because $\Delta u = \Delta v = 0$ we obtain:

$$0 = \Delta(u^2 - v^2) = 2\Big(u_x^2 + u_y^2 - (v_x^2 + v_y^2)\Big).$$

Hence $u_x^2 + u_y^2 = v_x^2 + v_y^2$. Again, let $f = u_x - iu_y$ and $g = v_x - iv_y$. As above, f and g are holomorphic on D. Moreover $|f|^2 = |g|^2$. Thus g is a rotation of f, say $g = e^{i\theta} f$.

Let $z_0 \in D$. Since D is simply connected, u and v have harmonic conjugates $\tilde{u}$ and $\tilde{v}$ respectively, satisfying $\tilde{u}(z_0) = \tilde{v}(z_0) = 0$. Let $F = u + i\tilde{u}$ and $G = v + i\tilde{v}$. Then, by Cauchy-Rieman, $F' = u_x + i\tilde{u}_x = u_x - iu_y = f$. Similiarly $G' = g$. Thus $G = e^{i\theta} F + c$ for some constant $c \in \mathbb{C}$. Taking real parts yields

$$v = \cos\theta\, u - \sin\theta\, \tilde{u} + a$$

for some real constant a. The converse is easy to check.

The above results are related to the following more general result:

Proposition 3. *Let h be an entire function and let $u : D \to \mathbb{R}$ and $v : D \to \mathbb{R}$ be two nonconstant harmonic functions in a simply connected domain D. Let $\tilde{u}$ be a harmonic conjugate of u in D. Then $h(u + iv) : D \to \mathbb{C}$ is harmonic if and only if $v = \pm\tilde{u} + a$ for a constant $a \in \mathbb{R}$.*

Proof Since h is holomorphic, we have, by Cauchy-Riemann, $h_y = ih_x$ and $h_x = h'$. Hence $h_{xx} = h'', h_{xy} = h_{yx} = ih''$ and $h_{yy} = -h''$. As above, let $f = u_x - iu_y$ and $g = v_x - iv_y$.Then

$$\Delta[h \circ (u + iv)] = h'' \circ (u + iv) \cdot [(|f|^2 - |g|^2) + 2i\mathrm{Re}\, f\overline{g}].$$

Obviously $h'' \circ q \not\equiv 0$ for any nonconstant continuous function q. Hence $h(u + iv)$ is harmonic if and only if $|f| = |g|$ and Re $f\overline{g} = 0$. By the paragraphs above we conclude that $f = i\lambda g$ for some $\lambda \in \mathbb{R}$. Hence $|\lambda| = 1$. Thus $u_x = \pm v_y$ and $-u_y = \pm v_x$. So v or $-v$ is a harmonic conjugate of u in D. Therefore $v = \pm\tilde{u} + a$.

To prove the converse, we have simply to note that the composition of a holomorphic function with a holomorphic or anti-holomorphic function is harmonic.

10638. *Proposed by Brian Conolly, Cambridge, UK.* For $0 \leq \lambda \leq 1$ and $m \geq 0$, let $S_m(\lambda) = \sum_{n\geq 1} e^{-\lambda n}(\lambda n)^{n-m}/n!$. Show that $S_0(\lambda) = \lambda/(1-\lambda)$, $S_1(\lambda) = 1$, $S_2(\lambda) = 1/\lambda - 1/2$, and $S_3(\lambda) = 1/\lambda^2 - 3/(4\lambda) + 1/6$.

solution of problem 10638 AMM 105 (1998), p. 69

In the following we present a solution to problem number 10638. We shall not only compute the functions $S_0, \cdots, S_3$, but we will give an explicit value for all $m \in \mathbb{N}$. To this end we need the following Lemma.

Lemma *Let $f(z) = ze^z$. Then f is invertible in a neighborhood of the origin in $\mathbb{C}$ and the inverse function has the Taylor representation*

$$f^{-1}(w) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-1)^{n-1}w^n,$$

which converges for $|w| < \frac{1}{e}$.

Proof By the residue theorem it is easy to see that whenever f is holomorphic and injective in a disque $D \subseteq \mathbb{C}$ (or even a simply connected domain), then

$$f^{-1}(w)n(\Gamma, f^{-1}(w)) = \frac{1}{2\pi i}\int_{\Gamma} z\,\frac{f'(z)}{f(z)-w}\,dz,$$

where Γ is an arbitrary cycle (=finite union of closed, piecewise C^1-curves) in D.

Applying this formula for $f(z) = ze^z$ and the disk $|z| < 2\delta$, δ small enough, we obtain :

$$\frac{d^n}{(dw)^n}f^{-1}(w) = \frac{n!}{2\pi i}\int_{|z|=\delta} z\,\frac{(z+1)e^z}{(ze^z-w)^{n+1}}\,dz.$$

Thus, for the power series $f^{-1}(w) = \sum_{n=0}^{\infty} a_n w^n$ we have $a_0 = 0$ and for $n \geq 1$:

$$a_n = \frac{1}{2\pi i}\int_{|z|=\delta} \frac{z+1}{z^n}e^{-nz}\,dz = \frac{1}{2\pi i}\sum_{k=0}^{\infty}(-1)^k \int_{|z|=\delta} \frac{z(nz)^k + (nz)^k}{z^n k!}\,dz = (-1)^{n-1}\frac{n^{n-1}}{n!}.$$

By d'Alembert's rule it is easy to check that the radius of convergence is $1/e$. ○

Proposition *For $0 < \lambda < 1$ and $m \in \mathbb{Z}$, let $g_m(\lambda) = \lambda^m S_m(\lambda)$, where*

$$S_m(\lambda) = \sum_{n=1}^{\infty} e^{-\lambda n}(\lambda n)^{n-m}/n!. \tag{1}$$

Then, for $m \in \{1, 2, \cdots\}$, g_m is a polynomial of degree m vanishing at the origin, say $g_m(\lambda) = -\sum_{n=1}^{m} b_{n,m}(-\lambda)^n$, and the coefficients $b_{n,m}$ are given by the recurrence relation

$$b_{n,m} = \frac{1}{n}(b_{n,m-1} + b_{n-1,m-1}), \quad b_{1,1} = 1. \tag{3}$$

Solving these difference equations yields

$$b_{n,m} = \sum_{j=1}^{n} \frac{1}{n!} \binom{n}{j} (-1)^{j-1} \left(\frac{1}{j}\right)^{m-n}. \tag{4}$$

Proof We note that, by Stirling's formula, the series $g_m(l)$ converges locally uniformly in $0 \leq l < 1$, but does not converge whenever $\lambda = 1$ and $m = 0$. Note that $g_m(0) = 0$. Due to local uniform convergence, it is easy to see that, in order to obtain $g'_m(l)$, one can differentiate the series for g_m term by term. This yields that for $m \in \mathbb{Z}$

$$g'_m(\lambda) = \frac{1-\lambda}{\lambda} g_{m-1}. \tag{5}$$

Later we shall show that $g_1(\lambda) = \lambda$. Hence, by induction on (5), it is clear that for $m = 1, 2, \cdots$ the function g_m is a polynomial vanishing at the origin, say $g_m(\lambda) = -\sum_{n=1}^{m} b_{n,m}(-\lambda)^n$. If we let $x = -\lambda$, then we obtain $\sum_{n=1}^{m} n b_{n,m} x^n = (1+x) \sum_{n=1}^{m} b_{n,m-1} x^n$. Comparing coefficients, finally yields (3).

This difference equation can be solved by the usual methods. May be Maple or Mathematica gives the solution. In any case, by the uniqueness of the solution, it suffices to show that (4) verifies the difference equation. Note also, that for $n > m$, the $b_{n,m}$ in (4) are 0. This follows from the fact that the p-th difference operator $D^p(a_n) = \sum_{j=0}^{p} \binom{n}{j} (-1)^j a_{n-j}$ vanishes identically whenever a_n is a polynomial (in n) of degree strictly less than p.

For the readers convenience, here are the coefficients for $m = 1, \cdots, 5$:

1				
1	$\frac{1}{2}$			
1	$\frac{3}{4}$	$\frac{1}{6}$		
1	$\frac{7}{8}$	$\frac{11}{36}$	$\frac{1}{24}$	
1	$\frac{15}{16}$	$\frac{85}{216}$	$\frac{25}{288}$	$\frac{1}{120}$

The case m=1 In that case we have

$$g_1(\lambda) = \lambda \sum_{n=1}^{\infty} e^{-\lambda n} (\lambda n)^{n-1}/n! = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} (\lambda e^{-\lambda})^n.$$

Let $w = -\lambda e^{-\lambda}$. Now, for $w \in \mathbb{C}, |w| < \frac{1}{e}$, the function $h(w) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} (-1)^{n-1} w^n$ is, by Lemma 1, nothing but the inverse function of the holomorphic function $f(z) = ze^z$, $|z| < \delta$ for sufficiently small $\delta > 0$. Thus $g_1(\lambda) = \lambda$.

The case m=0 By (5) we see that $1 = g'_1(\lambda) = \frac{1-\lambda}{\lambda} g_0(\lambda)$. Hence, $g_0(\lambda) = \frac{\lambda}{1-\lambda}$.

Using (5) it is also easy to derive, inductively, the values of g_m for negative integers m. For example we get:

$$g_{-1}(\lambda) = \frac{\lambda}{(1-\lambda)^3}, \qquad g_{-2}(\lambda) = \frac{\lambda}{(1-\lambda)^5}(1+2\lambda), \qquad g_{-3}(\lambda) = \frac{\lambda}{(1-\lambda)^7}(1+8\lambda+6\lambda^2).$$

In general, one can convince oneself that for $m \in \mathbb{Z}, m < 0$, $g_m(\lambda)$ has the form $g_m(\lambda) = \frac{\lambda}{(1-\lambda)^{-2m+1}} Q_m(\lambda)$, where Q is a polynomial of degree $-m-1$ with value 1 at the origin and satisfying the differential equations

$$Q_{m-1}(\lambda) = \lambda(1-\lambda)Q'_m(\lambda) + (1-2m\lambda)Q_m(\lambda).$$

Due to lack of time we were not able to solve this explicitely. May be Maple and Mathematica will be helpfull.

10624. ***Proposed by William F. Trench, Trinity University, San Antonio TX.*** **Suppose that $a_0 > a_1 > a_2 > \cdots$ and $\lim_{n\to\infty} a_n = 0$. Define**

$$S_n = \sum_{j=n}^{\infty}(-1)^{j-n}a_j = a_n - a_{n+1} + a_{n+2} - \cdots.$$

Show that $\sum a_n S_n < \infty$ if and only if $\sum a_n^2 < \infty$.

solution of problem 10624 AMM 104 (1997), p. 871

By Leibniz's criteria, we know that S_n actually converges and that $S_n \geq 0$ for every $n \in \mathbb{N}$. Since $S_n = a_n - S_{n+1}$, we see that $S_n \leq a_n$ and so $\sum a_n S_n \leq \sum a_n^2$. Thus the convergence of $\sum a_n^2$ implies the convergence of $\sum a_n S_n$.

Now assume that $\sum a_n S_n = \sum (S_n + S_{n+1})S_n \quad (1)$ converges. Since all the terms are positive, we deduce the convergence of the sums $\sum S_n^2$ and $\sum S_{n+1}S_n$. A shift of the variable yields that $\sum S_{n+1}^2$ converges. Hence $\sum S_{n+1}(S_{n+1} + S_n) \quad (2)$ converges. Summing (1) and (2) yields that $\sum a_n^2 = \sum (S_{n+1} + S_n)^2 = \sum (S_n + S_{n+1})S_n + \sum S_{n+1}(S_{n+1} + S_n)$ is convergent.

10605. *Proposed by Jonathan M. Borwein and C. G. Pinner, Simon Fraser University, Burnaby, BC, Canada.* Let r and m be positive integers and define

$$P_r(m) = \prod_{n \neq m} \frac{n^r - m^r}{n^r + m^r}.$$

(a) Show that $P_1(m) = 0$ and that

$$P_3(m) = (-1)^{m+1}\frac{2}{3}(m!)^2 \prod_{n=1}^{m} \frac{n+m}{n^3+m^3}.$$

(b) Show that $P_2(m) = (-1)^{m+1}\pi m/\sinh(\pi m)$ and that, more generally, $P_{2s}(m)$ is given by

$$(-1)^{m+1}\frac{2^\epsilon m\pi}{s}(\sinh m\pi)^{(-1)^s}\prod_{j=1}^{s-1}\left(\cosh\left(2\pi m \sin\left(\frac{j\pi}{2s}\right)\right) - \cos\left(2\pi m\cos\left(\frac{j\pi}{2s}\right)\right)\right)^{(-1)^j}$$

where $\epsilon = (1+(-1)^s)/2$.

solution of problem 10605 (b) AMM 104 (1997), p. 567

Write $\dfrac{n^{2s}-m^{2s}}{n^{2s}+m^{2s}} = \dfrac{1-(m/n)^{2s}}{1+(m/n)^{2s}}$. Let $y = \left(\dfrac{m}{n}\right)^2$. Then

$$\frac{1-y^s}{1+y^s} = \frac{\prod_{j=0}^{s-1}\left(1 - y\exp(-i\frac{2\pi j}{s})\right)}{\prod_{j=0}^{s-1}\left(1-y\exp(-i\frac{\pi+2\pi j}{s})\right)}.$$

Since $\varepsilon \in \mathbb{C}$ is an s-root of 1 [resp. (-1)] if and only if $\overline{\varepsilon}$ is an s-root, we obtain:

$$\frac{1-y^s}{1+y^s} = \frac{(1-y)(1+y)\prod_{j=1}^{p-1}\left|1-y\exp(-i\frac{2\pi j}{s})\right|^2}{\prod_{j=1}^{p-1}\left|1-y\exp(-i\frac{\pi(2j+1)}{s})\right|^2}$$

if $s = 2p$ and

$$\frac{1-y^s}{1+y^s} = \frac{(1-y)\prod_{j=1}^{p}\left|1-y\exp(-i\frac{2\pi j}{s})\right|^2}{(1+y)\prod_{j=1}^{p-1}\left|1-y\exp(-i\frac{\pi(2j+1)}{s})\right|^2}$$

if $s = 2p+1$.

This can be written by a single formula:

$$\frac{1-y^s}{1+y^s} = (1-y)(1+y)^{(-1)^s}\prod_{k=1}^{s-1}\left|1-y\exp(-i\frac{\pi k}{s})\right|^{2(-1)^k}. \tag{1}$$

In particular

$$\prod_{k=1}^{s-1}\left|1-exp(-i\frac{\pi k}{s})\right|^{2(-1)^k} = \lim_{y\to 1}\frac{\frac{1-y^s}{1+y^s}}{(1-y)(1+y)^{(-1)^s}} = \frac{s}{2\cdot 2^{(-1)^s}}. \tag{2}$$

It is easy to check that

$$P := \prod_{k=1}^{s-1} (2\pi^2 m^2)^{(-1)^k} = \begin{cases} 1, & \text{if } s \text{ is odd} \\ \frac{1}{2\pi^2 m^2}, & \text{if } s \text{ is even.} \end{cases} \tag{3}$$

Now use the infinite product representation of the function $\sin \pi z$. This gives:

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

and

$$\prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right) = \frac{\sin i\pi z}{i\pi z} = \frac{\sinh \pi z}{\pi z}.$$

Moreover we have by de l'Hôpital's rule that

$$\prod_{n \neq m}^{\infty} \left(1 - \frac{m^2}{n^2}\right) = \lim_{z \to m} \frac{\sin \pi z}{\pi z} \Big/ 1 - \left(\frac{z}{m}\right)^2 = \frac{(-1)^{m+1}}{2}.$$

Finally we need that $|\sin z|^2 = \frac{1}{2}(\cosh 2y - \cos 2x)$ for $z = x + iy$.
Put all this together to get from (1)

$$2^{(-1)^s} \prod_{n \neq m} \frac{n^{2s} - m^{2s}}{n^{2s} + m^{2s}} = \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \cdot \prod_{k=1}^{s-1} \left| \prod_{n \neq m} \left(1 - \left(\frac{m}{n} exp(-i\frac{\pi k}{2s})\right)^2\right) \right|^{2(-1)^k} =$$

$$= \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \prod_{k=1}^{s-1} \left| \frac{\sin\big(\pi m \, \exp(-i\frac{\pi k}{2s})\big)}{\pi m \left(1 - \exp(-i\frac{\pi k}{s})\right)} \right|^{2(-1)^k} =$$

$$= \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \frac{\prod_{k=1}^{s-1} \left[\frac{1}{2}\left(\cosh\left(2\pi m \sin \frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos \frac{\pi k}{2s}\right)\right)\right]^{(-1)^k}}{\prod_{k=1}^{s-1} \left|1 - \exp(-i\frac{\pi k}{s})\right|^{2(-1)^k} \cdot \prod_{k=1}^{s-1} (\pi m)^{2(-1)^k}} =$$

$$= \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \frac{\prod_{k=1}^{s-1} \left[\cosh\left(2\pi m \sin \frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos \frac{\pi k}{2s}\right)\right]^{(-1)^k}}{\prod_{k=1}^{s-1} \left|1 - \exp(-i\frac{\pi k}{s})\right|^{2(-1)^k} \cdot \prod_{k=1}^{s-1} (2\pi^2 m^2)^{(-1)^k}} =$$

$$= \frac{(-1)^{m+1} (\sinh \pi m)^{(-1)^s} \prod_{k=1}^{s-1} \left[\cosh\left(2\pi m \sin \frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos \frac{\pi k}{2s}\right)\right]^{(-1)^k}}{(\pi m)^{(-1)^s} \frac{s}{2^{(-1)^s}} \cdot P}$$

Clearly

$$\frac{1}{(\pi m)^{(-1)^s}\, s \cdot P} = 2^{(1+(-1)^s)/2} \left(\frac{\pi m}{s}\right).$$

Putting $\varepsilon = (1 + (-1)^s)/2$, we get the final equality:

$$P_{2s} = \prod_{n \neq m} \frac{n^{2s} - m^{2s}}{n^{2s} + m^{2s}} =$$

$$= (-1)^{m+1} \frac{2^{\varepsilon} m\pi}{s} (\sinh \pi m)^{(-1)^s} \prod_{k=1}^{s-1} \left[\cosh\left(2\pi m \sin \frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos \frac{\pi k}{2s}\right) \right]^{(-1)^k}.$$

If $s = 1$ we interpret the empty product as 1. This gives

$$P_2(m) = (-1)^{m+1} \pi m / \sinh(\pi m).$$

10588. ***Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.*** **Show that**

$$\prod_{j\geq 1} e^{-1/j}\left(1+\frac{1}{j}+\frac{1}{2j^2}\right)=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}},$$

where γ is Euler's constant.

solution of problem 10588/10595 AMM 104 (1997), p. 456

We show that

$$P=\prod_{n=1}^{\infty} e^{-\frac{1}{n}}\left(1+\frac{1}{n}+\frac{1}{2n^2}\right)=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}}.$$

Let

$$\Gamma(z)=\left[e^{\gamma z}z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{z/n}\right]^{-1}$$

be the Gamma function and let $\varepsilon=\frac{1}{2}(1+i)$. Then $\overline{\varepsilon}=1-\varepsilon$. Hence, as is well known,

$$\Gamma(\varepsilon)\Gamma(\overline{\varepsilon})=\Gamma(\varepsilon)\Gamma(1-\varepsilon)=\frac{\pi}{\sin\pi\varepsilon}.$$

Therefore

$$\frac{\sin\pi\varepsilon}{\pi}=e^{\gamma\varepsilon}\varepsilon\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon}{n}\right)e^{-\varepsilon/n}\times e^{\gamma\overline{\varepsilon}}\overline{\varepsilon}\prod_{n=1}^{\infty}\left(1+\frac{\overline{\varepsilon}}{n}\right)e^{-\overline{\varepsilon}/n}=$$

$$=e^{\gamma}\frac{1}{2}\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon}{n}\right)\left(1+\frac{\overline{\varepsilon}}{n}\right)e^{-1/n}=$$

$$=e^{\gamma}\frac{1}{2}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}+\frac{1}{2n^2}\right)e^{-1/n}.$$

Hence $P=\dfrac{2\sin\pi\varepsilon}{\pi e^{\gamma}}=\dfrac{2\cosh\pi/2}{\pi e^{\gamma}}$, which is the assertion.

6654. *Proposed by W. O. Egerland and C. E. Hansen, Aberdeen Proving Ground, Aberdeen, MD.*

Suppose ω is real, n is a positive integer greater than 1, and $a_1, a_2, \dots, a_n$ are complex numbers with $|a_k| < 1$ for $k = 1, 2, \dots, n$. Prove that the equation

$$e^{i\omega}(z-a_1)(z-a_2)\cdots(z-a_n) = z(1-\bar{a}_1 z)(1-\bar{a}_2 z)\cdots(1-\bar{a}_n z)$$

has at least $n-1$ roots on the unit circle.

Solution to Problem 6654

Amer. Math. Monthly **98** (1991), 273

by Raymond Mortini

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Let $B(z) = e^{i\omega}\prod_{j=1}^{n} \frac{a_j - z}{1-\bar{a}_j z}$ be a finite Blaschke product of degree $n \geq 2$ ($a_j \in \mathbb{D}$) and let $\varphi(z) = \frac{z_0 - z}{1-\bar{z}_0 z}$, $z_0 \in \mathbb{D}$.

Then $\varphi^{-1} \circ B \circ \varphi$, called a conjugate of B, is again a finite Blaschke product of degree $n \geq 2$. (This follows from Rouché's theorem which shows that f is n to 1 and from the maximum principle.)

It is also easy to see that the relation

$$(\star) \qquad 1 = B(z)\overline{B\left(\frac{1}{\bar{z}}\right)}$$

implies that z_0, $z_0 \neq 0$, is a fixed point of B if and only if $1/\bar{z}_0$ is a fixed point of B.

Assume now that for some $z_0 \in \mathbb{D}$ we have $B(z_0) = z_0$. Then $f(0) = 0$. Hence, by Schwarz's lemma, $|f(z)| < |z|$ in $\mathbb{D}$. Therfore f, and hence B, cannot have further fixed points in $\mathbb{D}$. Thus, by $(\star)$, B can have at most two fixed points outside the unit circle T. Because B has n (resp. $n+1$) fixed points in $\mathbb{C}$ whenever $0 \in \{a_1, \dots, a_n\}$ (resp. $0 \notin \{a_1, \dots, a_n\}$) we can conclude that B has either $n-1$ or $n+1$ fixed points on T.

Remark. One can also conclude that B has a unique fixed point in $\mathbb{D}$ if and only if B is conjugate to a finite Blaschke product of degree n with $f(0) = 0$.

6648. *Proposed by Walter Rudin, University of Wisconsin, Madison.*

Let Ω be the region obtained by removing the points $0, 1, \infty$ from the Riemann sphere. Find all nonconstant holomorphic functions defined on Ω which map Ω into itself.

Solution to Problem 6648

Amer. Math. Monthly **98** (1991), 63

by Raymond Mortini

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Answer: Let $\Omega = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Then there are, besides the constants, exactly the six functions

$$z, \quad \frac{1}{z}, \quad 1-z, \quad \frac{1}{1-z}, \quad \frac{z}{z-1}, \quad \frac{z-1}{z}$$

which map Ω holomorphically into Ω.

Proof: Let f be a nonconstant holomorphic map of Ω into Ω. Because f omits the two points $w = 0, 1$, Picard's theorem tells us that none of the points $z = 0, 1, \infty$ is an essential singularity. Thus f is a rational function R of degree (order) $n \in \mathbb{N}$. Because $R^{-1}(\{0,1\})$, which is a subset of $\{0, 1, \infty\}$, contains at least two different points, R can have at most one pole. This has then order n. By taking, if necessary, reflections, we may assume without loss of generality that ∞ is this pole. Thus R is a polynomial of degree n. The assumption on f now implies that both the points $z = 0$ and $z = 1$ must be n-fold w_0-points, where $w_0 \in \{0, 1\}$. This means that the derivative of the polynomial R has at least $2(n-1)$ zeros. This implies that $n = 1$.

Thus in the general case, f is a rational function of degree n, hence a Möbius transform. Considering all permutations of $(0, 1, \infty)$ and constructing the Möbius transforms S with $S(\{0, 1, \infty\}) = \{0, 1, \infty\}$ yields the assertion.

E 3329. *Proposed by Michel Balazard, Faculté des Sciences, Limoges, France.*

Suppose f and g are differentiable real-valued functions defined on $(-\infty, +\infty)$. Must there exist a differentiable real-valued function h defined on $(-\infty, +\infty)$ such that $h' = f'g'$?

Solution to Problem E 3329

Amer. Math. Monthly **96** (1989), 445

by Raymond Mortini

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This problem is well known, and its solution is **No.**

In fact, let $F(x) = \cos\frac{1}{x}$ for $x \neq 0$ and $F(0) = 0$. Then F admits a primitive f of the form

$$f(x) := \int_0^x 2t \sin\frac{1}{t}\, dt - x^2 \sin\frac{1}{x} \text{ for } x \neq 0$$

and $f(0) = 0$.

But F^2 does not have a primitive H, because otherwise

$$0 = F^2(0) = H'(0) = \lim_{x\to 0} \frac{1}{x} \int_0^x \cos^2\frac{1}{t}\, dt = \frac{1}{2},$$

which is a contradiction. Note that the last equality follows from the facts that

$$1 = \frac{1}{x}\int_0^x \left(\cos^2\frac{1}{t} + \sin^2\frac{1}{t}\right)\, dt$$

and that

$$\begin{aligned} \lim_{x\to 0}\frac{1}{x}\int_0^x \left(\cos^2\frac{1}{t} - \sin^2\frac{1}{t}\right)\, dt &= \lim_{x\to 0}\frac{1}{x}\int_0^x \cos\frac{2}{t}\, dt \\ &= \lim_{x\to 0}\frac{1}{\frac{x}{2}}\int_0^{\frac{x}{2}} \cos\frac{1}{s}\, ds \\ &= f'(0) = 0. \end{aligned}$$

E 3325. *Proposed by Walter Rudin, University of Wisconsin, Madison.*

Let us say that a function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

has property P_t if $\sum_2^\infty n|a_n| \leqslant t$. Prove:

(a) If P_t holds for some $t < \infty$, then f is continuous on the closed unit disc, i.e., on $\{z \in \mathbf{C} : |z| \leqslant 1\}$.

(b) If P_1 holds, then f is one-to-one on the closed unit disc.

(c) If $t > 1$, there exists a function f satisfying P_t which is not one-to-one in the open unit disc.

Solution to Problem E 3325

Amer. Math. Monthly **96** (1989), 445

by Raymond Mortini

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a) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfy

$$(\star) \qquad \sum_{n=2}^{\infty} n|a_n| \leq t$$

for some $t > 0$. Then $(\star)$ implies that the power series for f' and that for f converge uniformly (and absolutely) on $\overline{\mathbf{D}} = \{z : |z| \leq 1\}$. Hence f and f' have continuous extensions to $\overline{\mathbf{D}}$.

b) If $t = 1$, then $(\star)$ implies that for $z \in \overline{\mathbf{D}}$ we have

$$(1) \qquad |f'(z) - z| \leq 1.$$

In particular, we have $\operatorname{Re} f'(z) \geq 0$.

Let $z, w \in \overline{\mathbf{D}}$ and let $\xi(t) = z + t(w - z)$, $0 \leq t \leq 1$. By the identity theorem for power series, relation (1) implies that f' and hence $\operatorname{Re} f'$ does not vanish identically on the segment $[z, w]$ unless $f(z) \equiv z$. Thus we have for $z, w \in \overline{\mathbf{D}}$, $z \neq w$

$$\begin{aligned} |f(w) - f(z)| &= \left| \int_0^1 f'(\xi(t))(w - z)\, dt \right| \\ &\geq |w - z| \operatorname{Re} \int_0^1 f'(\xi(t))\, dt \\ &= |w - z| \int_0^1 \operatorname{Re} f'(\xi(t))\, dt \neq 0. \end{aligned}$$

Hence f is injective on $\overline{\mathbf{D}}$.

c) Let $t > 1$. Then the functions $f(z) = z - \frac{t}{2} z^2$ satisfy $(\star)$. Looking at the parabola $x\left(1 - \frac{t}{2}x\right)$, we see that its maximum is attained at $x = \frac{1}{t} \in (0, 1)$. Thus f cannot be injective.

Remark. **This problem is well known [see P. Duren, *Univalent Funktions*, Exercise 24, § 2, page 73]. Related to this problem is Exercise 12 in Rudin's book *Real and Complex Analysis*, 3rd Edition, § 14, page 294.**

2. Mathematics Magazine

2187. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.*

For $r > s \geq 0$, evaluate

$$\prod_{n=0}^{\infty}\left(1+\frac{\cosh 2^n s}{\cosh 2^n r}\right).$$

2186. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Evaluate

$$\int_0^1 \frac{\operatorname{arctanh}\left(x\sqrt{2-x^2}\right)}{x}\,dx.$$

2184. *Proposed by the Columbus State University Problem Solving Group, Columbus State University, Columbus, GA.*

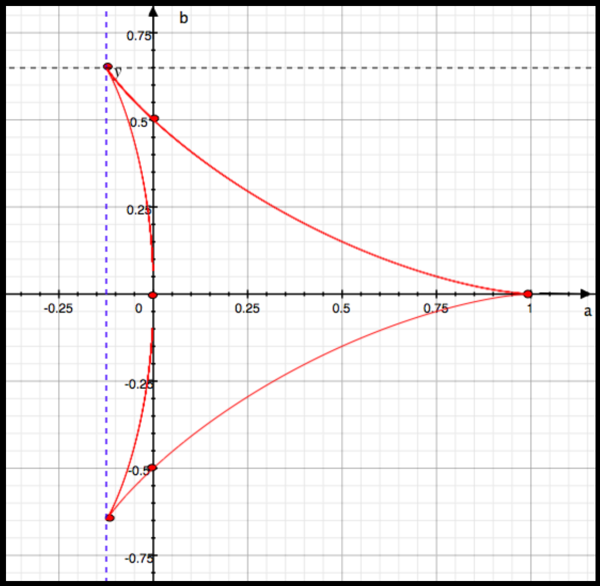
Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve

$$y = \frac{x}{x^2 + 1}$$

in exactly one point. (Be careful!)

3

29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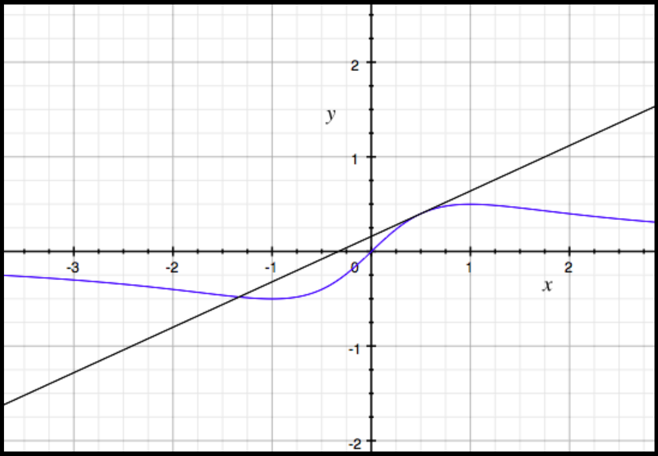
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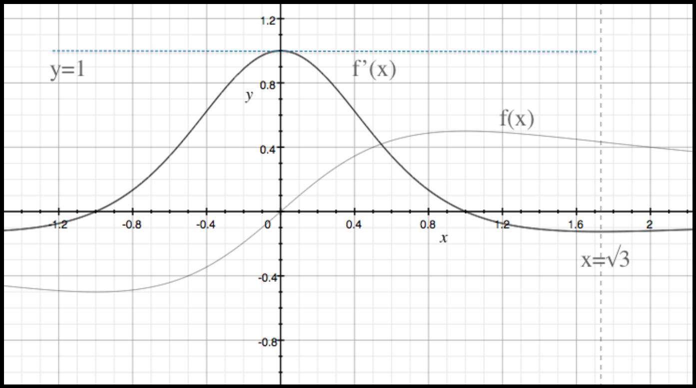
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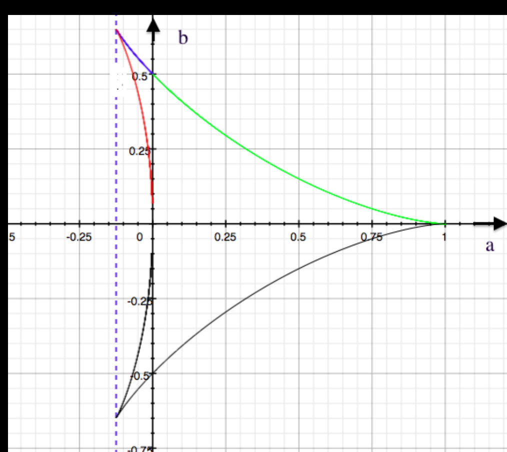
f'(x)
0.8
y
f(x)
0.4
-0.8
-0.4
0
0.4
0.8
1.2
1.6
2
x
x=√3
-0.4
-0.8

30

6

7



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4

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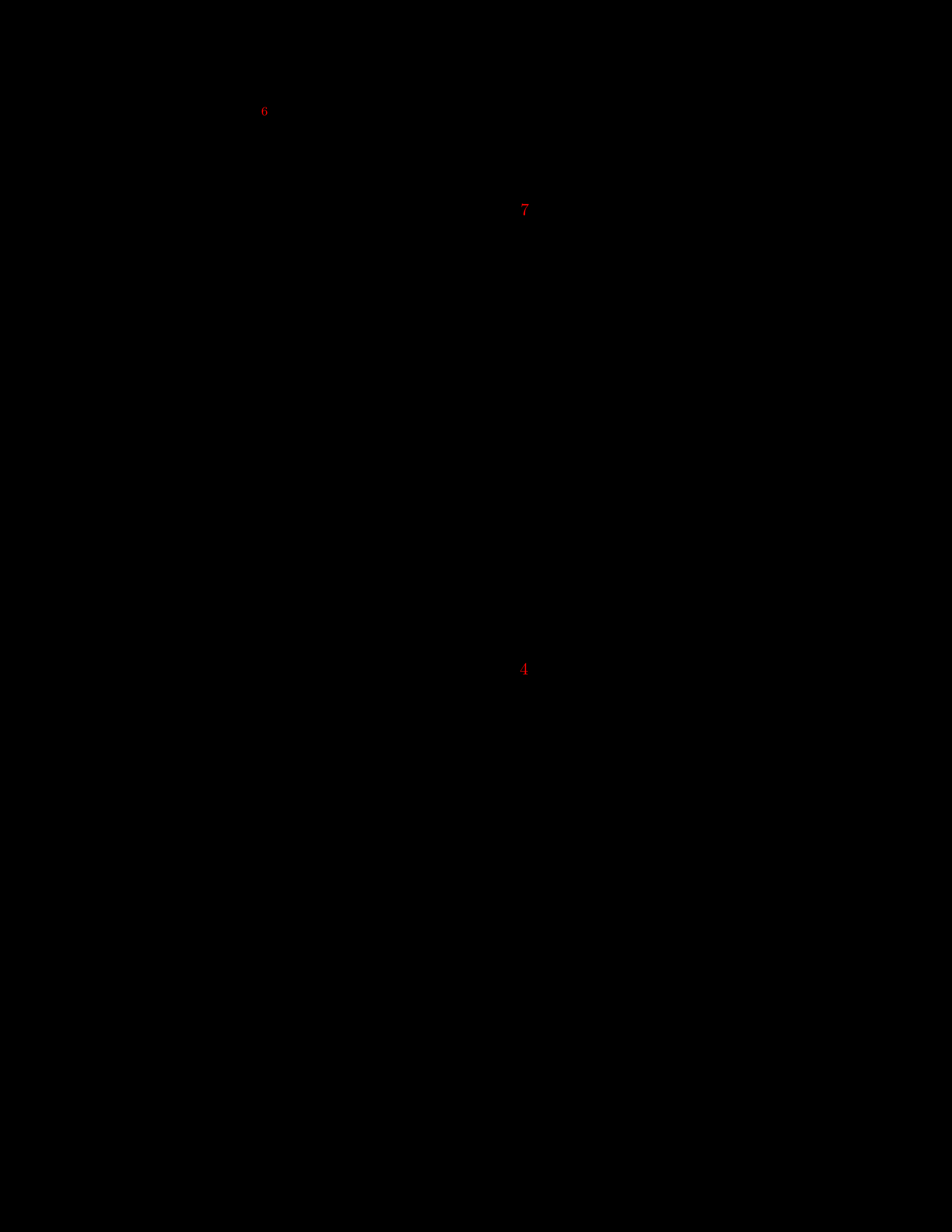
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1140. *Proposed by Raymond Mortini, Université du Luxembourg, Esch-sur-Alzette, Luxembourg and Rudolf Rupp, Technische Hochschule Nürnberg, Georg Simon Ohm, Nürnberg, Germany.*

Let m and n be nonnegative integers. Determine the value of

$$B(n,m) := \sum_{k=0}^{n} (-1)^k \binom{m+k}{k} \binom{m+n+1}{n-k}.$$

$\Gamma(a+k)$

$\Gamma(a+k+1)$

$\Gamma(b)$

$\Gamma(b)$

1

7

1

2

102

102

2185. *Proposed by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

Suppose n is a nonnegative integer. Let $P_n(x)$ be the nth degree polynomial defined by

$$P_n(x) = \frac{(-1)^n(1+x^2)^{n+1}}{n!}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Evaluate

$$\int_{-1}^{1} P_n(x)\,dx.$$

2181. *Proposed by Raymond Mortini, Université de Lorraine (emeritus), Metz, France, Peter Pflug, Carl von Ossietzky Universität Oldenburg (emeritus), Oldenburg, Germany, and Rudolf Rupp, Technische Hochschule Nürnberg Georg Simon Ohm, Nürnberg, Germany.*

Evaluate

$$\lim_{x\to\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k}{k!}\,\frac{(-1)^{m+1}}{(2(m+1))!}\,\frac{1}{2m+2+k}x^{2m+2+k}.$$

2176. *Proposed by Elton Bojaxhiu, Eppstein am Taunus, Germany and Enkel Hysnelaj, Sydney, Australia.*

Show that

$$\int_0^1 \frac{\log\left(x^2+x+1\right)}{x^2+1}\,dx = \frac{1}{6}\pi \log\left(\sqrt{3}+2\right) - \frac{C}{3},$$

where $C = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots$ is the Catalan constant.

[redacted]

31 32

2

1

6

4 8

7

8 8

6

$$\sum_{n=0}^{\infty} \frac{1}{(12n+3)^2}$$

$$-\sum_{n=0}^{\infty} \frac{1}{(12n+9)^2}$$

$$\frac{1}{9} \sum_{n=0}^{\infty} \left(\frac{1}{(4n+1)^2} - \frac{1}{(4n+3)^2} \right)$$

$$\frac{1}{9} C \quad -\frac{1}{3^2} \quad \frac{1}{9^2} \quad \frac{1}{3^2} - \frac{1}{9^2} + \frac{1}{15^2} - \cdots$$

$$\frac{1}{9} C \quad \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots \right)$$

$$\frac{1}{9} C$$

6

3

5

108 110

110

https://de.wikipedia.org/wiki/Catalansche_Konstante 113

https://math.stackexchange.com/questions/3044687/solving-the-integral-int-0-pi-2-log-left-frac2-sin2x2-sin2x-right-m/3044892#3044892 111

https://math.stackexchange.com/questions/4512956/is-it-hard-to-evaluate-int-0-infty-frac-ln-left1-xx2-right1x 113

https://math.stackexchange.com/questions/983044/integral-int-0-pi-12-ln-tan-x-dx 112

https://math.stackexchange.com/questions/4513080/computing-int-0-pi-ln-sin-x2-d-x-and-int-0-pi-ln-2-sin-x 111

https://math.stackexchange.com/questions/3800387/proving-int-01-frac-tanh-1-sqrtx1-x-sqrtx1-xdx-frac13?noredirect=1&lq=1 111

2171. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Evaluate the following sums in closed form.

(a) $\sum_{n=0}^{\infty}\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!}\right)$

(b) $\sum_{n=0}^{\infty}\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots (-1)^{n-1}\frac{x^{2n+1}}{(2n+1)!}\right)$

2167. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

Prove that

$$\lim_{n\to\infty} e^{n/2} \prod_{i=2}^{n} e^{i^2} \left(1 - \frac{1}{i^2}\right)^{i^4} = \pi \exp\left(-\frac{5}{4} + \frac{3\zeta(3)}{\pi^2}\right),$$

where $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$.

35 34

```
int(Pi·s^4·cot(Pi·s) + 2·s^5/(1 - s^2), s = 0 ..1);
```

$$\frac{2\pi^2\ln(\pi) - 3\pi^2 + 6\zeta(3)}{2\pi^2}$$

$$\lim_{s\to 1}\left(s^4\log(1-e^{-2\pi i s}) - \log(1-s^2)\right)$$

$$\lim_{s\to 0}\left(s^4\log(1-e^{-2\pi i s}) - \log(1-s^2)\right)$$

$$-i\frac{\pi}{2} + \log\pi \qquad 0$$

116

primitive of \pi s^4 \cot(\pi s)+\frac{2s^5}{1-s^2}

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD

Indefinite integral

$$\int\left(\pi\, s^4 \cot(\pi\, s) + \frac{2\, s^5}{1 - s^2}\right) d\,s = \frac{2\, i\, s^3\, \mathrm{Li}_2\left(e^{-2\, i\, \pi\, s}\right)}{\pi} + \frac{3\, s^2\, \mathrm{Li}_3\left(e^{-2\, i\, \pi\, s}\right)}{\pi^2} - \frac{3\, i\, s\, \mathrm{Li}_4\left(e^{-2\, i\, \pi\, s}\right)}{\pi^3} - \frac{3\, \mathrm{Li}_5\left(e^{-2\, i\, \pi\, s}\right)}{2\, \pi^4} + \frac{1}{5}\, i\, \pi\, s^5 - \frac{s^4}{2} + s^4 \log\left(1 - e^{-2\, i\, \pi\, s}\right) - s^2 - \log\left(1 - s^2\right) + \text{constant}$$

2147. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

Evaluate

$$\prod_{n=2}^{\infty} \frac{n^4 + 4}{n^4 - 1}.$$

2141. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.*

Evaluate

$$\mathrm{I} := \int_0^{\infty} \ln\left(1 + 2x^{-2}\cos\varphi + x^{-4}\right)\, dx.$$

36

2128. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $0 < a < b < 1$ and $\epsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1 - b^m)^n < \epsilon$ and $(1 - a^m)^n > 1 - \epsilon$.

2118. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

It is well known that the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

converges. Does the series

$$\sum_{k=1}^{\infty} e^{-\lfloor \ln k \rfloor} \sin k$$

converge or diverge?

solution of problem 2118 Math. Mag. 94 (2021), p. 150.

The series converges. This is an immediate consequence to Abel's theorem telling us that if (a_n) is a sequence of positive numbers with $a_n \searrow 0$, then the trigonometric series $S(t) := \sum_{n=0}^{\infty} a_n e^{int}$ converges for all $t \notin \{2k\pi : k \in \mathbb{Z}\}$ (see i.e. Appendix 4 in my encyclopedic monograph: R. Mortini, R. Rupp, Extension Problems and Stable Ranks, A Space Odyssey, Birkhäuser 2021, ca 2150 pages):

Just take $a_n = e^{-\lfloor \log n \rfloor}$, $t = 1$, and the imaginary part of $S(t)$. The proof is based on the Abel-Dirichlet rule, telling us that with $b_n = e^{int}$, and

$$|b_0 + b_1 + \cdots + b_m| = |1 + e^{it} + \cdots + e^{imt}| = \\ = \left| \frac{1 - e^{(m+1)it}}{1 - e^{it}} \right| \text{ if } e^{it} \neq 1.$$

we obtain for $t \notin 2\pi\mathbb{Z}$ that

$$|b_0 + b_1 + \cdots + b_m| \leq \frac{2}{|1 - e^{it}|} =: M. \tag{37}$$

Hence the series $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

2117. *Proposed by Ahmad Sabihi, Isfahan, Iran.*

Find all positive integer solutions to the equation

$$(m+1)^n = m! + 1.$$

Solution to problem 2117 in Math. Mag. 94 (2021), p. 150 by
Raymond Mortini, Rudolf Rupp and Amol Sasane

There are only the three solutions $(n, m) \in \{(1,1),(1,2),(2,4)\}$.
It is easy to check that these are solutions.
Now suppose that $n \geq m \geq 2$. Then (n, m) cannot be a solution since

$$(m+1)^n \geq (m+1)^m > m^m > m!, \text{ so } (m+1)^n > m! + 1.$$

Now, if $2 = n < m$, then

$$(m+1)^2 = m! + 1 \iff m + 2 = (m-1)!$$

which is obviously only satisfied for $m = 4$.

Next let $2 < n < m$. Then we see that if (n, m) is a solution to $(m+1)^n = m! + 1$, then m must be even. (Actually, by Wilson's theorem, $m+1$ divides $m!+1$ if and only $m+1$ is prime; but we do not need this result). In particular, $m \geq 4$. Note that the equation $(m+1)^n - 1 = m!$ under discussion is equivalent to

$$\sum_{k=0}^{n-1} (m+1)^k = (m-1)!. \tag{38}$$

<u>1°</u> $m = 4$. Then, due to (38), $6 = 3! = 1 + 5 + \cdots$ implying that $n = 2$. A contradiction to the assumption $2 < n < m$..

<u>2°</u> $m \geq 6$. Then $2 < m/2 < m - 1$. Hence the integer $m/2$ divides $(m-1)!$. Since $m/2 > 2$, additionally the number 2 divides $(m-1)!$. Thus $m = 2 \cdot (m/2)$ divides $(m-1)!$.

Now, (38) yields $n \equiv 0, \mod m$. That is, m divides n and so $m \leq n$. This is again a contradiction to the assumption $2 < n < m$.

2116. *Proposed by Fook Sung Wong, Temasek Polytechnic, Singapore.*

Evaluate

$$\int_0^{\infty} \frac{e^{\cos x}\cos(\alpha x+\sin x)}{x^2+\beta^2}\,dx,$$

where α and β are positive real numbers.

Solution to problem 2116 Math. Mag. 94 (2021), 150 by

Raymond Mortini, Rudolf Rupp

We use the Residue theorem for the meromorphic function f, given by $f(z):=\dfrac{e^{e^{iz}+i\alpha z}}{z^2+\beta^2}$,

and the positively oriented contour $\Gamma_R := S_R + H_R$, where S_R denotes the segment $[-R,R]$ and H_R the half-circle connecting $R, iR, -R$ for some $R>\beta>0$. Thus the simple pole $z_0 := i\beta$ is surrounded once in positive direction. The Residue theorem tells us that

(*)
$$\oint_{\Gamma_R} f(z)dz = 2\pi i \operatorname{Res}(f,z_0) = 2\pi i \frac{e^{e^{-\beta}-\alpha\beta}}{2i\beta} = \frac{\pi}{\beta} e^{e^{-\beta}-\alpha\beta}.$$

From $f(x)=\dfrac{e^{e^{ix}+i\alpha x}}{x^2+\beta^2}=\dfrac{e^{\cos(x)+i\sin(x)+i\alpha x}}{x^2+\beta^2}=\dfrac{e^{\cos(x)}\left[\cos(\alpha x+\sin(x))+i\sin(\alpha x+\sin(x))\right]}{x^2+\beta^2}$, we find that $f(-x)=\overline{f(x)}$ for all $x\ge 0$, hence

$$\int_{-R}^{R} f(x)dx = \int_0^R (f(x)+\overline{f(x)})dx = 2\int_0^R \frac{e^{\cos(x)}\cos(\alpha x+\sin(x))}{x^2+\beta^2}dx.$$

For $z\in H_R$, i.e. $z=\mathrm{Re}^{it}, 0\le t\le\pi$ we estimate

$$|f(z)| \le \frac{e^{\mathrm{Re}(e^{iR\cos(t)-R\sin(t)})+i\alpha R\cos(t)-\alpha R\sin(t))}}{R^2-\beta^2} = \frac{e^{e^{-R\sin(t)}\cos(R\cos(t))-\alpha R\sin(t)}}{R^2-\beta^2} \le \frac{e}{R^2-\beta^2}.$$

The standard estimate for integrals now shows that

$$\left|\int_{H_R} f(z)dz\right| \le \pi R\cdot\frac{e}{R^2-\beta^2} \to 0 \text{ for } R\to+\infty.$$

Passing to the limit in (*) then gives the value of the integral in question:

$$\boxed{\int_0^{\infty} \frac{e^{\cos(x)}\cos(\alpha x+\sin(x))}{x^2+\beta^2}dx = \frac{\pi}{2\beta}e^{e^{-\beta}-\alpha\beta}}.$$

1075. *Proposed by Raymond Mortini, Université de Lorraine and IECL, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Assume that there exist $a, b \in \mathbb{R}$ such that $f(a) \le f(x) \le f(b)$ for all $x \in \mathbb{R}$. Is it true that, for every $d > 0$, there exists a horizontal segment of length d with endpoints on the graph of f?

Solution to Quicky 1075 in Math. Mag. 90 (2017), 384

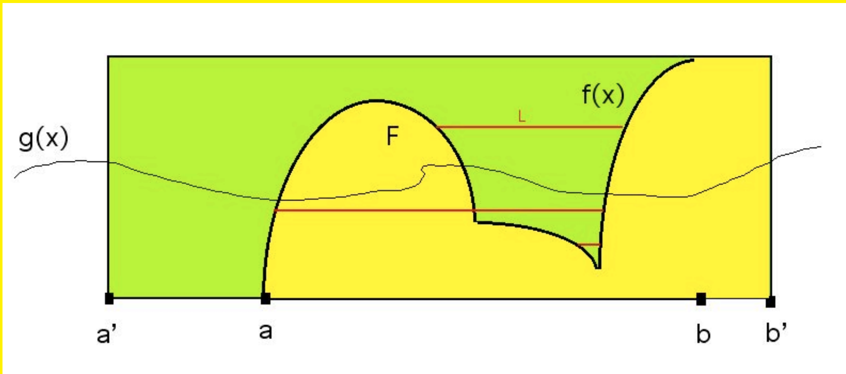

FIGURE 8. Intersecting curves

Yes. We have to show that for every $d \in \mathbb{R}$, there is $x_0 \in \mathbb{R}$ such that $f(x_0) = f(x_0 - d)$.

1) Non-elementary geometric approach.
Put $g(x) := f(x-d)$ and choose $a, b \in \mathbb{R}$ such that $m = f(a)$, $M = f(b)$ and $m < f(x) < M$ for $x \in]a, b[$. We may assume that $a < b$. Of course, $m \le g(x) \le M$. Let $a' < a$ and $b > b'$. Then $x \mapsto (x, g(x)), a' \le x \le b'$ is a curve in the rectangle $R := [a', b'] \times [m, M]$ starting at the left of the graph $F := \{(x, f(x)) : a \le x \le b\}$ of f and ending at the right (here we need that the Jordan arc F is a cross-cut of R). Thus this curve meets the graph: that is there is $a' \le x_0 \le b'$ such that $(x_0, g(x_0)) \in F$. Hence, there is $a \le x_1 \le b$ such that $(x_0, g(x_0)) = (x_1, f(x_1))$. Consequently, $x_0 = x_1$ and so $f(x_0) = f(x_0 - d)$.

2) Analytic approach. Let $H := f - g$. Then $H(a) = m - g(a) \le 0$ and $H(b) = M - g(b) \ge 0$. If $g(a) = m$ or $g(b) = M$, then we are done. So we may assume that $H(a) < 0$ and $H(b) > 0$. Hence, by the intermediate value theorem, there is $x_0 \in]a, b[$ such that $H(x_0) = 0$. We conclude that $f(x_0) = g(x_0) = f(x_0 - d)$.

Let us point out that the assertion does not hold whenever merely $\inf_{\mathbb{R}} f$ and $\sup_{\mathbb{R}} f$ exist: just look at $f(x) = \arctan x$. Motivation for the problem came from the paper: Peter Horak, Partitioning $\mathbb{R}^n$ into connected components. Am. Math. Mon. 122, No. 3, 280-283 (2015), where periodic functions were considered.

1947. *Proposed by Raymond Mortini and Jérôme Noël, Université de Lorraine, Metz, France.*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n}{2}.$$

Solution to problem 1947 Math. Mag. 87 (2014), 230 by
Raymond Mortini, Jérôme Noël

$$\sum_{k=0}^{n} |\cos k| \geq 1 + \sum_{k=1}^{n} (\cos k)^2 = 1 + \sum_{k=1}^{n} \frac{\cos(2k)+1}{2} = 1 + \frac{n}{2} + \frac{1}{2}\mathrm{Re}\left(\sum_{k=1}^{n} e^{2ik}\right).$$

Now

$$\sum_{k=1}^{n} e^{2ik} = e^{2i}\frac{1-e^{2in}}{1-e^{2i}} = e^{2i}\frac{e^{in}}{e^{i}}\frac{\sin n}{\sin 1} = e^{i(n+1)}\frac{\sin(n)}{\sin 1}.$$

Hence

$$\sum_{k=0}^{n} |\cos k| \geq 1 + \frac{n}{2} + \frac{\cos(n+1)\sin n}{2\sin 1} \geq 1 + \frac{n}{2} - \frac{1}{2\sin 1}$$
$$= \frac{n}{2} + \underbrace{\left(1 - \frac{1}{2\sin 1}\right)}_{>0} \geq \frac{n}{2},$$

because $2\sin 1 > 1$ (note that $\pi/4 < 1 < \pi/3$ implies $1 < \sqrt{2} < 2\sin 1 < \sqrt{3}$).

Let us remark that in the very first step it was important to begin the sum at $k = 1$ in order to have the summand 1. Otherwise we would have obtained

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n+1}{2} + \frac{\cos n \sin(n+1)}{2\sin 1} \geq \frac{n+1}{2} - \frac{1}{2\sin 1}$$
$$= \frac{n}{2} + \frac{1}{2}\left(1 - \frac{1}{\sin 1}\right),$$

an estimate that is less than $n/2$.

1871. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ.*

Let f, g be two differentiable real functions such that $g(x) \neq 0$ for all real numbers x. Suppose that c is a real number such that

$$f(c)\int_a^b g(x)\,dx \neq g(c)\int_a^b f(x)\,dx,$$

for all pairwise distinct real numbers a and b. Prove that $(f/g)'(c) = 0$.

solution of problem 1871 Math. Mag. 84 (2011), p. 229.

The solution is a based on the following Lemma:

Lemma 9. *Let F be a continuous, real-valued function on $\mathbb{R}\times\mathbb{R}$. Suppose that F is not zero outside the diagonal D and not constant 0 on D. Then either $F \geq 0$ or $F \leq 0$ everywhere.*

Proof. Let $P^+ = \{(x,y) \in \mathbb{R}^2; x < y\}$ and $P^- = \{(x,y) \in \mathbb{R}^2; x > y\}$.
Case 1: if $F(x_0,y_0) < 0$ and $F(x_1,y_1) > 0$ for some points $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$ in P^+, then F must have a zero on the segment S joining P_0 and P_1 in P^+ (since the image of S under F is an interval).

Case 2: if $F(P_0) < 0$ and $F(P_1) > 0$ for some $P_0 \in P^+$ and $P_1 \in P^-$, then we may choose an arc A (piecewise parallel to the axis) such that $F \neq 0$ on $A \cap D$, which is a singleton. By the the intermediate value theorem, there is a zero of F on the arc A, but outside D.

Case 3: if $F(Q_0) < 0$ and $F(Q_1) > 0$ for some $Q_0, Q_1 \in D$, then there are $P_0 \in P^+$ and $P_1 \in P^-$ such that $F(P_0) < 0$ and $F(P_1) > 0$. Hence we are in the second case.

Thus, all cases yield a contradiction to the assumption. Hence, in the image space, 0 is a global extremum. □

Solution to the problem Without loss of generality, we may assume that $g > 0$. Let

$$H(a,b) = \begin{cases} \dfrac{\int_a^b f(x)dx}{\int_a^b g(x)dx} & \text{if } a \neq b \\[2ex] \dfrac{f(a)}{g(a)} & \text{if } a = b. \end{cases}$$

We claim that H is continuous on $\mathbb{R}\times\mathbb{R}$. In fact, it suffices to prove continuity at the diagonal. So let $(a_0,a_0) \in D$. Then, for $(a,b) \in \mathbb{R}^2 \setminus D$, there is $\xi \in \,]a,b[$ such that $\int_a^b f(x)dx/(b-a) = f(\xi) \to f(a_0)$ if $(a,b) \to (a_0,a_0)$. Thus $\lim H(a,b) = H(a_0)$.

By assumption, $H(a,b) \neq f(c)/g(c)$ whenever (a,b) is outside the diagonal in $\mathbb{R}^2$.

Case 1: $H \equiv f(c)/g(c)$ on the diagonal D. Then the function $x \mapsto f(x)/g(x)$ has derivative 0 everywhere, and so satisfies the assertion of the problem.

Case 2: H not constant $f(c)/g(c)$ on D. Then, by Lemma 9 applied to $F = H - f(c)/g(c)$, we see that $H \geq f(c)/g(c)$ on $\mathbb{R}\times\mathbb{R}$ or $H \leq f(c)/g(c)$ on $\mathbb{R}\times\mathbb{R}$. In particular, c is an extrema of the function $x \mapsto f(x)/g(x)$ and so the differentiability of f/g implies that $(f/g)'(c) = 0$.

1867. *Proposed by Ángel Plaza and César Rodríguez, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(t)\,dt = 1$ and n a positive integer. Show that

1. there are distinct $c_1, c_2, \ldots, c_n$ in $(0, 1)$ such that
$$f(c_1) + f(c_2) + \cdots + f(c_n) = n,$$
2. there are distinct $c_1, c_2, \ldots, c_n$ in $(0, 1)$ such that
$$\frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \cdots + \frac{1}{f(c_n)} = n.$$

solution of problem 1867 Math. Mag. 84 (2011), 150

If $f \equiv 1$, then the assertions are trivially true (just take any n points in $]0,1[$). If $f \not\equiv 1$, then there exist points at which f is strictly less than 1 and points where f is strictly bigger than one (note that this is the only occasion where we have used the hypothesis that $\int_0^1 f(t)dt = 1$). Hence, due to intermediate value theorem, there is at least one point at which f takes the value 1. In particular, if $h = f$ or $h = 1/f$, and noticing that the image of $[0,1]$ under f is an interval containing the point 1 in its interior, there exist $b \in [0,1]$ with $M := h(b) > 1$ and a sequence (a_i) with $h(a_i) < 1$ and $\lim h(a_i) = 1$. By compactness, we may assume that (a_i) is converging to some $a \in [0,1]$. Hence $h(a) = 1$ and
$$m(\delta) := \min\{h(x) : x \in [a-\delta, a+\delta] \cap [0,1]\} \to 1 \text{ if } \delta \to 0.$$
For later purposes, we note that $m(\delta) < 1$. Choose δ so small that
$$(n-1)(1-m(\delta)) \le M - 1.$$
Then
$$n - M = (n-1) - (M-1) \le (n-1)m(\delta).$$
Now choose $n-1$ distinct points $x_1, \ldots, x_{n-1}$ in $[a-\delta, a+\delta] \cap\,]0,1[$ such that
$$m(\delta) < h(x_j) < 1.$$
Then $A := \sum_{j=1}^{n-1} h(x_j)$ satisfies
$$(n-1)m(\delta) \le A < n-1.$$
Thus $n - M \le A$ and so $1 < n - A \le M$. Again, by the intermediate value theorem, there is $x_n \in]0,1[$ such that $h(x_n) = n - A$. Hence
$$\sum_{j=1}^{n} h(x_j) = n.$$
Note that $x_n \notin \{x_1, \ldots, x_{n-1}\}$.

Alternate proof concerning the existence of the c_j

Let $F(x) = \int_0^x f(t)dt$ be the primitive of f vanishing at the origin. Let $x_j = j/n, j = 0, 1, \ldots, n$. Then, by the mean-value theorem of differential calculus, there exist $c_j \in]x_{j-1}, x_j[\subseteq]0,1[$ such that
$$1 = F(1) - F(0) = \sum_{j=1}^{n}(F(x_j) - F(x_{j-1})) = \sum_{j=1}^{n} F'(c_j)(x_j - x_{j-1}) = \frac{1}{n}\sum_{j=1}^{n} f(c_j).$$

1863. *Proposed by Duong Viet Thong, Department of Economics and Mathematics, National Economics University, Hanoi, Vietnam.*

Let f be a continuously differentiable function on $[a, b]$ such that $\int_a^b f(x)\,dx = 0$. Prove that

$$\left|\int_a^b xf(x)\,dx\right| \le \frac{(b-a)^3}{12}\max\{|f'(x)| : x \in [a,b]\}.$$

solution of problem 1863, Math. Mag. 84 (2011), 64.

We use Carathéodory's definition of differentiability: A function $f : I \to \mathbb{R}$ is differentiable at a point $x_0 \in I$, $I \subseteq \mathbb{R}$ an interval, if there exists a function $g = g_{x_0} : I \to \mathbb{R}$ continuous at x_0 such that

$$f(x) = f(x_0) + (x - x_0)g(x);$$

just define $g_{x_0}(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x \neq x_0 \end{cases}$.

Now if $f \in C^1[a, b]$, then g_{x_0} is continuous and, by Rolle's theorem, $g_{x_0}(x) = f'(\xi)$ for some $\xi \in]a, b[$, ξ depending on x_0 and x. Hence

$$\sup_{a\le s\le b} |g_{x_0}(s)| \le \max_{a\le t\le b} |f'(t)| =: M.$$

Let $c = (a + b)/2$. Then, using the hypotheses that $\int_a^b f(x)dx = 0$ and the fact that $\int_a^b (x - c)dx = 0$ we obtain the following equalities:

$$J := \int_a^b xf(x)dx = \int_a^b (x-c)f(x)dx =$$
$$\int_a^b (x-c)(f(x)-f(c))dx = \int_a^b (x-c)^2 g_c(x)dx.$$

Thus

$$|J| \le \int_a^b (x-c)^2 M dx = \frac{1}{3}\left[(x-c)^3\right]_a^b M =$$
$$\frac{2}{3}\left(\frac{b-a}{2}\right)^3 M = \frac{1}{12}(b-a)^3 M.$$

If $f(x) = x$ and $a = -1, b = 1$ then $\int_{-1}^1 f(x)dx = 0$ and $\int_{-1}^1 xf(x)dx = 1/3 = (b-a)^3/12$.

1860. ***Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bârlad, Romania.***

Let α be a complex number such that $|\alpha| > 1$ and let n be an integer such that $n > 2$. Prove that at least $n - 2$ roots of the equation $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ have norm equal to 1.

solution of problem1860, Math. Mag. 83 (2010), 392.

We use the Schwarz-Pick Lemma telling us that holomorphic selfmaps of the unit disk are contractions with respect to the (pseudo)-hyperbolic metric ρ and that $\rho(f(z), f(w)) = \rho(z, w)$ for some pair $(z, w) \in \mathbb{D}^2$, $z \neq w$ implies that f is a conformal selfmap of $\mathbb{D}$ (hence of the form $e^{i\theta}\frac{b-z}{1-\bar{b}z}$) and so a (pseudo)-hyperbolic isometry.

Note that $z^n + \alpha z^{n-1} + \overline{a} z + 1 = 0$ for some $z \in \mathbb{D}$ if and only if $z^{n-1} = -\frac{\overline{a}z+1}{a+z}$. Now suppose that there are two solutions z, w in $\mathbb{D}$. Let $f(z) = -\frac{\overline{a}z+1}{a+z}$. Then

$$\rho(z, w) = \rho(f(z), f(w)) = \rho(z^{n-1}, w^{n-1}).$$

But this would imply that z^{n-1} is a bijection of $\mathbb{D}$ onto itself; a contradiction since $n \geq 3$.

Thus the equation $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ has at most one solution in $\mathbb{D}$. Since z is a solution if and only if $\frac{1}{\overline{z}}$ is a solution, we see that this polynomial of degree n must have at least $n - 2$ solutions (multiplicities counting) on the unit circle.

Next we note that $u \in \mathbb{T}$ s a solution of modulus one of $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ if and only if u is a fixed point on $\mathbb{T}$ of the selfmap $\varphi(z) = f^{-1}(z^{n-1})$ of $\mathbb{D}$. Since the derivative of φ does not vanish at boundary fixed points, we conclude that there are at least $n - 2$ *distinct* solutions of unit modulus.

6. Crux Mathematicorum

4924. *Proposed by Yagub N. Aliyev.*

Let n be a positive integer. Find all possible values of $x \geq 0$ for which the inequality

$$1+\frac{n}{B} \leq \left(1+\frac{1}{B}\right)\left(1+\frac{1}{B+x}\right)\left(1+\frac{1}{B+2x}\right)\dots\left(1+\frac{1}{B+(n-1)x}\right),$$

holds true for all $B>0$. For which $x \geq 0$ is the reverse inequality true?

4930. *Proposed by Toyesh Prakash Sharma.*

For positive integers a, b, c, show that

$$a^b b^c c^a \leq \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}.$$

4925. *Proposed by Ivan Hadinata.*

Determine all possible real numbers a for which there exists a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y) + xf(f(y)) = f(f(x)) + f(y) + axy$$

for all $x, y \in \mathbb{R}$.

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4929. *Proposed by Seán M. Stewart.*

Evaluate

$$\int_0^1 \frac{\log(1+\sqrt{1-u^2})}{1+u}\,du.$$

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For simplicity we will use $\alpha = \sin a$ so let's consider:

$$I(a) := \int_0^{\frac{\pi}{2}} \frac{\ln(1+\sin a \sin x)}{\sin x} dx$$

Note that $\sin a$ is always inside $[-1, 1]$ so it's equivalent to $|\alpha| \leq 1$. Also put $x \to \pi - x$, then average the two integrals to see that:

$$2\int_0^{\frac{\pi}{2}} \frac{\ln(1+\sin a \sin x)}{\sin x} dx = \int_0^{\pi} \frac{\ln(1+\sin a \sin x)}{\sin x} dx$$

$$\Rightarrow I(a) = \frac{1}{2}\int_0^{\pi} \frac{\ln(1+\sin a \sin x)}{\sin x} dx \Rightarrow I'(a) = \frac{1}{2}\int_0^{\pi} \frac{\cos a}{1+\sin a \sin x} dx$$

$$\overset{\tan\frac{x}{2}=t}{=} \int_0^{\infty} \frac{\cos a}{1+\sin a \frac{2t}{1+t^2}} \frac{1}{1+t^2} dt = \int_0^{\infty} \frac{\cos a}{(t+\sin a)^2 + \cos^2 a} dt$$

$$= \arctan\left(\frac{t+\sin a}{\cos a}\right)\Bigg|_0^{\infty} = \frac{\pi}{2} - a$$

Now we integrate to get back:

$$I(a) = \int \left(\frac{\pi}{2} - a\right) da = \frac{\pi a}{2} - \frac{a^2}{2} + C$$

$$I(0) = 0 \Rightarrow C = 0 \Rightarrow I(a) = \frac{a}{2}(\pi - a)$$

https://math.stackexchange.com/questions/549028/deriving-maclaurin-series-for-frac-arcsin-x-sqrt1-x2 137, 138, 139
https://math.stackexchange.com/questions/1448822/taylor-expansion-for-arcsin2x 137
https://math.stackexchange.com/questions/3262856/integral-int-0-frac-pi2-ln1a-sin-x-cscxdx 137, 138, 139

4920. *Proposed by Ángel Plaza.*

If $k > 1$ and $n \in \mathbb{N}$, evaluate $\displaystyle\int_0^1 \frac{\log(1 + x^k + x^{2k} + \cdots + x^{nk})}{x} dx.$

4918. *Proposed by Yagub Aliyev.*

Let $L = \lim_{\lambda \to +\infty} \frac{\lambda^{x^2}}{\int_a^b \lambda^{t^2} dt}$.

a) Show that if $0 \le a \le x < b$, then $L = 0$.

b) Show that if $0 \le a < x = b$, then $L = +\infty$.

4915. *Proposed by Michel Bataille.*

Let $S_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+n+1)}$, where n is a nonnegative integer. Find real numbers a, b, c such that $\lim_{n\to\infty} \left(n^3 S_n - (an^2 + bn + c)\right) = 0$.

9

4914. *Proposed by Ivan Hadinata.*

Let $\mathbb{R}_{\geq 0}$ be the set of all non-negative real numbers. Find all possible monotonically increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$f(x^2 + y + 1) = xf(x) + f(y) + 1, \quad \forall x, y \in \mathbb{R}_{\geq 0}.$$

45

46

4913. *Proposed by Albert Natian.*

Suppose the continuous function f satisfies the integral equation

$$\int_0^{xf(7)} f\left(\frac{tx^2}{f(7)}\right) dt = 3f(7)\,x^4.$$

Find $f(7)$.

4910. *Proposed by Paul Bracken.* Let m and n be non-negative integers and let

$$J_{m,n} = \int_0^{\infty} \left(\left(\frac{\sin t}{t} \right)^m - \left(\frac{\sin t}{t} \right)^n \right) \frac{dt}{t^2}.$$

Prove that the $J_{m,n}$ are rational multiples of π.

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4909. *Proposed by Michel Bataille.*

For each positive integer n, let $P_n(x) = (x-1)^{2n+1}(x^2-(2n+1)x-1)$. Show that the equation $P_n(x) = 1$ has a unique solution x_n in the interval $(0, \infty)$. Prove that $\lim_{n\to\infty} (x_n - 2n) = 1$ and find $\lim_{n\to\infty} n(x_n - 2n - 1)$.

4905. *Proposed by Aravind Mahadevan.*

In a right-angled triangle, the acute angles x and y satisfy the following equation:

$$\tan x + \tan y + \tan^2 x + \tan^2 y + \tan^3 x + \tan^3 y = 70.$$

Find x and y.

[redacted]

4904. *Proposed by Ivan Hadinata.*

Find all pairs (x, y) of prime numbers x and y such that $x \geq y$, $x + y$ is prime and $x^x + y^y$ is divisible by $x + y$.

[redacted]

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[redacted]

4903. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1}{2n+3}-\cdots\right)-\frac{1}{4n}\right].$$

4900. *Proposed by Daniel Sitaru.*

For a positive integer m, let H_m denote the m-th harmonic number, that is, $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. For m, n, p, q positive integers, prove that

$$H_m + H_n + H_p + H_q \leq 3 + H_{mnpq}.$$

$\left(\frac{1}{p+1} + \cdots + \frac{1}{2p}\right)$ $\left(\frac{1}{(q-1)p+1} + \cdots + \frac{1}{qp}\right)$

$p \cdot \frac{1}{2p}$ $p \cdot \frac{1}{qp}$

49

49

4896. *Proposed by Ivan Hadinata.*

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$ the following equation holds:

$$f(f(x) + yf(z) - 1) + f(z + 1) = zf(y) + f(x + z).$$

[redacted]

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4894. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty} \frac{H_{n-1}H_{n+1}}{n(n+1)},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number and $H_0 = 0$.

$$\left(\frac{H_{n+1}}{n} - \frac{H_{n+2}}{n+1}\right)$$

$$\frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\left(H_2 - \lim_{N\to\infty} \frac{H_{N+2}}{N+1}\right)$$

$$\frac{3}{2} + \frac{1}{4}$$

sum (HarmonicNumber[n]-1/n)*(HarmonicNumber[n]+1/(n+1))/(n*(n+1)), n=1 to infinity

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Infinite sum

$$\sum_{n=1}^{\infty} \frac{\left(H_n - \frac{1}{n}\right)\left(H_n + \frac{1}{n+1}\right)}{n\,(n+1)} = 3$$

$$\sum_{n=2}^{\infty} \frac{\left(H_n - \frac{1}{n}\right)\left(H_n + \frac{1}{n+1}\right)}{n\,(n+1)} = \frac{7}{4} = 1.75$$

Approximated sum

$$\sum_{n=1}^{\infty} \frac{H_{n-1}\, H_{n+1}}{n\,(n+1)} \approx 2.75949$$

Approximated sum

$$\sum_{n=1}^{\infty} \frac{H_{n-1}\, H_{n+1}}{n\,(n+1)} \approx$$

2.9135100717114100051500066686563672253196126664946473290321059642\
270541454027040860966879348133249579785186812243965

4893. *Proposed by Albert Natian.*

Find all continuous real functions f on $[-1,1]$ that satisfy the integral equation

$$x^2+\int_1^{\frac{1}{x}} f\left(x^2 t\right) dt = 1.$$

The statement of the problem is a bit ambiguous, as problems arise for $x=0$. Note that for $0<x\leq 1$ and $1\leq t\leq 1/x$ one has

$$0\leq x^2 t\leq x^2\frac{1}{x}=x\leq 1,$$

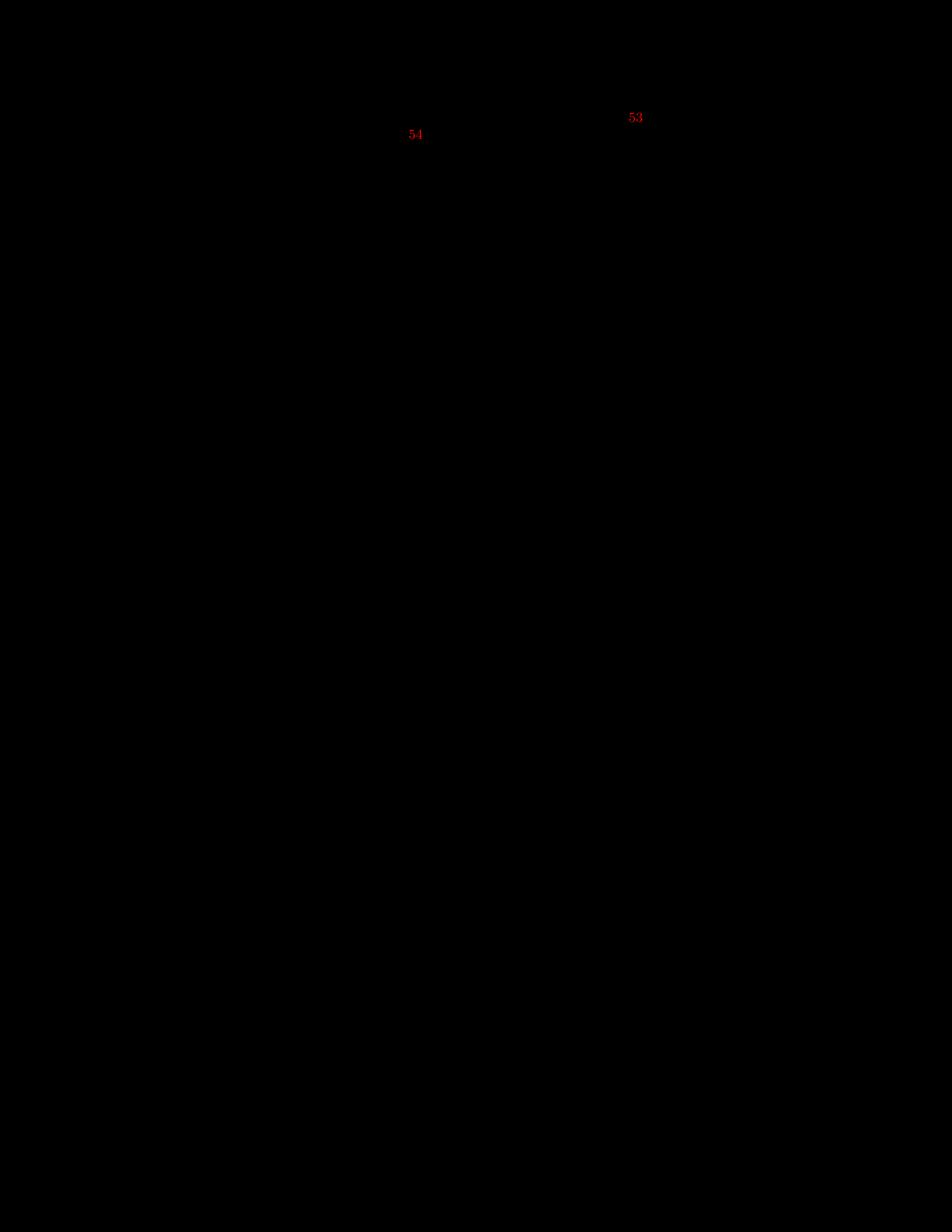
so that the integral $\int_1^{1/x} f(x^2t)\,dt$ is well defined for $0<x\leq 1$. Moreover, for $-1\leq x<0$ and $1/x\leq t\leq 0$, one has

$$-1\leq x=x^2\frac{1}{x}\leq x^2 t\leq 0,$$

and so the integral $\int_1^{1/x} f(x^2t)\,dt=-\int_0^1 f(x^2t)\,dt-\int_{1/x}^0 f(x^2t)\,dt$ is well defined for $-1\leq x<0$, too. If $x=0$, though, then the symbol $\int_1^{1/x}$ is not well defined as $1/0^+=\infty$ and $1/0^-=-\infty$. Actually no function can be a solution to $x^2+\int_1^{1/x} f(x^2t)dt=1$ also at this point, as $\int_1^{\pm\infty} f(0)dt$ is divergent if $f(0)\neq 0$, and if $f(0)=0$, then $\int_1^{\pm\infty} 0\,dt=0$ but $0+0\neq 1$.
Thus we need to interprete at $x=0$ this functional equation as

$$\lim_{x\to 0}\left(x^2+\int_1^{1/x} f(x^2t)dt\right)=1.$$

$x\in[-1,1]\setminus\{0\}=:X$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4889. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Find all non-constant continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{y-x}\int_x^y f(g(t))\mathrm{d}t = f\left(\frac{x+y}{2}\right), \quad \forall x, y \in \mathbb{R},\ x \neq y. \tag{1}$$

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4862. *Proposed by Michel Bataille.*

Let m be a nonnegative integer. Find

$$\lim_{n\to\infty} \frac{1}{2^n n^m} \sum_{k=0}^{n} \binom{m+k}{k} \binom{m+n+1}{n-k}.$$

58

4870*. *Proposed by Borui Wang.*

Define the series $\{a_n\}$ by the following recursion: $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ for $n > 0, q > 0$. Find the constant number $c(q)$ such that

$$\lim_{n\to\infty} (a_n - \sqrt{c(q) \cdot n}) = 0.$$

4866. *Proposed by Ivan Hadinata.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equation

$$f(xy + f(f(y))) = xf(y) + y$$

holds for all real numbers x and y.

62

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62

4857. *Proposed by Toyesh Prakash Sharma.*

Let a, b, c be positive real numbers such that $a+b+c=\frac{3}{2}$. Show that

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

Solution to problem 4857 Crux Math. 49 (5) 2023, 323

Raymond Mortini

Since the function $\log x$ is concave on $]0,\infty[$, we have

$$\log\left(\frac{A+B+C}{3}\right) \geq \frac{\log A + \log B + \log C}{3} =: R.$$

Here we take

$$A := a^a b^b, B := b^b c^c, C = c^c a^a.$$

Now the function $f(x) := 2x\log x$ is convex on $]0,\infty[$, since $f''(x) = 2/x \geq 0$. Hence

$$\begin{aligned} 2a\log a + 2b\log b + 2c\log c &= f(a)+f(b)+f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \\ &= 3f\left(\frac{1}{2}\right) = -3\log 2. \end{aligned}$$

Hence $3R \geq -3\log 2$, equivalently $R \geq \log(1/2)$ from which we deduce that $A+B+C \geq 3/2$. In other words

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

4855. *Proposed by Ivan Hadinata.*

Find all pairs of positive integers (a, b) such that $a^b - b^a = a - b$.

Solution to problem 4855 Crux Math. 49 (5) 2023, 323

Raymond Mortini, Rudolf Rupp

We claim that all solutions $(a, b) \in \mathbb{N} \times \mathbb{N}$ are given by

$$\boxed{(1, v), (u, 1), (t, t), (2, 3), (3, 2)}$$

where $u, v, t \in \mathbb{N} := \{1, 2, \cdots\}$ can be arbitrarily chosen.

It is easily seen that these are solutions. Now let (a, b) be a solution. Then (b, a) is a solution, too. If $b = a$, or if $b = 1$, then nothing remains to be shown. So we may assume that $a > b > 1$. Let $\log x$ be the natural logarithm. Now the function $f : x \mapsto x/\log x$ is strictly increasing for $x \geq e$ and strictly decreasing for $1 < x \leq e$ with $\min_{x>0} f(x) = e$. So if $a > b \geq 3 > e$,

$$\frac{a}{\log a} > \frac{b}{\log b}$$

or equivalently,

$$b^a > a^b.$$

Hence $a^b - b^a < 0$, but $a - b > 0$. So this case, where $a > b \geq 3$, does not occur. So it remains to consider the case $a > b = 2$. If $a = 3$, then we actually have the solution $(3, 2)$. If $a \geq 4$, then

$$\frac{a}{\log a} \geq \frac{4}{\log 4} = \frac{2}{\log 2},$$

and so

$$a^2 - 2^a \leq 0 < a - 2.$$

Thus this case $a \geq 4 > 2 = b$ does not occur, either. As all cases have been considered, we obtain the assertion.

4854. *Proposed by Michel Bataille.*

Let n be a positive integer and let $\theta_k = \frac{k\pi}{n+1}$. For $r, s \in \{1, 2, \dots, n\}$, evaluate

$$\sum_{j=1}^{n} (\sin\theta_{jr} + \sin\theta_{js})^2.$$

Solution to problem 4854 Crux Math. 49 (5) 2023, 323

Raymond Mortini, Rudolf Rupp

We prove that for $1 \le r, s \le n$,

$$\boxed{S := \sum_{j=1}^{n}\left(\sin\left(j\frac{r\pi}{n+1}\right) + \sin\left(j\frac{s\pi}{n+1}\right)\right)^2 = \begin{cases} n+1 & \text{if } r \neq s \\ 2(n+1) & \text{if } r = s \end{cases}}$$

We first show that

$$\sum_{j=1}^{n}\cos\left(j\frac{2\rho\pi}{n+1}\right) = \begin{cases} -1 & \text{if } \rho \in \mathbb{Z}\setminus(n+1)\mathbb{Z} \\ n & \text{if } \rho \in (n+1)\mathbb{Z} \end{cases} \tag{65}$$

and that for odd $\rho \in \mathbb{Z}$

$$\sum_{j=1}^{n}\cos\left(j\frac{\rho\pi}{n+1}\right) = 0. \tag{66}$$

To see this, we will use that $\cos x = \mathrm{Re}(e^{ix})$, and that

$$\sum_{j=1}^{n} e^{ijt} = e^{it}\sum_{j=0}^{n-1} e^{ijt} = e^{it}\frac{1-e^{int}}{1-e^{it}} = \frac{e^{it}-e^{i(n+1)t}}{1-e^{it}}. \tag{67}$$

Now put $t = 2\rho\pi/(n+1)$ whenever $\rho \in \mathbb{Z}\setminus(n+1)\mathbb{Z}$. The latter guarantees that the denominator does not vanish. Hence

$$\sum_{j=1}^{n} e^{ij\frac{2\rho\pi}{n+1}} = \frac{e^{i\frac{2\rho\pi}{n+1}}-1}{1-e^{i\frac{2\rho\pi}{n+1}}} = -1.$$

Now if $\rho \in (n+1)\mathbb{Z}$, then,

$$\sum_{j=1}^{n} e^{ij\frac{2\rho\pi}{n+1}} = n.$$

Thus (65) holds. If ρ is odd, then, by putting $t = \rho\pi/(n+1)$ in (67), we obtain

$$\sum_{j=1}^{n} e^{ij\frac{\rho\pi}{n+1}} \quad = \quad \frac{e^{i\frac{\rho\pi}{n+1}}+1}{1-e^{i\frac{\rho\pi}{n+1}}} = i\,\cot\left(\frac{1}{2}\,\frac{\rho\pi}{n+1}\right).$$

This is a purely imaginary number, so its real part is 0. This yields (66).
From (65) we easily deduce that for $r \in \{1, 2, \dots, n\}$

$$\sum_{j=1}^{n}\sin^2\left(j\frac{r\pi}{n+1}\right) = \frac{n+1}{2}. \tag{68}$$

In fact, using that $\sin^2 x = \frac{1-\cos 2x}{2}$, we obtain from (65)

$$\begin{aligned}\sum_{j=1}^{n} \sin^2\left(j\frac{r\pi}{n+1}\right) &= \sum_{j=1}^{n} \frac{1-\cos\left(j\frac{2r\pi}{n+1}\right)}{2} \\ &= \frac{n}{2} - \frac{1}{2}\sum_{j=1}^{n}\cos\left(j\frac{2r\pi}{n+1}\right) \\ &= \frac{n+1}{2}.\end{aligned}$$

We are now ready to calculate the value of S.

• *Case 1* $\boxed{r = s.}$ Then

$$\begin{aligned}S = \sum_{j=1}^{n}\left(\sin\left(j\frac{r\pi}{n+1}\right) + \sin\left(j\frac{r\pi}{n+1}\right)\right)^2 &= 4\sum_{j=1}^{n}\sin^2\left(j\frac{r\pi}{n+1}\right) \\ &\underset{(68)}{=} 4\frac{n+1}{2} = 2(n+1).\end{aligned}$$

• *Case 2* $\boxed{r \neq s.}$ Since $r, s \in \{1, 2, \dots, n\}$, r and s do not belong to $(n+1)\mathbb{Z}$. Note that due to $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$,

(69) $$(\sin x + \sin y)^2 = \sin^2 x + \sin^2 y + \cos(x-y) - \cos(x+y).$$

Hence

$$\begin{aligned}S &= \sum_{j=1}^{n}\sin^2\left(j\frac{r\pi}{n+1}\right) + \sum_{j=1}^{n}\sin^2\left(j\frac{s\pi}{n+1}\right) + \sum_{j=1}^{n}\cos\left(j\pi\frac{r-s}{n+1}\right) - \sum_{j=1}^{n}\cos\left(j\pi\frac{r+s}{n+1}\right) \\ &\underset{(68)}{=} n+1 + \sum_{j=1}^{n}\cos\left(j\pi\frac{r-s}{n+1}\right) - \sum_{j=1}^{n}\cos\left(j\pi\frac{r+s}{n+1}\right) \\ &= n+1 + S_1 - S_2.\end{aligned}$$

Several cases have to be analyzed now:

a) $r-s$ is even, say $r-s = 2\rho$, where $\rho \in \mathbb{Z}$. Then $r+s$ is even, too. Since $0 < |r-s| \leq n-1$ and $0 < r+s \leq 2n < 2(n+1)$, we again have two subcases:

a1) $r+s \notin \mathbb{Z}(n+1)$ (equivalently $r+s \neq n+1$): Then by (65),

$$S = n+1+(-1)-(-1) = n+1.$$

a2) $r+s = n+1 \in \mathbb{Z}(n+1)$: Then n is odd, say $n = 2m+1$ for some $m \in \{0,1,2,\dots\}$, and so

$$S_2 = \sum_{j=1}^{2m+1}\cos(j\pi) = \underline{(-1)+(+1)} + \cdots + \underline{(-1)+(+1)} + (-1) = -1.$$

Hence

$$S = n+1+(-1)-(-1) = n+1.$$

b) $r-s$ is odd. Then $r+s$ is odd, too. Again we have two subcases:

b1) $r+s \neq n+1$: Then by (66),

$$S = n+1+0-0 = n+1.$$

b2) $r+s = n+1$. Then n is even, say $n = 2m$ with $m \in \{1,2,\dots\}$, and so

$$S_2 = \sum_{j=1}^{2m}\cos(j\pi) = \underline{(-1)+(+1)} + \cdots + \underline{(-1)+(+1)} = 0.$$

Hence

$$S = n+1+0-0 = n+1.$$

4844. *Proposed by Seán M. Stewart.*

Suppose n is a positive integer. Show that the value of the improper integral

$$\int_0^\infty \frac{x^{n-1}e^{-x}}{\sqrt{x}}\left(\sum_{k=0}^{n-1}\binom{2k}{k}\frac{x^{-k}}{2^{2k}(n-k-1)!}\right)dx$$

is independent of n.

Solution to problem 4844 Crux Math. 49 (5) 2023, 273

Raymond Mortini, Rudolf Rupp

For $n \geq 1$, let

$$I_n := \int_0^\infty \frac{x^{n-1}e^{-x}}{\sqrt{x}}\left(\sum_{k=0}^{n-1}\binom{2k}{k}\frac{x^{-k}}{2^{2k}(n-k-1)!}\right)\,dx.$$

We show that

$$\boxed{I_n = \sqrt{\pi}}.$$

We use the following well-known formulas, where Γ is the Gamma function:

(70) $$\int_0^\infty x^{s-1}e^{-x}\,dx = \Gamma(s), \quad \Gamma(s+1) = s\,\Gamma(s), s > 0$$

(71) $$\Gamma\left(m+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\cdots\frac{2m-1}{2} = \sqrt{\pi}\,\frac{\prod_{k=1}^m(2k-1)}{2^m} = \frac{(2m)!}{m!4^m}\sqrt{\pi}$$

So, with $m = n-k-1$,

$$\begin{aligned} I_n &= \sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{2^{2k}(n-k-1)!}\int_0^\infty x^{(n-k-1/2)-1}e^{-x}\,dx \\ &= \sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{2^{2k}(n-k-1)!}\,\Gamma(n-k-1/2) \\ &= \sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{2^{2k}(n-k-1)!}\,\frac{(2(n-k-1))!}{(n-k-1)!4^{n-k-1}}\sqrt{\pi} \\ &= \frac{1}{4^{n-1}}\sum_{k=0}^{n-1}\frac{(2k)!}{(k!)^2}\frac{(2(n-k-1))!}{((n-k-1)!)^2}\sqrt{\pi} = \frac{1}{4^{n-1}}\sum_{k=0}^{n-1}\binom{2k}{k}\binom{2(n-k-1)}{n-k-1}\sqrt{\pi}. \end{aligned}$$

This is related to the coefficient in the Cauchy product of

$$\sum_{n=0}^\infty\binom{2n}{n}x^n = \sum_{n=0}^\infty\binom{-1/2}{n}(-1)^n4^nx^n = \frac{1}{\sqrt{1-4x}},$$

with itself and which converges for $|x| < 1/4$, or if we take $x = y/4$,

$$\sum_{n=0}^\infty\frac{1}{4^n}\binom{2n}{n}y^n = \frac{1}{\sqrt{1-y}}.$$

In fact, for $|y| < 1$,

$$\sum_{m=0}^\infty y^m = \frac{1}{1-y} = \frac{1}{\sqrt{1-y}}\frac{1}{\sqrt{1-y}} = \sum_{m=0}^\infty\left(\sum_{k=0}^m\frac{\binom{2k}{k}}{4^k}\frac{\binom{2(m-k)}{m-k}}{4^{m-k}}\right)y^m$$

The coefficients being unique, we deduce that for every $m = 0, 1, \cdots$

$$\sum_{k=0}^m\frac{\binom{2k}{k}}{4^k}\frac{\binom{2(m-k)}{m-k}}{4^{m-k}} = 1.$$

Hence, with $m = n - 1$, we conclude that $I_n = \sqrt{\pi}$.

4850. *Proposed by George Stoica.*

Let R be a finite field of characteristic 2 and $n \geq 2$. Then the sum of all invertible $n \times n$ matrices over R is the $n \times n$ zero matrix over R.

Solution to problem 4850 Crux Math. 49 (5) 2023, 274, first version [14]

Raymond Mortini, Rudolf Rupp

Let $n \geq 2$. We show that for *any* finite field the sum S of all invertible $n \times n$ matrices is the $n \times n$ zero matrix O_n.

For $n \geq 1$, let $\mathcal{M}_n$ be the set of all $n \times n$ matrices and let $\mathcal{U}_n$ be the set of all invertible $n \times n$ matrices. Since the field has only a finite number of elements, $\mathcal{U}_n$ has only a finite number of elements. So $S := \sum_{U \in \mathcal{U}_n} U$ is a well defined element in $\mathcal{M}_n$. We will show that for every $\tilde{U} \in \mathcal{U}_n$,

$$S \cdot \tilde{U} = S.$$

Fix an invertible matrix $\tilde{U} \in \mathcal{U}_n$ and consider the map

$$\iota : \begin{cases} \mathcal{M}_n & \to \mathcal{M}_n \\ X & \mapsto X \cdot \tilde{U} \end{cases}.$$

Then ι is a bijection of $\mathcal{M}_n$ onto itself. The inverse is given by $\iota^{-1}(Y) = Y \cdot \tilde{U}^{-1}$, since

$$\iota \circ \iota^{-1}(Y) = \iota(Y \cdot \tilde{U}^{-1}) = (Y \cdot \tilde{U}^{-1}) \cdot \tilde{U} = Y$$

and

$$\iota^{-1} \circ \iota(X) = \iota^{-1}(X \cdot \tilde{U}) = (X \cdot \tilde{U}) \cdot \tilde{U}^{-1} = X.$$

Moreover, and this is the main point here, ι maps $\mathcal{U}_n$ bijectively onto itself. Thus (and here we have not yet used that $n \neq 1$)

$$S = \sum_{U \in \mathcal{U}_n} \iota(U) = \iota\Big(\sum_{U \in \mathcal{U}_n} U\Big) = \iota(S) = S \cdot \tilde{U}. \tag{72}$$

Now we use that $n \geq 2$. Take for $\tilde{U}$ and $1 \leq i < j \leq n$ the elementary matrices

$$E_{ij} = (\vec{e}_1, \ldots, \underbrace{\vec{e}_j}_{i\text{-th col}}, \ldots, \underbrace{\vec{e}_i}_{j\text{-th col}}, \ldots, \vec{e}_n),$$

which interchange for $X \cdot E_{ij}$ the i-th and j-th column of X. Thus $S \cdot E_{ij} = S$ implies that all the columns of S are the same. Say $S = (\vec{s}, \ldots, \vec{s})$. Next we consider the matrix

$$E = \begin{pmatrix} 1 & 1 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{pmatrix}.$$

Note that the action $X \cdot E$ of E on a matrix X is to replace the second column of X by the sum of the first and second column. Since E is invertible, we obtain from (72) that $S \cdot E = S$ and so

$$\vec{s} + \vec{s} = \vec{s}.$$

Hence $\vec{s} = \vec{0}$. Consequently $S = O_n$.

Remark We may also consider the case $n = 1$. Note that the smallest field is given by $\mathbb{F}_2 := \{0, 1\}$, with $1 \neq 0$, where 0 is the neutral element for addition and 1 the one for multiplication. This necessarily has characteristic 2. Here $S = 1$. If the finite field is not field-isomorphic to $\mathbb{F}_2$, it has more than two elements, and so there is an (invertible) element u different from 1. Now by (72), $S = Su$, hence $S(1 - u) = 0$. Since $1 - u \neq 0$, hence invertible,we conclude that $S = 0$.

[14] This was tacitly replaced by another problem later on.

4835. *Proposed by George Stoica.*

Prove that the four complex numbers z_i, $i = 1, \ldots, 4$, are the consecutive vertices of a cyclic quadrilateral (or are collinear) in the complex plane if and only if the number $\dfrac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$ is real.

Solution to problem 4835 Crux Math. 49 (4) 2023, 213

Raymond Mortini, Rudolf Rupp

This is a standard result/exercise in old monographs on function theory/complex analysis and is for instance in [1, p. 70] (see figure (**??**).

Using a not so sophisticated wording, we will show that four distinct points z_j $(j = 1, \ldots, 4)$ in the plane belong to a circle or a line if and only if their cross-ratio (bi-rapport, Doppelverhältnis)

$$DV(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_1 - z_4} \Big/ \frac{z_3 - z_2}{z_3 - z_4}$$

is a real number.

In particular, being real, will be independent of the "order" of the points on the circle, respectively line.

Our proof will be done in the extended complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (also called the one-point compactification of $\mathbb{C}$). Let us recall some terminology here. If L is a line in $\mathbb{C}$, then $L \cup \{\infty\}$ is called an extended line. As usual we call the elements of the set of circles and extended lines in $\widehat{\mathbb{C}}$ "generalized circles".

We also use an extension of the definition of the cross-ratio to points in $\widehat{\mathbb{C}}$. This is done by taking limits. For instance

$$D(z_1, z_2, z_3, \infty) = \frac{z_1 - z_2}{z_3 - z_2}. \tag{73}$$

Finally, let us recall the following results:

i) There is a unique linear-fractional map (or in modern terminology, a Möbius transform) $T(z) := (az + b)/(cz + d)$, $ad - bc \neq 0$, viewed as map from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ mapping three distinct points z_2, z_3, z_4 in $\widehat{\mathbb{C}}$ to $0, 1, \infty$, namely $T(z) = DV(z, z_2, z_3, z_4)$.

ii) The cross ratio is invariant under linear-fractional maps:

$$DV(T(z_1), T(z_2), T(z_3), T(z_4)) = DV(z_1, z_2, z_3, z_4).$$

Note that the latter is an immediate consequence of i).

iii) The class of generalized circles is invariant under Möbius transforms.

Now we are ready to confirm the statement above:

Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$, consider the map $S(z) := DV(z, z_2, z_3, z_4)$. Suppose that these z_j belong to a generalized circle E. Now S maps E to the extended real line $\mathbb{R} \cup \{\infty\}$, since $z_2 \to 0$, $z_3 \to 1$ and $z_4 \to \infty$. In particular, $w_j := S(z_j) \in \mathbb{R} \cup \{\infty\}$ for $j = 1, \ldots, 4$. Since $DV(w_1, w_2, w_3, w_4)$ is real, the invariance result shows that $DV(z_1, z_2, z_3, z_4)$ is real.

Conversely, suppose that $DV(z_1, z_2, z_3, z_4)$ is real. Note that $S(z_j) \in \{0, 1, \infty\} \subseteq \mathbb{R} \cup \{\infty\}$ for $j = 2, 3, 4$. Now the image of the extended real line by the inverse Möbius transform S^{-1} is a generalized circle, E. Of course E contains the points z_2, z_3 and z_4. But, by (73), and the assumption, we have

$$S(z_1) = DV(S(z_1), S(z_2), S(z_3), S(z_4)) = DV(z_1, z_2, z_3, z_4) \in \mathbb{R}.$$

Hence $z_1 = S^{-1}(S(z_1)) \in E$. In other words, all the z_j belong either to a circle or a line.

This can be shortened, without the explicit use of the cross ratio. Actually, just iii) is relevant here: Consider the Möbius transform

$$M(z) := \frac{z - z_4}{z - z_2}\,\frac{z_3 - z_2}{z_3 - z_4}.$$

Then z_4, z_3, z_2 are mapped to $0, 1, \infty$, and so the (unique) generalized circle E determined by z_4, z_3, z_2 is mapped to the extended real line. Thus the point z_1 belongs to E if and only if $M(z_1) \in \mathbb{R}$. In other words, all the z_j belong either to a circle or a line if and only if $\frac{z_1 - z_4}{z_1 - z_2}\,\frac{z_3 - z_2}{z_3 - z_4} \in \mathbb{R}$.

References

[1] K. Knopp, *Elemente der Funtionentheorie* Sammlung Göschen, Berlin, Leipzig 1937
175

4836. *Proposed by Mohammad Bakkar.*

Prove the following formula:

$$\frac{\pi^3}{32} = \prod_{n=1, 2n+1 \notin \mathcal{P}}^{\infty} \frac{4n(n+1)}{(2n+1)^2},$$

where $\mathcal{P}$ is the set of prime numbers.

Solution to problem 4836 Crux Math. 49 (4) 2023, 214

Raymond Mortini, Rudolf Rupp

We first calculate the missing part

$$P := \prod_{\substack{n=1 \\ 2n+1 \in \mathcal{P}}}^{\infty} \frac{4n(n+1)}{(2n+1)^2}.$$

Put $p := 2n+1$. Then $n = (p-1)/2$ and so, in view of the Euler formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{p \in \mathcal{P}} \frac{1}{1-p^{-2}},$$

we have

$$P = \prod_{\substack{p \in \mathcal{P} \\ p \neq 2}} \frac{p^2-1}{p^2} = \frac{4}{3}\,\frac{6}{\pi^2} = \frac{8}{\pi^2}.$$

To calculate

$$R := \prod_{n=1}^{\infty} \frac{4n(n+1)}{(2n+1)^2},$$

we use partial products and Stirling's formula $\lim\limits_{n\to\infty} \dfrac{n^n e^{-n}\sqrt{2\pi n}}{n!} = 1$.

$$\begin{aligned}
P_N := \prod_{n=1}^{N} \frac{4n(n+1)}{(2n+1)^2} &= \frac{4^N\ N!(N+1)!}{\left(\frac{(2N+1)!}{\prod_{n=1}^{N}(2n)}\right)^2} = \frac{4^N N!(N+1)!}{(2N+1)!^2}\ \frac{\left(2^N N!\right)^2}{1} \\
&= \frac{4^{2N} N!^4 (N+1)}{(2N+1)!^2} \\
&\sim \frac{4^{2N}\ N^{4N} e^{-4N} 4\pi^2 N^2 (N+1)}{(2N+1)^{4N+2} e^{-4N-2}\ 2\pi(2N+1)} \\
&= \pi e^2\ \frac{(2N)^{4N} N^2 (N+1)}{(2N+1)^{4N}(2N+1)^3} \\
&= 2\pi e^2\ \frac{1}{\left[\left(1+\frac{1}{2N}\right)^{2N}\right]^2}\,\frac{N^2(N+1)}{(2N+1)^3} \\
&\to 2\pi e^2\ \frac{1}{e^2}\ \frac{1}{8} = \frac{\pi}{4}.
\end{aligned}$$

Hence $\displaystyle\prod_{\substack{n=1 \\ 2n+1 \notin \mathcal{P}}}^{\infty} \frac{4n(n+1)}{(2n+1)^2} = \frac{\pi/4}{8/\pi^2} = \frac{\pi^3}{32}.$

A second way to derive the value of P is as follows:

For $z \in \mathbb{C}$ we have

$$\begin{aligned}
\sin(\pi z) &= \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right) e^{\frac{z}{n}} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \\
&= \pi z(1-z)e^z \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right) e^{\frac{z}{n+1}} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \\
&= \pi z(1-z)e^z \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)\left(1+\frac{z}{n}\right) e^{\frac{z}{n+1}-\frac{z}{n}} \\
&= \pi z(1-z)e^z \, e^{\sum_{n=1}^{\infty}\left(\frac{z}{n+1}-\frac{z}{n}\right)} \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)\left(1+\frac{z}{n}\right) \\
&= \pi z(1-z) \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)\left(1+\frac{z}{n}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
P &= \prod_{n=1}^{\infty}\frac{4n(n+1)}{(2n+1)^2} = \prod_{n=1}^{\infty}\frac{2n}{2n+1}\frac{2n+2}{2n+1} \\
&= \prod_{n=1}^{\infty}\frac{1}{1+\frac{1}{2n}}\;\frac{1}{1-\frac{1}{2(n+1)}} = \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2(n+1)}\right)} \\
&= \left.\frac{\pi z(1-z)}{\sin(\pi z)}\right|_{z=1/2} \\
&= \frac{\pi}{4}.
\end{aligned}$$

4828. *Soumis par Narendra Bhandari.*

Démontrer que

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{\sec(x+y)\sec(x-y)}{\sec x\sec y}dxdy=\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2.$$

Solution to problem 4828 Crux Math. 49 (3) 2023, 157

Raymond Mortini, Rudolf Rupp

Let

$$I:=\int_0^{\pi/4}\underbrace{\int_0^{\pi/4}\frac{\cos x\ \cos y}{\cos(x+y)\ \cos(x-y)}\,dy}_{:=I(x)}\,dx.$$

Now fix the variable x. Since

$$\cos(x+y)\ \cos(x-y)=\cos^2 y-\sin^2 x,$$

we obtain

$$\begin{aligned}
I(x) &= \cos x\int_0^{\pi/4}\frac{\cos y}{(1-\sin^2 x)-\sin^2 y}\,dy\\
\overset{u:=\sin y}{} &= \cos x\int_0^{\sqrt{2}/2}\frac{du}{\cos^2 x-u^2}\\
&= \frac{1}{2}\left(\log(\cos x+u)-\log(\cos x-u)\right)\Big|_{u=0}^{\sqrt{2}/2}\\
&= \frac{1}{2}\log\left(\frac{\cos x+1/\sqrt{2}}{\cos x-1/\sqrt{2}}\right).
\end{aligned}$$

Hence (using Fubini),

(74) $$I=\frac{1}{2}\int_0^{\pi/4}\log\left(\frac{\sqrt{2}\cos x+1}{\sqrt{2}\cos x-1}\right)\,dx.$$

The value of this integral is known to be the Catalan number C (see formula (18) in [1]). An independent proof is below: using that $\cos a+\cos b=2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$ and $\cos(a-b)=-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$, we obtain

$$\begin{aligned}
\log\left(\frac{\sqrt{2}\cos x+1}{\sqrt{2}\cos x-1}\right) &= \log\left(\frac{\cos x+\cos\pi/4}{\cos x-\cos\pi/4}\right)\\
&= -\log\tan\left(\frac{x+\pi/4}{2}\right)-\log\tan\left(\frac{-x+\pi/4}{2}\right).
\end{aligned}$$

A change of the variable $x+\pi/4=2y$, respectively $-x+\pi/4=2y$, and a standard integral representation of C yields

$$I=-\frac{1}{2}\int_{\pi/8}^{\pi/4}\log\tan y\ (2dy)-\frac{1}{2}\int_0^{\pi/8}\log\tan y\ (2dy)=-\int_0^{\pi/4}\log\tan y\ dy=C.$$

A proof of this standard representation can be given for instance by using power series or Fourier series:

$$h(z):=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)=\sum_{n=0}^{\infty}\frac{1}{2n+1}z^{2n+1}.$$

Its Taylor coefficients belong to ℓ^2 and so the associated Fourier series

$$h^*(e^{it}):=\sum_{n=0}^{\infty}\frac{1}{2n+1}e^{i(2n+1)t}$$

converges in the $L^2(]0,\pi[)$-norm to

$$h(e^{it}) = \frac{1}{2}\log(i\cot(t/2)) = i\,\frac{\pi}{4} - \frac{1}{2}\log\tan(t/2)$$

(Actually the series $h^*(e^{it})$ converges pointwise for $z = e^{it}$ with $0 < t < \pi$ by the Abel-Dirichlet rule, but we do not need this.)

Taking real parts, and using that $\int\sum = \sum\int$ (note that Fourier series converge in the L^2-norm, hence in the L^1 norm), we may conclude that

$$-\int_0^{\pi/4} \log\tan y\, dy = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{1}{2n+1}\cos(2n+1)t\, dt = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)^2}.$$

References

[1] Seán M. Stewart, A Catalan constant inspired integral odyssey, Math. Gaz. 104, No. 561, 449-459 (2020), 179

4830. *Proposed by Goran Conar.*

Let $a_i \in (0, \frac{1}{2})$, $i \in \{1, 2, \ldots, n\}$ be real numbers such that $\sum_{i=1}^{n} a_i = 1$. Prove that the following inequalities hold:

$$n\sqrt{\frac{n-1}{n+1}} \leq \sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_i}} < (n+1)\sqrt{\frac{n-1}{n+1}}.$$

Solution to problem 4830 Crux Math. 49 (3) 2023, 158

Raymond Mortini, Rudolf Rupp

First we claim that on $[0, 1/2]$ the function $f(x) = \sqrt{\frac{1-x}{1+x}}$ is convex. In fact,

$$f'(x) = -\frac{1}{\sqrt{\frac{1-x}{1+x}}(1+x)^2}$$

and

$$f''(x) = \frac{1-2x}{(1-x)(x+1)^3\sqrt{\frac{1-x}{1+x}}} \geq 0.$$

Since the graph of a convex function lies below the secant determined by $(a, f(a)), (b, f(b))$, we obtain that $f(x) \leq 1 - 2(1 - 3^{-1/2})x$, where $a = 0$ and $b = 1/2$. Since $1 - 3^{-1/2} \geq 1/3$, we deduce that for $0 \leq x \leq 1/2$

$$f(x) \leq 1 - (2/3)x$$

, and so

$$\sum_{i=1}^{n} f(a_i) \leq n - (2/3)\sum_{i=1}^{n} a_i = n - 2/3.$$

But for $n \geq 2$, we have

$$n - 2/3 < (n+1)\sqrt{\frac{n-1}{n+1}} = \sqrt{n^2 - 1},$$

since

$$n^2 - 1 - (n - 2/3)^2 = 4/3n - 13/9 \geq 8/3 - 13/9 = 11/9 > 0.$$

This upper bound in the problem appears to be artificial. We did not see a way to derive this in a natural way. To prove the reverse inequality, we use Jensen's inequality and obtain

$$\frac{1}{n}\sum_{i=1}^{n} f(a_i) \geq f\left(\frac{\sum_{i=1}^{n} a_i}{n}\right) = f(1/n).$$

Hence

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_i}} \geq n\sqrt{\frac{1-\frac{1}{n}}{1+\frac{1}{n}}} = n\sqrt{\frac{n-1}{n+1}}.$$

4826. *Proposed by Paul Bracken.*

Let H_n is the n-th harmonic number $H_n = \sum_{k=1}^{n} 1/k$. Evaluate the following sum in closed form

$$S = \sum_{k=1}^{\infty} \frac{H_k}{k(k+1)(k+2)}.$$

Solution to problem 4826 Crux Math. 49 (3) 2023, 157

Raymond Mortini, Rudolf Rupp

We claim that

$$\boxed{S = \frac{\pi^2}{12} - \frac{1}{2}}.$$

Just write

$$\begin{aligned}\frac{H_k}{k(k+1)(k+2)} &= \frac{1}{2}\left(H_k\Big(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\Big)\right)\\ &= \frac{1}{2}\left(\frac{H_k}{k(k+1)} - \frac{H_{k+1}}{(k+1)(k+2)} + \frac{1}{(k+1)^2(k+2)}\right).\end{aligned}$$

Now

$$\frac{1}{(k+1)^2(k+2)} = \frac{1}{k+2} - \frac{k+1-1}{(k+1)^2} = \left(\frac{1}{k+2} - \frac{1}{k+1}\right) + \frac{1}{(k+1)^2}.$$

Since the Cesaro means of the sequences $(1/k)$ converge to 0, that is $H_k/k \to 0$, we conclude that

$$S = \frac{1}{2}\frac{H_1}{2} - \frac{1}{2}\frac{1}{1+1} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k+1)^2} = \frac{\pi^2}{12} - \frac{1}{2}.$$

4825. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Calculate

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)}.$$

Solution to problem 4825 Crux Math. 49 (3) 2023, 157

Raymond Mortini, Rudolf Rupp

We prove that

$$\boxed{I := \sum_{n=1}^{\infty} \frac{O_n}{n(n+1)} = \log 4.}$$

First we note that

$$\frac{O_n}{n(n+1)} = O_n\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{O_n}{n} - \frac{O_{n+1}}{n+1} + \frac{1}{(2n+1)(n+1)}.$$

Since the Cesaro means of the null sequence $(1/(2n+1))$ converge to 0, we obtain

$$I = \frac{O_1}{1} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)(n+1)} = 1 + 2\log 2 - 1 = \log 4.$$

The value of the series $S := \sum_{n=1}^{\infty} \frac{1}{(2n+1)(n+1)}$ can be determined as follows:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{(2n+1)(n+1)} &= \sum_{n=1}^{N} \left(\frac{2}{2n+1} - \frac{1}{n+1}\right) \\
\text{splitting into even an odd} &= \sum_{n=1}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n+1}\right) + \sum_{n=1}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n}\right) + \sum_{n=N+1}^{2N+1} \frac{1}{n} \\
&= -1 + \sum_{n=1}^{2N+1} (-1)^{n+1} \frac{1}{n} + \sum_{n=N+1}^{2N+1} \frac{1}{n} \\
&\underset{N\to\infty}{\longrightarrow} -1 + \log 2 + \log 2.
\end{aligned}$$

Note that the well-known assertion $\lim_{N\to\infty} \sum_{n=N+1}^{2N} \frac{1}{n} = \log 2$ is a direct consequence of the fact that the Euler-Mascheroni constant γ is given by

$$\gamma = \lim(H_n - \log n),$$

where $H_n := \sum_{i=1}^{n} \frac{1}{i}$, since

$$H_{2N} - H_N = (H_{2N} - \log(2N) - \gamma) + (\log N + \gamma - H_N) + \log 2 \to \log 2.$$

4822. *Proposed by Anton Mosunov.*

The n-th Chebyshev polynomial of the first kind is defined by means of the recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{for } n \geq 2.$$

Prove that for all $n \geq 2$,

$$\frac{1}{3} < \int\limits_1^{+\infty} \frac{dx}{T_n(x)^{2/n}} < \frac{1}{3}\sqrt[n]{4}.$$

Solution to problem 4822 Crux Math. 49 (3) 2023, 156

Raymond Mortini, Rudolf Rupp

Substituting $x = \cosh t$ we obtain $T_n(\cosh t) = \cosh(nt)$. In particular, T_n has no zeros on $[1, \infty[$. Hence

$$\begin{aligned} I := \int_1^\infty \frac{dx}{T_n(x)^{2/n}} &= \int_0^\infty \frac{\sinh t}{\big(\cosh(nt)\big)^{2/n}}\, dt = \int_0^\infty \frac{e^t - e^{-t}}{2\left(\dfrac{e^{nt}+e^{-nt}}{2}\right)^{2/n}}\, dt \\ &= 2^{-1+2/n} \int_0^\infty \frac{1-e^{-2t}}{e^t\big(1+e^{-2nt}\big)^{2/n}}\, dt. \end{aligned}$$

Hence

$$\begin{aligned} I &< 2^{-1+2/n} \int_0^\infty \frac{1-e^{-2t}}{e^t}\, dt = 2^{-1+2/n}\left[-e^{-t} + \frac{1}{3}e^{-3t}\right]_0^\infty \\ &= 2^{-1+2/n}\frac{2}{3} = \frac{1}{3}\sqrt[n]{4}. \end{aligned}$$

Moreover

$$I > 2^{-1+2/n} \int_0^\infty \frac{1-e^{-2t}}{e^t(1+1)^{2/n}}\, dt = 2^{-1} \int_0^\infty \frac{1-e^{-2t}}{e^t}\, dt = 2^{-1}\frac{2}{3} = \frac{1}{3}.$$

4816. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $a, b, k \geq 0$. Calculate

$$\lim_{n\to\infty} \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2 x^{2n}}\, dx.$$

Solution to problem 4816 Crux Math. 49 (2) 2023, 101

Raymond Mortini, Rudolf Rupp

We show that for $a, b, k \geq 0$ (k not necessary an integer)

$$\boxed{I_n := \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2 x^{2n}}\, dx \overset{n\to\infty}{\longrightarrow} \sqrt{b} + \frac{\sqrt{a}}{k+1/2}.}$$

Write

$$f_n(x) = x^{k-1/2}\sqrt{a + bn^2 x^{2n+1}}.$$

If $a = 0$, then

$$I_n = \int_0^1 \sqrt{b} n x^{n+k} dx = \frac{n\sqrt{b}}{n+k+1} \to \sqrt{b}.$$

For $a > 0$, let

$$d_n(x) := x^{k-1/2}\Big(\sqrt{a + bn^2 x^{2n+1}} - \sqrt{bn^2 x^{2n+1}}\Big).$$

Then

$$0 \leq d_n(x) = x^{k-1/2}\,\frac{a}{\sqrt{a + bn^2 x^{2n+1}} + \sqrt{b} n x^{n+1/2}} \leq \frac{a}{\sqrt{a}} x^{k-1/2}.$$

Hence d_n is dominated by an $L^1[0,1]$ function and so, by using that $nx^n \to 0$ for $0 < x < 1$,

$$\lim_n \int_0^1 d_n(x)dx = \int_0^1 \lim_n d_n(x)dx = \int_0^1 \sqrt{a} x^{k-1/2} = \frac{\sqrt{a}}{k+1/2}.$$

Consequently,

$$\begin{aligned}
\int_0^1 f_n(x)dx &= \int_0^1 d_n(x)dx + \sqrt{b}\int_0^1 n x^{k-1/2} x^{n+1/2} dx \\
&= \int_0^1 d_n(x)dx + \sqrt{b}\,\frac{n}{k+n+1} \\
&\underset{n\to\infty}{\longrightarrow} \frac{\sqrt{a}}{k+1/2} + \sqrt{b}.
\end{aligned}$$

4819. *Proposed by Daniel Sitaru.*

Let $f:[0,1]\to[0,1]$ be a continuous function and $0<a\le b<1$.
Prove that:

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \ge \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt+\int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

Solution to problem 4819 Crux Math. 49 (2) 2023, 102

Raymond Mortini, Rudolf Rupp

Note that the harmonic mean $x_0 := 2ab/(a+b)$ is less than or equal to the arithmetic mean $y_0 := (a+b)/2$. We show that the inequality holds for arbitrary x_0, y_0 with $0<x_0<y_0<1$. So let F be that primitive of f on $[0,1]$ with $F(0)=0$. We shall prove that

$$\boxed{2\int_{x_0}^{y_0} tf(t)\,dt \ge (F(y_0)-F(x_0))(F(y_0)+F(x_0))},$$

from which the desired inequality immediately follows. By partial integration,

$$2\int_{x_0}^{y_0} t\,f(t)\,dt = 2\int_{x_0}^{y_0} tF'(t)\,dt = 2(y_0F(y_0)-x_0F(x_0)) - 2\int_{x_0}^{y_0} F(t)dt. \tag{75}$$

For $0\le x,y\le 1$, put

$$H(x,y) := 2yF(y)-2xF(x)-2\int_x^y F(t)dt-(F(y)^2-F(x)^2).$$

We have to show that $H(x_0,y_0)\ge 0$. Since $0\le f\le 1$, $F(x)\le\int_0^x 1\,dt = x$. Hence

$$\frac{\partial H}{\partial x}(x,y) = -2\big(F(x)+xf(x)\big)+2F(x)+2F(x)f(x) = 2\big(F(x)-x\big)f(x)\le 0.$$

Consequently, by using that $H(y,y)=0$, we obtain $\xi\in]x_0,y_0[$ with

$$H(x_0,y_0) = H(x_0,y_0)-H(y_0,y_0) = \underbrace{\frac{\partial H}{\partial x}(\xi,y_0)}_{\le 0}\underbrace{(x_0-y_0)}_{\le 0}\ge 0.$$

4817. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $abc = 1$. Prove that the following inequality holds

$$\frac{a^7+a^3+bc}{a+bc+1}+\frac{b^7+b^3+ca}{b+ca+1}+\frac{c^7+c^3+ab}{c+ab+1}\geq 3\,.$$

When does equality occur?

Solution to problem 4817 Crux Math. 49 (2) 2023, 102

Raymond Mortini, Rudolf Rupp

Let $E :=]0,\infty[\times]0,\infty[\times]0,\infty[$ and let $H : E \to]0,\infty[$ be given by

$$H(a,b,c)=\frac{a^7+a^3+bc}{a+bc+1}+\frac{b^7+b^3+ca}{b+ca+1}+\frac{c^7+c^3+ab}{c+ab+1}.$$

Put $L := \{(a,b,c) \in E : abc = 1\}$. To be shown is that $\inf_L H = 3$ and that this lower bound is obtained exactly at $(1,1,1)$. To this end, consider for $x > 0$ the function

$$f(x) := \frac{x^7+x^3+x^{-1}}{x+x^{-1}+1}=\frac{x^8+x^4+1}{x^2+x+1}=x^6-x^5+x^3-x+1.$$

Then f is convex on $[0,\infty[$. In fact,

$$f'(x)=6x^5-5x^4+3x^2-1 \text{ and } f''(x)=30x^4-20x^3+6x=2x(15x^3-10x^2+3).$$

Now $f''(x) = 2x\big(5x^2(3x-2)+3\big)$. Then, clearly, $f''(x) \geq 0$ if $x \geq 2/3$. Since

$$\max_{[0,2/3]} x^2(2-3x)=32/3^5\leq 3/5,$$

we deduce that $f''(x) \geq 0$ on $[0,2/3]$, too. Due to Jensen's inequality, for $(a,b,c) \in L$

$$H(a,b,c) \quad = \quad f(a)+f(b)+f(c)=3\,\frac{f(a)+f(b)+f(c)}{3}\geq 3\,f\Big(\frac{a+b+c}{3}\Big)$$

Since f is convex for $x \geq 0$, $f(x) \geq f(1) + f'(1)(x-1) = 1 + 3(x-1) = -2 + 3x$. Why we take evaluation at 1? Because it works! It is an a posteriori choice, since the minimal value is taken at $(a,b,c) = (1,1,1)$. Thus we obtain the estimate

$$H(a,b,c)\geq 3\,(-2+(a+b+c))=-6+3\left(a+b+\frac{1}{ab}\right).$$

We can even avoid Jensen's inequality:

$$H(a,b,c)=f(a)+f(b)+f(c)\geq(-2+3a)+(-2+3b)+(-2+3c)=-6+3(a+b+c).$$

Since $a + b + \frac{1}{ab} \geq 3$ (see below) we deduce that for $abc = 1$ we have $H(a,b,c) \geq -6+9 = 3$. As $H(1,1,1) = 3$, we are done.

The inequality $g(a,b) := a + b + \dfrac{1}{ab} \geq 3$ is well known. It can for instance be shown by using differential calculus:

$$g_a(a,b)=1-\frac{1}{ab^2}=0 \iff ab^2=1 \text{ and } g_b(a,b)=1-\frac{1}{ba^2}=0 \iff ba^2=1.$$

In other words, $ab(a-b) = 0$. Hence $a = b = 1$ is the only stationary point. Thus $g(1,1) = 3$ is the minimum, since the limit of g at the boundary $ab = 0$ is ∞.

4811. *Proposed by Nguyen Viet Hung.*

Find all positive integers n such that $\sqrt{n^3+1}+\sqrt{n+2}$ is a positive integer.

Solution to problem 4811 Crux Math. 49 (2) 2023, 101

Raymond Mortini, Rudolf Rupp

We show that $n=2$ is the only solution. In fact $\sqrt{2^3+1}+\sqrt{2+2}=3+2=5$. Now, for $x,y\geq 0$, one has $\sqrt{x}+\sqrt{y}\in\mathbb{N}$ if and only if x and y are perfect squares. To see this, just note that

$$\sqrt{x}+\sqrt{y}=\frac{x-y}{\sqrt{x}-\sqrt{y}}$$

implies that $\sqrt{x}+\sqrt{y}\in\mathbb{Q}$ if and only $\sqrt{x}-\sqrt{y}\in\mathbb{Q}$ and so, by adding (respectively substracting), $\sqrt{x}$ and $\sqrt{y}$ are rational. Thus $\sqrt{x}=p/q$ for some $p,q\in\mathbb{N}$ with no common divisor. Hence $x^2=p^2/q^2\in\mathbb{N}$, and so $q=1$.

Due to a classical result by L. Euler, the Diophantine equation $n^3+1=m^2$ has in $\mathbb{N}=\{0,1,2,\dots\}$ only the solutions $(m,n)=(1,0)$ and $(m,n)=(3,2)$ (see for instance [1], a reference provided to the first author by Amol Sasane). Thus $n=2$ is the only positive integer also satisfying $\sqrt{n+2}\in\mathbb{N}$.

References

[1] https://mathoverflow.net/questions/39561/is-there-an-elementary-way-to-find-the-integer-solutions-to-x2-y3-1 188

4810. *Proposed by Goran Conar.*

Let $a_1, a_2, \dots, a_n > 0$ be real numbers such that $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$, $n > 1$. Prove that

$$\frac{a_2^2 + a_3^2 + \cdots + a_n^2}{(a_2 + a_3 + \cdots + a_n)^3} + \frac{a_1^2 + a_3^2 + \cdots + a_n^2}{(a_1 + a_3 + \cdots + a_n)^3} + \cdots + \frac{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}{(a_1 + a_2 + \cdots + a_{n-1})^3} \geq \frac{n\sqrt{n}}{(n-1)^2}.$$

Solution to problem 4810 Crux Math. 49 (1) 2023, 45

Raymond Mortini, Rudolf Rupp

We first show that whenever $\sum_{j=1}^n a_j^2 = 1$, then

(76)
$$\frac{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j^2}{\Big(\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j\Big)^3} \geq \frac{1}{\sqrt{1-a_i^2}}\;\frac{\sqrt{n-1}}{(n-1)^2}.$$

In fact, using Cauchy-Schwarz, we immediately obtain

$$\frac{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j^2}{\Big(\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j\Big)^3} \geq \frac{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j^2}{\Big(\big(\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j^2\big)(n-1)\Big)^{3/2}} = \frac{1}{\sqrt{1-a_i^2}}\;\frac{\sqrt{n-1}}{(n-1)^2}.$$

Next we prove that whenever $\sum_{j=1}^n a_j^2 = 1$, then

(77)
$$\sum_{i=1}^{n} \frac{1}{\sqrt{1-a_i^2}} \geq n\sqrt{\frac{n}{n-1}}.$$

In fact, consider the convex function $f(x) = \frac{1}{\sqrt{1-x}}$. By Jensen's inequality (or one of the possible defintions of convexity), if $\sum_{j=1}^n t_j = 1$ where $(0 \leq t_j \leq 1)$, then

$$f\Big(\sum_{j=1}^{n} t_j x_j\Big) \leq \sum_{j=1}^{n} t_j f(x_j).$$

Here we choose $x_i = a_i^2$, and $t_j = 1/n$. Note that $\frac{\sum_{i=1}^n a_i^2}{n} = 1/n$. Hence

$$\sum_{i=1}^{n} \frac{1}{\sqrt{1-a_i^2}} = n\sum_{i=1}^{n} \frac{1}{n} f(x_i) \geq n f\big(\frac{1}{n}\sum_{i=1}^{n} x_j\big) = \frac{n}{\sqrt{1-\frac{1}{n}}} = n\sqrt{\frac{n}{n-1}}.$$

Now putting (76) and (77) together yields

$$\sum_{i=1}^{n} \frac{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j^2}{\Big(\sum\limits_{\substack{j=1\\ j\neq i}}^{n} a_j\Big)^3} \geq \frac{\sqrt{n-1}}{(n-1)^2}\; n\sqrt{\frac{n}{n-1}} = \frac{n\sqrt{n}}{(n-1)^2}.$$

4803. *Proposed by Nguyen Viet Hung.*

Find all non-negative integers a, b, c and pairs (p, q) of prime numbers satisfying

$$p^{2a} + q^{2b} = (2c+1)^2.$$

Solution to problem 4803 Crux Math. 49 (1) 2023, 44

Raymond Mortini, Rudolf Rupp

It turns out that the triple $(3,4,5)$ satisfying $3^2+4^2=5^2$ is relevant here. Only one solution to the problem with $p \le q$ exists: $p=2, q=3$ and $a=2, b=1, c=2$. To sum up:

$$\boxed{2^{2\cdot 2} + 3^{2\cdot 1} = (2\cdot 2+1)^2}$$

To see this, we use of course the well known parametrizations of the solutions to $A^2+B^2=C^2$, which are given by

$$(*)\ A = 2mn, B = m^2 - n^2 \text{ and } C = m^2 + n^2,\ m, n \in \mathbb{N}.$$

The conditions to be dealt with are

$$i)\ 2mn = p^a,\ ii)\ m^2 - n^2 = q^b \text{ and } iii)\ m^2 + n^2 = 2c + 1.$$

• First we note that $(a,b) = (0,0)$ is not admissible as $1+1=2$ is even. Now if $b=0$ and $a \neq 0$, then by i) p necessarily must be an even prime, that is $p=2$. Hence

$$2^{2a} + 1 = (2c+1)^2.$$

By (*), $1 = m^2 - n^2$ and $2^{2a} = 2mn$. Consequently m and n are powers of 2. Hence $m^2 - n^2$ is an even number; and not 1. Thus $ab \neq 0$.

• So let $ab > 0$. Since p is prime, m and n can only be powers of 2 by (i). Due to iii), telling us that m^2+n^2 is an odd number, not both m and n can be proper powers of 2. Since $m \ge n$ (by ii)), we necessarily have $n=1$ and $m = 2^x$ with $x \neq 0$. By ii),

$$q^b = m^2 - 1 = (2^x)^2 - 1 = (2^x - 1)(2^x + 1).$$

This implies that $q \neq 2$ (as the right hand side is odd). Since the difference of the factors is 2, $q \ge 3$ cannot divide both factors. Thus we can only have that the factor $2^x - 1$ equals 1.

Hence $x = 1$ and $q^b = 3$, yielding $b = 1$ and $q = 3$. Finally by i), $p^a = 2mn = 2 \cdot 2^1 \cdot 1 = 2^2$. So $p = 2$ and $a = 2$. Finally, $c = 2$ as $3^2 + 4^2 = 5^2$.

4809. *Proposed by Daniel Sitaru.*

Let $a, b > 0$. Find

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \left(\int_0^1 \frac{x^k}{ax+b} dx \right)^{-1} \left(\int_0^1 \frac{x^k}{bx+a} dx \right)^{-1}.$$

Solution to problem 4809 Crux Math. 49 (1) 2023, 45

Raymond Mortini, Rudolf Rupp

We show, more generally, that whenever $f, g : [0,1] \to [0, \infty[$ are continuous and $f(1)g(1) \neq 0$, then

$$\boxed{\lim_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\, dx \right)^{-1} \left(\int_0^1 x^k g(x)\, dx \right)^{-1} = \frac{1}{3}\, \frac{1}{f(1)} \frac{1}{g(1)}.}$$

Hence the limit in the problem is $(a+b)^2/3$.

Proof Let $M := \max\{|f(x)| : 0 \leq x \leq 1\}$. Given ε with $0 < \varepsilon < \frac{1}{2}\min\{f(1), g(1)\}$, choose $\delta > 0$ so that $|f(x) - f(1)| \leq \varepsilon$ for $\delta \leq x \leq 1$. Moreover, let n_0 be so large that $\delta^{k+1} \leq \varepsilon/(2M)$ for $k \geq n_0$. Then

$$\begin{aligned} \left| \int_0^1 x^k f(x)\, dx - \frac{1}{k+1} f(1) \right| &= \left| \int_0^1 x^k (f(x) - f(1))\, dx \right| \\ &\leq 2M \int_0^\delta x^k + \int_\delta^1 x^k |f(x) - f(1)|\, dx \\ &\leq 2M \frac{\delta^{k+1}}{k+1} + \varepsilon \int_0^1 x^k dx \\ &\leq \frac{\varepsilon}{k+1} + \frac{\varepsilon}{k+1}. \end{aligned}$$

Therefore

$$\left(\frac{1}{k+1} f(1) + \frac{2\varepsilon}{k+1} \right)^{-1} \leq \left(\int_0^1 x^k f(x)\, dx \right)^{-1} \leq \left(\frac{1}{k+1} f(1) - \frac{2\varepsilon}{k+1} \right)^{-1}.$$

We conclude that

$$\sum_{k=n_0}^{n} \frac{k+1}{f(1)+2\varepsilon}\, \frac{k+1}{g(1)+2\varepsilon} \leq \sum_{k=n_0}^{n} \left(\int_0^1 x^k f(x)\, dx \right)^{-1} \left(\int_0^1 x^k g(x)\, dx \right)^{-1} \leq \sum_{k=n_0}^{n} \frac{k+1}{f(1)-2\varepsilon}\, \frac{k+1}{g(1)-2\varepsilon}.$$

Hence, by using that $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$, we deduce that

$$\limsup_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\, dx \right)^{-1} \left(\int_0^1 x^k g(x)\, dx \right)^{-1} \leq \frac{1}{3}\, \frac{1}{g(1)-2\varepsilon}\, \frac{1}{f(1)-2\varepsilon}$$

and

$$\liminf_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\, dx \right)^{-1} \left(\int_0^1 x^k g(x)\, dx \right)^{-1} \geq \frac{1}{3}\, \frac{1}{f(1)+2\varepsilon} \frac{1}{g(1)+2\varepsilon}.$$

from which we conclude that

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\, dx \right)^{-1} \left(\int_0^1 x^k g(x)\, dx \right)^{-1} = \frac{1}{3}\, \frac{1}{f(1)} \frac{1}{g(1)}.$$

Remark The lower estimate show that the limit is infinite if $f(1)g(1) = 0$.

4805. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $ab + bc + ca = 4abc$. Prove

$$\frac{1}{\sqrt[a]{a}} + \frac{1}{\sqrt[b]{b}} + \frac{1}{\sqrt[c]{c}} \geq 4\sqrt[3]{\frac{4}{3}}.$$

Solution to problem 4805 Crux Math. 49 (1) 2023, 44

Raymond Mortini, Rudolf Rupp

First we note that $ab + bc + ca = 4abc$ is equivalent to

$$(*) \qquad \ell(a,b,c) := \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 4.$$

If $a = b = c$, then this condition is satisfied if $a = 3/4$. Let

$$g(a,b,c) = a^{-1/a} + b^{-1/b} + c^{-1/c}.$$

It suffices to show that the minimum M of g under condition (*) is obtained for $a = b = c$. Note that $M := g(3/4, 3/4, 3/4) = 3(3/4)^{-4/3} = 4(4/3)^{1/3} \sim 4.402$.

The gradient of the Lagrange function

$$H(a,b,c,\lambda) = g(a,b,c) + \lambda(\ell(a,b,c) - 4)$$

is zero if

$$\lambda = a^{-1/a}\,(1 - \log a) = b^{-1/b}\,(1 - \log b) = c^{-1/c}\,(1 - \log c).$$

Since the function $x \mapsto x^{-1/x}\,(1 - \log x)$ is strictly decreasing on $]0, \infty[$, the only solution is where $a = b = c$. The existence of the minimum is shown as follows (note that

$$E := \{(a,b,c) : a,b,c > 0, \ell(a,b,c) = 4\}$$

is not compact. Condition (*) implies that $a, b, c \geq 1/4$. Let $L := \inf_E g$. Then

$$L \geq 3 \min_{[1/4,\infty[} x^{-1/x} = 3e^{-1/e} \geq 3 \times 0.692 = 2.076.$$

If this infimum is not taken on E, then there is $a_n \to \infty$ (or $b_n \to \infty$, or $c_n \to \infty$) such that $(a_n, b_n, c_n) \in E$ and $g(a_n, b_n, c_n) \to L$. In particular $a_n^{-1/a_n} \to 1$. We may assume that $b_n \to b_0$ and $c_n \to c_0$ (since otherwise $b_n \to \infty$ and so $c_n \to 1/4$, as well as $L = 1 + 1 + 4^4 > M$, a contradiction). Hence $L = \inf_{E'}(1 + b^{-1/b} + c^{-1/c})$, where

$$E' = \{(b,c) : b, c > 0, 1/b + 1/c = 3\}.$$

In particular, $b \geq 1/3$. Thus (by using Lagrange again, yielding $x = 2/3$)

$$L = \inf_{]1/3,\infty[} 1 + x^{-1/x} + \left(\frac{3x-1}{x}\right)^{\frac{3x-1}{x}} \overset{x=2/3}{=} 1 + 2(3/2)^{3/2} \sim 4.674 > M.$$

A contradiction. Consequently $(a_n, b_n, c_n) \to (\alpha, \beta, \gamma) \in E$ and so the infimum is a minimum. Hence

$$g(a,b,c) \geq g(\alpha, \beta, \gamma) = L = M.$$

Here is a second proof, based on the article [1] (which unfortunately contains many typos (poor proofreading? Poor referee job?). The function $f(x) := x^x$ is convex. Let $T_u(x) := f'(u)(x-u) + f(u)$ be the tangent to the graph of f at the point $(u, f(u))$. Then $f(x) \geq T_u(x)$. Next, let $x_1 = 1/a$, $x_2 = 1/b$ and $x_3 = 1/c$. Then with $u := S = (x_1 + x_2 + x_3)/3$,

$$\sum_{j=1}^{3} f(x_j) \geq \sum_{j=1}^{3} T_S(x_j) = \sum_{j=1}^{3} (f'(S)(x_j - S) + f(S)) = f'(S) \sum_{j=1}^{3} (x_j - S) + 3f(S)$$

$$= f'(S)\sum_{j=1}^{3} x_j - 3Sf'(S) + 3f(S) = 3f(S).$$

Since $S = (1/a + 1/b + 1/c)/3 = 4/3$, we obtain with $1/a + 1/b + 1/c = 4$ that

$$(1/a)^{1/a} + (1/b)^{1/b} + (1/c)^{1/c} = \sum_{j=1}^{3} f(x_j) \geq 3f(4/3) = 3(4/3)^{4/3} = 4(4/3)^{1/3}.$$

References

[1] Manasseh Ahmed, The Tangent Line Trick, CRUX 49 (2023), 38–43

192

4801. *Proposed by Michel Bataille.*

Find all functions $f:(0,\infty)\to\mathbb{R}$ such that

$$f\left(x+\frac{1}{y}\right)=yf(xy+y)$$

for all $x,y>0$.

Solution to problem 4801 Crux Math. 49 (1) 2023, 44

Raymond Mortini, Rudolf Rupp

We show that all solutions are given by $f(x)=\dfrac{C}{1+x}$ for $C\in\mathbb{R}$.

• It is straightforward to check that these are solutions:

$$f\left(x+\frac{1}{y}\right)=\frac{c}{1+x+\frac{1}{y}}=\frac{cy}{y+yx+1}=yf(xy+y).$$

• Suppose that $f:]0,\infty[\to\mathbb{R}$ is a solution. Let $y=\frac{1}{1+x}$. Then

(78)
$$f(2x+1)=f(x+\frac{1}{y})=yf(y(1+x))=\frac{1}{1+x}f(1).$$

Next, let $y=\frac{1}{x}$. Hence, by using (78),

$$f(2x)=f(x+\frac{1}{y})=yf(y(1+x))=\frac{1}{x}f(1+\frac{1}{x})=\frac{1}{x}f(1+2\frac{1}{2x})\overset{(78)}{=}\frac{1}{x}\frac{f(1)}{1+\frac{1}{2x}}=\frac{2f(1)}{2x+1}.$$

Now let $X:=2x$ and $C:=2f(1)$. Then $f(X)=\dfrac{2f(1)}{X+1}=\dfrac{C}{1+X}$.

4772. *Proposed by Mihaela Berindeanu.*

Find all functions $f:(0,\infty)\longrightarrow(0,\infty)$ such that $f(kx+f(y))=\dfrac{y}{k}\cdot f(xy+1)$ for all $x,y\in(0,\infty)$, where $k>0$ is a real and fixed parameter.

Solution to problem 4772 Crux Math. 48 (8) 2022, 483

Raymond Mortini, Rudolf Rupp

For $k=1$, this problem was given for instance in the Middle European Mathematical Olympiad (MEMO) in 2012 in Switzerland (see [**?**] and [**?**]) and we follow those published solutions.

We claim that for $a>0$, all solutions $f:\mathbb{R}^+\to\mathbb{R}^+$ of [15]

(79) $$f(ax+f(y))=\frac{y}{a}\,f(xy+1)$$

are given by $f(x)=a/x$. First, it is straightforward to see that this is a solution. Now we proceed as in [**?**, **?**]. Let f be a solution.

Step 1 Consider for $y>0$, $y\neq a$, the auxiliary function

$$g(y):=\frac{a-yf(y)}{a-y}$$

(this function is formally obtained by solving in $\mathbb{R}\times\mathbb{R}^+$ the equation $ax+f(y)=xy+1$, which gives $x=x_y=\frac{1-f(y)}{a-y}$ for $y\neq a$, and so $ax+f(y)=\frac{a-yf(y)}{a-y}=g(y)$. It will turn out that $x=-1/y$ and $g\equiv 0$).

Now for every $y>0$ with $y\neq a$ and $x_y>0$, we have that $g(y)\leq 0$, since otherwise f is well-defined at $g(y)>0$ and so $f(g(y))=\frac{y}{a}f(g(y))$, yielding that $y=a$, a contradiction.

Step 2

Case 1 If there would exist $y_0>1$ such that $f(y_0)<a/y_0$, then with $x_0:=1-\frac{1}{y_0}>0$ we have $x_0y_0+1=y_0$,

$$u_0:=ax_0+f(y_0)=a-\frac{a}{y_0}+f(y_0)<a,$$

and

$$f(u_0)=f(ax_0+f(y_0))=\frac{y_0}{a}\,f(y_0)<1.$$

Then $x_{u_0}:=\frac{1-f(u_0)}{a-u_0}>0$ and so

$$g(u_0)=ax_{u_0}+f(u_0)=x_{u_0}u_0+1>0.$$

But by Step 1, $g(u_0)\leq 0$, a contradiction.

Case 2 If there would exist $y_1>1$ such that $f(y_1)>a/y_1$, then by the same reasoning as above, with $x_1:=1-\frac{1}{y_1}$ and

$$u_1:=ax_1+f(y_1)>a,$$

we have $f(u_1)>1$ and so $g(u_1)>0$, again. A contradiction.

We conclude that $f(y)=a/y$ for every $y>1$. To deal with the remaining case, take $x=1/a$ and $0<y\leq 1$. Then by (79),

(80) $$f(1+f(y))=\frac{y}{a}f\left(\frac{y}{a}+1\right).$$

[15] We prefer to use the letter a instead of k, as for us k always belongs to $\mathbb{N}$.

As both $1 + f(y)$ and $\frac{y}{a} + 1$ are bigger than 1, we deduce from (80) that

$$\frac{a}{1 + f(y)} = \frac{y}{a} \frac{a}{\frac{y}{a} + 1} = \frac{ay}{y + a}.$$

Hence $f(y) = a/y$.

4779. *Proposed by Marian Ursărescu.*

Let $0 < a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and with $f(a) = f(b)$. Prove that there exist distinct $c_1, c_2 \in (a, b)$ such that

$$\sqrt{b} f'(c_1) + \sqrt{a} f'(c_2) = 0.$$

Solution to problem 4779 Crux Math. 48 (8) 2022, 483

Raymond Mortini

If f is constant, then $f' \equiv 0$ and we may choose any numbers $a < c_1 < c_2 < b$ to satisfy

$$\sqrt{b} f'(c_1) + \sqrt{a} f'(c_2) = 0. \tag{81}$$

Otherwise, f takes its distinct extremal values on $[a, b]$. We may assume that $M := \max_{[a,b]} f > f(a)$ (if not, $M = f(a)$ and so $\min_{[a,b]} f < f(a)$ and we consider $-f$). Say $M = f(x_0)$ for some $x_0 \in]a, b[$. Then $f'(x_0) = 0$, and due to continuity of f', there are $a < x_1 < x_2 \le x_0$ with $f'(x) > 0$ for $x \in]x_1, x_2[$, but $f'(x_2) = 0$; we may choose

$$x_2 = \inf\{t \le x_0 : f' \equiv 0 \text{ on } [t, x_0]\}.$$

By a similar argument, there are $x_0 \le y_2 < y_1$ such that $f'(y_2) = 0$, but $f'(x) < 0$ for $x \in]y_2, y_1[$. By the intermediate value theorem for continuous functions, here for f', there exists a small $\varepsilon > 0$ such that f' takes every value from $[0, \varepsilon]$ on $]x_1, x_2]$ and every value from $[-\varepsilon, 0]$ on $[y_2, y_1[$. Now choose $c_1 \in]x_1, x_2[$ so that $\frac{\sqrt{b}}{\sqrt{a}} f'(c_1) \in]0, \varepsilon[$ (this is possible since $\lim_{x \nearrow x_2} f'(x) = 0$). Hence there exists $c_2 \in]y_2, y_1[$ with

$$f'(c_2) = -\frac{\sqrt{b}}{\sqrt{a}} f'(c_1).$$

Thus $\sqrt{b} f'(c_1) + \sqrt{a} f'(c_2) = 0$ and $c_1 < c_2$.

Remark I do not see the role played by the special coefficients $\sqrt{a}$ and $\sqrt{b}$. The whole works for any $0 < s_1 < s_2 < \infty$.

4771. *Proposed by Michel Bataille.*

Let I be an open interval containing 0 and 1 and let $f: I \to \mathbb{R}$ be a differentiable, strictly increasing, convex function. If $f'(1) < 2f(1)$, prove that there exist positive real numbers a, b such that

$$\int_0^1 (f(x))^{2n+1}\,dx \sim a \cdot \frac{b^n}{n} \qquad \text{as } n \to \infty$$

and express a and b as a function of $f(1)$ and $f'(1)$.

Solution to problem 4771 Crux Math. 48 (8) 2022, 483

Raymond Mortini, Rudolf Rupp

The problem is a bit ambiguous, due to an undefined $\sim$ symbol. Let

$$L_n := \int_0^1 f(x)^{2n+1}\,dx \quad \text{and} \quad R_n = a \cdot \frac{b^n}{n}$$

Is it $L_n - R_n \to 0$? Or $L_n/R_n \to 1$? Or $cL_n \leq R_n \leq CL_n$ for almost every n and some positive constants c, C? Note that, a priori, it is not even clear that $L_n > 0$.

We are going to show the following:

Example 12.

$$\boxed{\lim_{n\to\infty} \frac{n+1}{f(1)^{2n+1}} \int_0^1 f(x)^{2n+1}\,dx = \frac{f(1)}{2f'(1)}.}$$

Hence, with $a := \dfrac{f(1)^2}{2f'(1)}$ and $b = f(1)^2$ we get that $L_n/R_n \to 1$.

Proof. Since f is assumed to be increasing, we see that $f'(x) \geq 0$ for $0 \leq x \leq 1$. To exclude that for some points $x_0 \in]0,1]$, $f'(x_0) = 0$, we need the convexity [16] of f: in fact, let T be the tangent to the graph of f at $(x_0, f(x_0))$; then $T(x) = f(x_0) + f'(x_0)(x - x_0)$. The convexity of f implies that the graph of f lies above T. In particular, if $f'(x_0) = 0$, then, due to f being strictly increasing, $f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon)$ would contradict this fact. We conclude that $f'(x) > 0$ for every $x \in]0,1]$.

To calculate our limit, we let $0 < s < 1$ and write the integral $\dfrac{n+1}{f(1)^{2n+1}} L_n$ as $I_n(s) + J_n(s)$, where

$$I_n(s) = \frac{n+1}{f(1)^{2n+1}} \int_0^s f(x)^{2n+1}\,dx \quad \text{and} \quad J_n(s) = \frac{n+1}{f(1)^{2n+1}} \int_s^1 f(x)^{2n+1}\,dx.$$

Claim 1 There is a function $h(s)$ with $0 < h(s) < 1$, such that

$$\frac{f(1)}{2f'(1)}\Big(1 - h^{2n+2}(s)\Big) \leq J_n(s) \leq \frac{f(1)}{2f'(s)}. \tag{82}$$

To see this, note that f convex and C^1 imply that f' is increasing (by the way, a fact equivalent to f being convex). By the mean-value theorem, and for $s < x \leq 1$, there is $c_x \in]s, 1[$ with $f'(c_x) = \frac{f(1)-f(x)}{1-x}$. Hence

$$f'(s) \leq f'(c_x) \leq f'(1)$$

and so

$$f'(s) \leq \frac{f(1) - f(x)}{1 - x} \leq f'(1).$$

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[16] Note that f merely being strictly increasing, does not exclude the existence of zeros of f': $f(x) = (x - 1/2)^3$.

In other words

$$f(1) - f'(1) + f'(1)x \le f(x) \le f(1) - f'(s) + f'(s)x. \tag{83}$$

Now for $f(x) = Ax + B$ with $A \neq 0$ we have

$$\int_s^1 (Ax+B)^{2n+1}\, dx = \frac{(A+B)^{2n+2} - (As+B)^{2n+2}}{A(2n+2)}.$$

Applying this to (83) yields

$$\begin{aligned}
\frac{n+1}{f(1)^{2n+1}} \int_s^1 f(x)^{2n+1}\, dx &\le \frac{n+1}{f(1)^{2n+1}} \frac{(f'(s)+f(1)-f'(s))^{2n+2} - (f'(s)s + f(1) - f'(s))^{2n+2}}{f'(s)(2n+2)} \\
&= \frac{1}{2f'(s)} \frac{f(1)^{2n+2} - (f'(s)s + f(1) - f'(s))^{2n+2}}{f(1)^{2n+1}} \\
&= \frac{f(1)}{2f'(s)} \left(1 - \left(1 - \frac{f'(s)}{f(1)}(1-s) \right)^{2n+2} \right) \\
&\le \frac{f(1)}{2f'(s)}
\end{aligned}$$

because $0 \le 1 - \frac{f'(s)}{f(1)}(1-s) < 1$ for $s \in [s_1, 1]$. Similarily,

$$\begin{aligned}
\frac{n+1}{f(1)^{2n+1}} \int_s^1 f(x)^{2n+1}\, dx &\ge \frac{n+1}{f(1)^{2n+1}} \frac{(f'(1)+f(1)-f'(1))^{2n+2} - (f'(1)s + f(1) - f'(1))^{2n+2}}{f'(1)(2n+2)} \\
&= \frac{1}{2f'(1)} \frac{f(1)^{2n+2} - (f'(1)s + f(1) - f'(1))^{2n+2}}{f(1)^{2n+1}} \\
&= \frac{f(1)}{2f'(1)} \left(1 - \left(1 - \frac{f'(1)}{f(1)}(1-s) \right)^{2n+2} \right) \\
&=: \frac{f(1)}{2f'(1)} \left(1 - h(s)^{2n+2} \right),
\end{aligned}$$

with $h(s) := 1 - \frac{f'(1)}{f(1)}(1-s)$. Note that $0 < h(s) < 1$ for $s \in [s_2, 1]$.

This finishes the proof of Claim 1.

Claim 2 $\lim_{n\to\infty} I_n(s) = 0$ for every $0 < s < 1$.

To this end, we need to show that $\max_{[0,1]} |f| = f(1)$ and that the maximum is *only* obtained at 1 (note that f may take negative values). In fact, since f is increasing, $f(0) \le f(x) \le f(1)$ for every $x \in [0,1]$. If $f(0) \ge 0$, nothing has to be proven. So let $f(0) < 0$. Then, by the mean value theorem on $[0,1]$ there is $0 < c_x < 1$ such that

$$f(x) = f(0) + f'(c_x)x \le f(0) + f'(1)x \le f(0) + f'(1)$$

(note that f' is increasing). Using that $0 \le f'(1) < 2f(1)$ [17], we obtain $f(1) < f(0) + 2f(1)$. Hence $f(0) > -f(1)$. As f is strictly increasing, we also have $f(0) < f(1)$, and so $|f(0)| < f(1)$. Moreover, $|f(x)| \neq f(1)$ for any $x \in\ [0,1[$.

We conclude that

$$|I_n(s)| = \left| \frac{n+1}{f(1)^{2n+1}} \int_0^s f(x)^{2n+1} \right| \le (n+1)s \left(\frac{\max_{[0,s]} |f(x)|}{f(1)} \right)^{2n+1} =: (n+1)M^{2n+1},$$

where $0 < M = M(s) < 1$. As $\sum_{n=1}^{\infty} (n+1)M^{2n+1}$ converges, $I_n(s) \to 0$ as $n \to \infty$.

[17] It is only here that we use this assumption.

We are now ready to determine the limit of $\frac{n+1}{f(1)^{2n+1}} \int_0^1 f(x)^{2n+1}\,dx$. To this end, fix $\varepsilon > 0$ and choose $s_3 = s_3(\varepsilon) \in]0,1[$ so that for all $s \in [s_3, 1]$

$$\left|\frac{f(1)}{2f'(s)} - \frac{f(1)}{2f'(1)}\right| < \varepsilon.$$

Now for $s_0 := \max\{s_1, s_2, s_3\}$, depending on ε, we obtain from Claim 1 that

$$\frac{f(1)}{2f'(1)}\Big(1 - h^{2n+2}(s_0)\Big) \le J_n(s_0) \le \frac{f(1)}{2f'(s_0)} \le \frac{f(1)}{2f'(1)} + \varepsilon.$$

Since $0 < h(s_0) < 1$, there is $n_0 = n_0(\varepsilon, s_0)$ such that

$$0 < h(s_0)^{2n+2} < \varepsilon \text{ for all } n \ge n_0.$$

Thus, for $n \ge n_0$

$$\frac{f(1)}{2f'(1)}(1-\varepsilon) \le J_n(s_0) \le \frac{f(1)}{2f'(1)} + \varepsilon.$$

By Claim 2, there is $n_1 \ge n_0$ (depending on ε) such that $|I_n(s_0)| < \varepsilon$ for $n \ge n_1$. We conclude that for these $n \ge n_1$

$$\frac{n+1}{f(1)^{2n+1}} L_n = I_n(s_0) + J_n(s_0) \begin{cases} \le & \varepsilon + \frac{f(1)}{2f'(1)} + \varepsilon \\ \ge & -\varepsilon + \frac{f(1)}{2f'(1)}(1-\varepsilon). \end{cases}$$

Hence

$$\left|\frac{n+1}{f(1)^{2n+1}} L_n - \frac{f(1)}{2f'(1)}\right| \le \max\left\{2\varepsilon, \varepsilon\left(1 + \frac{f(1)}{2f'(1)}\right)\right\}.$$

□

Remark The function $f(x) = x - 1/2$ shows that the assertion may fail if $f'(1) = 2f(1)$, since in this case $L_n = 0$. On the other hand, it may hold, too if $f'(1) = 2f(1)$. In fact, if $f(x) = e^{2x}$, then $f'(1) = 2f(1)$ and

$$L_n = \frac{e^{4n+2} - 1}{4n+2} \quad \text{and} \quad R_n = \frac{e^4}{4e^2} \cdot \frac{e^{4n}}{n} = \frac{e^{4n+2}}{4n},$$

nevertheless $L_n/R_n \to 1$. What is the reason for this? Well, an analysis of the proof shows that the condition $f'(1) < 2f(1)$ can be replaced by the assumption that the maximum of $|f|$ is *only* obtained at 1. This makes the class of functions with the wished assymptotic behavior of the integrals $\int_0^1 f(x)^{2n+1}\,dx$ much larger.

4780. *Proposed by Floricǎ Anastase.*

Let $0 < a < b, m = \frac{a+b}{2}$ and $f\colon [a,b] \to \mathbb{R}$ differentiable with derivative continuous on $[a,b]$ such that $f(m) = 0$. Prove that

$$2a^3 \int_{-a}^{a} (f'(x))^2\, dx \geq 3\left(\int_{-a}^{a} f(x)\, dx\right)^2.$$

Solution to problem 4780 Crux Math. 48 (8) 2022, 484

Raymond Mortini, Rudolf Rupp

The assertion is not compatible with the hypotheses. So we prove the following two results:

Example 13. *Let $a > 0$ and $f \in C^1[-a,a]$. If $f(0) = 0$, then*

$$\int_{-a}^{a} (f'(x))^2\, dx \geq \frac{3}{2a^3}\left(\int_{-a}^{a} f(x)\, dx\right)^2.$$

Example 14. *Let $0 < a, b < \infty$ and $f \in C^1[a,b]$. If $f((a+b)/2) = 0$, then, with $C = \frac{12}{(b-a)^3}$,*

$$\int_{a}^{b} (f'(x))^2\, dx \geq C\left(\int_{a}^{b} f(x)\, dx\right)^2.$$

Proof of Example 1. Let p be a polynomial. Then, using The Cauchy-Schwarz inequality

$$I := \left(\int_0^a (f'p)(x)\, dx\right)^2 \leq \left(\int_0^a (f'(x))^2\, dx\right)\left(\int_0^a p(x)^2\, dx\right)$$

Using partial integration,

$$I = \left((f(x)p(x)\Big|_0^a - \int_0^a f(x)p'(x)\, dx\right)^2$$

Now choose $p(x) = x - a$. Then $\int_0^a p(x)^2\, dx = \frac{1}{3}(x-a)^3\Big|_0^a = \frac{1}{3}a^3$. Hence, by noticing that $p(a) = f(0) = 0$,

$$I = \left(\int_0^a f(x)\, dx\right)^2 \leq \left(\int_0^a (f'(x))^2\, dx\right)\frac{1}{3}a^3$$

If we choose $p(x) = x + a$, then $p(-a) = 0$, and we similarily obtain the appropriate estimation for $\int_{-a}^0 f(x)dx$. Hence, using that $(x+y)^2 \leq 2(x^2+y^2)$,

$$\left(\int_{-a}^{a} f(x)\, dx\right)^2 \leq \frac{2}{3}a^3 \int_{-a}^{a} (f'(x))^2\, dx$$

□

Proof of Example 2. Just use the affine transformation ϕ given by $\phi(x) = x + \frac{a+b}{2}$. Then $\phi(-\frac{b-a}{2}) = a$ and $\phi(\frac{b-a}{2}) = b$, as well as $\phi(0) = \frac{a+b}{2}$. Let $c := (b-a)/2$. Hence, with $F(t) := f(\phi(t))$ for $-c \leq t \leq c$ we obtain

$$\int_a^b (f'(x))^2\, dx = \int_{-c}^{c} (F'(t))^2\, dt \geq \frac{3}{2c^3}\left(\int_{-c}^{c} F(t)\, dt\right)^2 = \frac{12}{(b-a)^3}\left(\int_a^b f(x)\, dx\right)^2.$$

□

Of course Example 1 is a special case of Example 2. Is C best possible? Let

$$q(x) = \begin{cases} \dfrac{(x-a)^2}{2} - \dfrac{(b-a)^2}{8} & \text{if } a \le x \le (a+b)/2 \\ \dfrac{(x-b)^2}{2} - \dfrac{(a-b)^2}{8} & \text{if } (a+b)/2 \le x \le b. \end{cases}$$

Then q is continuous on $[a,b]$, $q((a+b)/2) = 0$ and

$$\int_a^b (q'(x))^2 \, dx = \frac{12}{(b-a)^3} \left(\int_a^b q(x) \, dx \right)^2 .$$

Unfortunately, q is not C^1. How to modify?

4777. *Proposed by Goran Conar, modified by the Editorial Board.*

Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \geq 1$ such that $\sum_{i=1}^n \frac{1}{x_i} = 1$. Prove

$$\frac{n}{1/2+n^2} < \sum_{i=1}^n \frac{1}{\frac{1}{2}+x_i^2} < \frac{2}{3}.$$

Solution to problem 4777 Crux Math. 48 (8) 2022, 484

Raymond Mortini, Rudolf Rupp

The assertion is not correct. In fact, let $\mathbf{x} := (x_1, \ldots, x_n)$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$,

$$S := \Big\{(x_1, \ldots, x_n) \in (\mathbb{R}^+)^n : \sum_{i=1}^n \frac{1}{x_i} = 1\Big\},$$

and

$$f(\mathbf{x}) := \sum_{i=1}^n \frac{1}{\frac{1}{2}+x_i^2}.$$

We prove that for $n \geq 2$,

$$\boxed{\frac{n}{1/2+n^2} = \min_{\mathbf{x}\in S} f(\mathbf{x}) < \sup_{\mathbf{x}\in S} f(\mathbf{x}) = \frac{2}{3},}$$

and that for $n = 1$, $S = \{1\}$ and so

$$\boxed{f(x) = \frac{1}{\frac{1}{2}+x^2} = f(1) = \frac{2}{3}.}$$

Proof Wlog $n \geq 2$. First we note that $\sum_{i=1}^n 1/x_i = 1$ for $x_i \in \mathbb{R}^+$ implies that $x_i \geq 1$ for every i. Now $\max\limits_{1\leq x<\infty} \dfrac{x}{1+2x^2} = \dfrac{1}{3}$, since the function is decreasing on $[1, \infty[$. Hence, for $\mathbf{x} \in S$,

$$\sum_{i=1}^n \frac{1}{\frac{1}{2}+x_i^2} = \sum_{i=1}^n \frac{x_i}{1+2x_i^2}\frac{2}{x_i} < \frac{2}{3}\sum_{i=1}^n \frac{1}{x_i} = \frac{2}{3},$$

since for $n \geq 2$, no x_i can be 1. If for $k > n$

$$\mathbf{x}_k = \Big(x_1^{(k)}, \ldots, x_n^{(k)}\Big) := \Big(\frac{1}{1-(n-1)/k}, k, \ldots, k\Big),$$

then $\sum_{i=1}^n (1/x_i^{(k)}) = 1$, $\mathbf{x}_k \to (1, \infty, \ldots, \infty)$ and $f(\mathbf{x}_k) \to 2/3$. Hence $\sup_S f = 2/3$.

To prove the assertion on the minimum, we use Lagrange. It is preferable to work with the new variable $y_j := 1/x_j$ (to get a compact definition set, guarantying the existence of the global extrema). So let

$$S' = \Big\{(y_1, \ldots, y_n) \in \mathbb{R}^n,\ y_j \geq 0 : \sum_{j=1}^n y_j = 1\Big\}$$

and

$$g(y_1, \ldots, y_n) := \sum_{i=1}^n \frac{y_i^2}{1+\frac{1}{2}y_i^2}.$$

Then S' is compact and $\inf f_S = \inf g_{S'} = \min g_{S'} =: m$. Say $g(\mathbf{x}') = m$ for some $\mathbf{x}' \in S'$. In order to apply Lagrange, we need to show that $\mathbf{x}'$ is an interior point of S' (in symbols, $\mathbf{x}' \in (S')^\circ$). Let $\mathbf{y}' := (1/n, \ldots, 1/n)$. Then $\mathbf{y}' \in (S')^\circ$. Now on $\partial S'$ at least one of the coordinates of these points $\mathbf{y} := (y_1, \ldots, y_n) \in \partial S'$ is 0. Say, $y_n = 0$. But then $\sum_{i=1}^{n-1} y_i = 1$ and (via induction on n, starting with the trivial case of one-tuples)

$$g(\mathbf{y}) \geq \frac{n-1}{1/2 + (n-1)^2} > \frac{n}{1/2 + n^2} = g(\mathbf{y}').$$

Hence the absolute minimum of g on S' does not belong to the boundary.

By Lagrange's theorem, there exists $\lambda \in \mathbb{R}$ and $(y_1, \ldots, y_n) \in S'$ such that

$$\nabla\Big(g(y_1, \ldots, y_n) + \lambda(1 - \sum_{i=1}^{n} y_i)\Big) = \mathbf{0}.$$

That is, for every $i \in \{1, \ldots, n\}$,

$$\lambda = \frac{2y_i}{(1 + \frac{1}{2}y_i^2)^2}. \tag{84}$$

Unfortunately, the function $y \mapsto q(y) := \frac{2y}{(1+\frac{1}{2}y^2)^2}$ is not injective on $[0, 1]$ (note that the derivative vanishes at $y = \pm\sqrt{2/3}$). So we must discuss several cases (see figure 9):

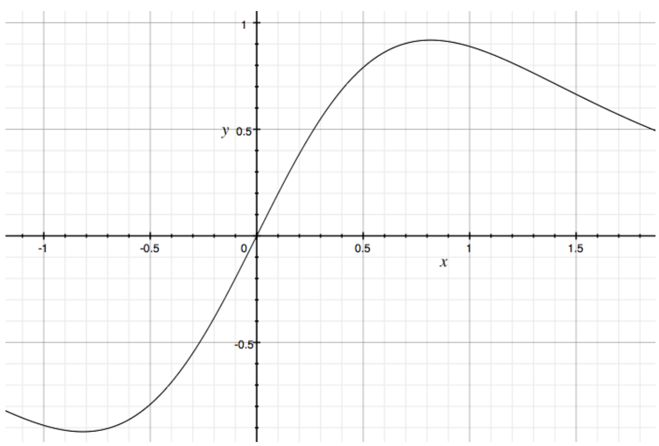

FIGURE 9. Non injectivity of q on $[0, 1]$

(i) If $8/9 = q(1) \leq \lambda < \max_{[0,1]} q$, then the equation $q(y) = \lambda$ has two solutions $0 < y_1, y_2 \leq 1$.

(ii) If $\lambda = \max_{[0,1]} q$ or if $0 \leq \lambda < q(1) = 8/9$, then the equation $q(y) = \lambda$ has exactly one solution $0 \leq y_0 \leq 1$.

(iii) In all other cases, there is no solution with $y \geq 0$.

We first show that the case (i) does not yield minimal solutions. In fact, for fixed $\lambda \in [q(1), \max_{[0,1]} q[$, equation (84) has 2^n solutions of the form $P := (\underbrace{a, \ldots, a}_{k\text{-times}}, \underbrace{b, \ldots, b}_{(n-k)\text{-times}})$ and their permutations, where $k = 0, \ldots, n$ and $0 \leq a \leq b \leq 1$. Note that

$$q(1/n) = \frac{8n^3}{(1 + 2n^2)^2} \leq q(1/2) < q(1) = \frac{8}{9} < q(\sqrt{2/3}). \tag{85}$$

Hence $1/n \leq 1/2 < \min\{a, b\}$ (see figure 9).

Let $A := (1/n, \ldots, 1/n)$. Then $A \in S'$. Since the function $y \mapsto y^2/(1 + \frac{1}{2}y^2)$ is increasing on $[0, \infty[$, we deduce that

$$g(P) = k\frac{a^2}{1 + \frac{1}{2}a^2} + (n-k)\frac{b^2}{1 + \frac{1}{2}b^2} > g(A),$$

so P does not yield a minimum. Thus only the second case occurs. That is, we need to consider only a solution of (84) of the form $(y_1, \ldots, y_n) = (a, \ldots, a)$ with $0 < a \leq 1$. Using the constraint condition $\sum_{i=1}^n y_i = 1$, we obtain that $a = 1/n$, hence $(y_1, \ldots, y_n) = (1/n, \ldots, 1/n)$. Consequently, $\mathbf{x}' = (1/n, \ldots, 1/n)$ is the unique point where g takes its absolute minimum on S'. We conclude that

$$\min g_{S'} = \frac{n}{\frac{1}{2} + n^2}.$$

For completeness, we observe that $M := \max_{S'} g$ necessarily is obtained on the boundary of S' (for instance, $M = g(1, 0, \ldots, 0) = 2/3$), as Lagrange only yields a single stationary point of the Lagrange function in $(S')^\circ$.

A second way to see that case (i) does not occur goes as follows:

We first show that the case (i) does not yield minimal solutions. In fact, for fixed $\lambda \in [q(1), \max_{[0,1]} q[$, equation (84) has 2^n solutions of the form $P := (\underbrace{a, \ldots, a}_{k\text{-times}}, \underbrace{b, \ldots, b}_{(n-k)\text{-times}})$ and their permutations, where $k = 0, \ldots, n$ and $0 \leq a \leq b \leq 1$. Note that $q(1/2) = (8/9)^2$ and that $n \geq 2$. Thus

$$q(1/n) \leq q(1/2) < q(1) \leq \lambda < q(\sqrt{2/3}). \tag{86}$$

Hence $1/n \leq 1/2 < \min\{a, b\} = a$ (see figure 9). Since for such a point $P = (y_1, \ldots, y_n)$ we have

$$\sum_{i=1}^n y_i = ka + (n-k)b > k\frac{1}{2} + (n-k)\frac{1}{2} = \frac{n}{2} \geq 1,$$

P does not belong to S'; that is such a solution of the system (84) of equations does not satisfy the constraint $P \in S'$.

4763. *Proposed by William Weakley.*

Let K be a field and let S be a nonempty subset of K that is closed under subtraction.

a) For all K and S, characterize the functions $f : S \to K$ such that

$$f(x)f(y) = f(x-y) \text{ for all } x, y \in S.$$

b) As K and S vary, what finite cardinalities can the set of such functions have?

Partial Solution to problem 4763 Crux Math. 48 (7) 2022, 421

Raymond Mortini, Rudolf Rupp

Here we give our thoughts on this not very precisely formulated problem.

First we note that $S \subseteq K$ necessarily is an additive subgroup of the field K. Note that $\{0,1\} \subseteq K$. In particular $0 = x - x \in S$ and with $x \in S$ we have $-x = 0 - x \in S$.

If

$$(FE) \qquad f(x-y) = f(x)f(y) \text{ for all } x, y \in S$$

then we get the following:

(1) $y = x \Longrightarrow f(0) = f(x)^2$

(2) $y = 0 \Longrightarrow f(x) = f(x)f(0) \Longrightarrow f(x)(1 - f(0)) = 0$

Case 1 There exists $x_0 \in S$ with $f(x_0) = 0$. Then, by (1), $f(0) = 0$ and so $f(x) = 0$ for all $x \in S$.

Case 2 f has no zeros. Then (2) implies that $f(0) = 1$.

We claim that $f(2x) = 1$ for every $x \in S$ (note that $\mathbb{Z}S \subseteq S$).

In fact, $f(x) = f(2x - x) = f(2x)f(x)$, hence $f(2x) = 1$.

We conclude that for $S = \mathbb{R}$ e.g., the constant function $f(y) = 1$ is the only solution, as every $y \in \mathbb{R}$ writes as $y = 2x$ for some x.

Next we show that f is even and that $f(x) \in \{-1, 1\}$. In fact, by (FE), for $x = 0$,

$$f(-y) = f(0)f(y) = f(y) \text{ for every } y \in S.$$

Hence $1 = f(2u) = f(u - (-u)) = f(u)f(-u) = f(u)^2$ for any $u \in S$.

If $S = \mathbb{Z}$, then we have three solutions: $f \equiv 0$, $f \equiv 1$ but also

$$f(n) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd.} \end{cases}$$

In fact by the claim above, $f(2m) = 1$ for every $m \in \mathbb{Z}$. Now let $\sigma := f(1)$. We already know that $\sigma = \pm 1$. Now for every $m \in \mathbb{Z}$,

$$\sigma = f(1) = f((2m+1) - 2m) = f(2m+1)f(2m) = f(2m+1).$$

Let $P := P_f := \{x \in S : f(x) = 1\}$ and $R := \{x \in S : f(x) = -1\}$. Then P is a subgroup of S since $x, y \in P$ implies that $x - y \in P$, because $f(x-y) = f(x)f(y) = 1 \cdot 1 = 1$.

As shown above, $2S \subseteq P \subseteq S$ and $2S$ is a subgroup of S. Here $S = 2S$ if and only if all the translation operators $\tau_x : S \to S, y \mapsto x - y$ have a fixed point.

Also note that R has the following property:

$$(PR) \qquad (R - R) \subseteq P \text{ and } (R - P) \cup (P - R) \subseteq R.$$

Conversely, if P is a proper subgroup of S and $R := S \setminus P$ such that (PR) holds, then the function g given by

$$g(x) = \begin{cases} 1 & \text{if } x \in P \\ -1 & \text{if } x \in R \end{cases},$$

satisfies the functional equation (FE) $g(x-y) = g(x)g(y)$ for $x, y \in S$.

Note that P may be strictly bigger than $2S$: in fact, let $K = \mathbb{C}$, $S := \mathbb{Z} + i\mathbb{Z}$, $P = 2\mathbb{Z} + i\mathbb{Z}$ and $R = S \setminus P$. Then S, P, R satisfy (PR), but $P := 2S$ does not satisfy (PR).

If $S = K$ is a field of characteristic 2, then $P_f = R = S$ (note that $1 = -1$), and so only the constant functions 1 and 0 satisfy (FE).

4747. *Proposed by Stanescu Florin.*

Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2f(x)+f(y))=f(f(x^3))+y$$

for all $x,y\in\mathbb{R}$.

Solution to problem 4747 Crux Math. 48 (2022), 282 by
Raymond Mortini, Rudolf Rupp

We claim that all solutions of the functional equation

$$f(x^2f(x)+f(y))=f(f(x^3)))+y,\ \ x,y\in\mathbb{R}$$

are given by $f(x)=x$ and $f(x)=-x$.

Claim1 f is injective:
Put $x=0$. Then

$$f(f(y))=f(f(0))+y. \tag{87}$$

Now if $f(y_1)=f(y_2)$, then by (87)

$$f(f(y_1))=f(f(0))+y_1 \text{ and } f(f(y_2))=f(f(0))+y_2$$

Hence $y_1=y_2$.

Claim 2 f is surjective:
Let $w\in\mathbb{R}$. Then, by (87),

$$w=f(f(0))+(w-f(f(0)))=f(f(w-f(f(0))).$$

Claim 3 $f(0)=0$:
Take $y=0$: then $f(x^2f(x)+f(0))=f(f(x^3))$. Since f is bijective, we conclude that $x^2f(x)+f(0)=f(x^3)$. Now put $x=1$: then $1^2\,f(1)+f(0)=f(1)$. Hence $f(0)=0$.

Claim 4 $f\circ f=id$ (that is, f is an involution).
This follows from (87).
Hence our equation becomes

$$f(x^2f(x)+f(y))=x^3+y\ \ (x,y\in\mathbb{R}). \tag{88}$$

In particular, for $y=0$,

$$f(x^2f(x))=x^3 \text{ or equivalenty } x^2f(x)=f(x^3). \tag{89}$$

Claim 5 f is additive:
In fact, the surjectivity of f and $x\mapsto x^3$ now imply that $x\mapsto x^2f(x)$ is surjective, too. Hence

$$f(\underbrace{x^2f(x)}_{=a}+\underbrace{f(y)}_{=b})=x^3+y=f(x^2f(x))+y=f(a)+f(b)$$

yields the additivity of f.
Claim 6 $f(-x)=-f(x)$.
Just use that with $f(0)=0$ and f additive,

$$0=f(0)=f(x+(-x))=f(x)+f(-x).$$

Claim 7 Let $f(a)=1$. Then $a=\pm1$.

Recall that by (5) and (6), $f(a+b) = f(a) + f(b)$ for $a, b \in \mathbb{R}$ and $f(mx) = mf(x)$ for every $m \in \mathbb{Z}$. Hence, by (89), for $x = a + b$,

$$(a+b)^2 f(a+b) = f((a+b)^3).$$

Expansion yields:

$$\begin{aligned} (a^2+b^2+2ab)(f(a)+f(b)) &= f(a^3)+3f(a^2b)+3f(ab^2)+f(b^3) \iff \\ \underline{a^2f(a)}+b^2f(a)+2abf(a)+a^2f(b)+\underline{b^2f(b)}+2abf(b) &= \underline{f(a^3)}+3f(a^2b)+3f(ab^2)+\underline{f(b^3)} \iff \\ b^2f(a)+2abf(a)+a^2f(b)+2abf(b) &= 3f(a^2b)+3f(ab^2). \end{aligned}$$

• Let $b = 1$ and note that $a = f(1)$. Then

$$1 + 2a + a^3 + 2a^2 = 3f(a^2) + 3 = 3a^3 + 3 \iff$$

$$2a^3 - 2a^2 - 2a + 2 = 0 \iff a^2(a-1) - (a-1) = 0 \iff (a-1)(a^2-1) = 0 \iff a \in \{-1, 1\}.$$

Claim 8 If the additive function f satisfies $x^2 f(x)) = f(x^3)$, then $f(x) = f(1)\,x$. To see this, we consider four cases:

• Let $a = 2$, $f(1) = \pm 1$ and $b = x$. Then

(90) $$\boxed{\pm x^2 \pm 4x - 4f(x) + 2xf(x) - 3f(x^2) = 0}.$$

• Let $a = 1$, $f(1) = \pm 1$ and $b = x$. Then,

(91) $$\boxed{\pm x^2 \pm 2x - 2f(x) + 2xf(x) - 3f(x^2) = 0}.$$

Calculating (90)-(91), yields $\pm 2x - 2f(x) = 0$. Hence $f(x) = \pm x = f(1)x$.

One can also prove Claim 8 without using Claim 7, and then deducing Claim 7 from Claim 8 if additionally we assume that f is an involution.

In

(92) $$b^2f(a) + 2abf(a) + a^2f(b) + 2abf(b) = 3f(a^2b) + 3f(ab^2).$$

choose $a = 1$, resp. $a = 2$ and $b = x$. Then $f(2) = f(2 \cdot 1) = 2f(1)$ and so

(93) $$x^2f(1) + 2f(1)x + f(x) + 2xf(x) - 3f(x) - 3f(x^2) = 0$$

(94) $$2x^2f(1) + 8f(1)x + 4f(x) + 4xf(x) - 12f(x) - 6f(x^2) = 0$$

Hence, by calculating (93)-$\frac{1}{2}$(94), we obtain

(95) $$-2f(1)x - f(1)x + 3f(x) = 0.$$

Hence $f(x) = f(1)x$. Using (89), that is $f(x^2f(x)) = x^3$, we have

$$f(1)x^2f(1)x = x^3.$$

Hence $f(1)^2 = 1$ and so $f(1) = \pm 1$.

4657. *Proposed by George Stoica.*

Let us consider the equation $f(x) + f(2x) = 0$, $x \in \mathbb{R}$.

(i) Prove that, if f is continuous at 0, then $f(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Construct a function f, discontinuous at every $x \in \mathbb{R}$, that solves the given equation.

Solution to problem 4657 Crux Math. 47 (2021), 301 by

Raymond Mortini, Rudolf Rupp

a) Suppose that the function f satisfies $f(x) + f(2x) \equiv 0$ on $\mathbb{R}$. Then the continuity of f at $x = 0$ implies that $f \equiv 0$. In fact, fix $x \in \mathbb{R} \setminus \{0\}$. By induction, $f(x/2^n) = (-1)^n f(x)$. By taking limits, the continuity at 0 implies that for n even we get $f(0) = f(x)$ and for n odd, we get $f(0) = -f(x)$. Hence $2f(x) = f(0) - f(0) = 0$, and so $f \equiv 0$.

b) Define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(0) = 0$, $f(x) = 1$ if $1 \leq x < 2$ and x rational, $f(x) = -1$ if $1 < x < 2$ and x irrational. If $n \in \mathbb{N}$ and $2^n \leq x < 2^{n+1}$, put $f(x) = (-1)^n f(x/2^n)$. If $\frac{1}{2^{n+1}} \leq x < \frac{1}{2^n}$, put $f(x) = (-1)^n f(2^n x)$. If $x < 0$, then let $f(x) = f(-x)$. Then f is discontinuous everywhere and, by construction, $f(x) + f(2x) = 0$.

c) All solutions to $f(x) + f(2x) \equiv 0$ on $\mathbb{R}$:

Let $g : [-2, -1[\cup[1, 2[\to \mathbb{R}$ be an arbitrary function. Put

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ (-1)^n g(x/2^n) & \text{if } 2^n \leq |x| < 2^{n+1} \\ (-1)^n g(2^n x) & \text{if } \frac{1}{2^{n+1}} \leq |x| < \frac{1}{2^n}. \end{cases}$$

This functional equation and its companion $f(x) = f(2x)$ appear multiple times:

https://math.stackexchange.com/questions/3144431/if-fx-f2x-is-continuous-is-f-continuous-or-not

https://math.stackexchange.com/questions/3374236/limit-question-unknown-function

https://math.stackexchange.com/questions/1046961/finding-continuous-functions

https://math.stackexchange.com/questions/1039622/continuous-functions-satisfying-fxf2x-0

https://math.stackexchange.com/questions/3480985/find-an-example-of-function-where-lim-x-to-0fxf2x-0-but-lim-x-to

https://math.stackexchange.com/questions/2757365/find-f0-if-fxf2x-x-space-space-forall-x

https://math.stackexchange.com/questions/2579482/function-satisfying-lim-limits-x-to-0fx-f2x-but-doesnt-have-lim-li

https://math.stackexchange.com/questions/3524310/if-fx-f2x-then-f-is-differentiable

https://math.stackexchange.com/questions/277313/proving-a-function-is-constant-in-mathbbr-if-fx-f2x-and-f-is-continuo

https://math.stackexchange.com/questions/2821984/functional-equation-satisfying-f2x-fx

4636. *Proposed by Mihaela Berindeanu.*

Solve the following equation over the set of real numbers:

$$(3^x+7)^{\log_4 3}-(4^x-7)^{\log_3 4}=4^x-3^x-14.$$

Solution to problem 4636 Crux Math. 47 (2021), 200 by

Raymond Mortini, Rudolf Rupp

The equation

$$(3^x+7)^{\log_4 3}-(4^x-7)^{\log_3 4}=4^x-3^x-14$$

has on $\mathbb{R}$ the unique solution $x=2$. In fact, first note that $a:=\log_4 3=\frac{\log 3}{\log 4}>0$ and $\log_3 4=1/a$. Then with $A:=3^x+7$ and $B:=4^x-7$ we have to solve $A^a-B^{1/a}=B-A$ or equivalently,

$$A^a+A=(B^{1/a})^a+B^{1/a}.$$

Since the function $x\mapsto x^a+x$ is strictly increasing, we deduce that $A=B^{1/a}$. In other words, $3^x+7=(4^x-7)^{1/a}$, or equivalently

$$\log 4\log(4^x-7)=\log 3\log(3^x+7). \tag{96}$$

The curve $y(x)=\log 4\log(4^x-7)-\log 3\log(3^x+7)$ is defined for $x>\log 7/\log 4:=x_0$ with $\lim_{x\to x_0}y(x)=\infty$ and its derivative

$$y'(x)=\log^2 4\frac{1}{1-7x^{-4}}-\log^2 3\frac{1}{1+7x^{-3}}$$

is strictly decreasing with $\lim_{y\to x_0}y'(x)=\infty$ and $\lim_{x\to\infty}y'(x)=(\log^2 4-\log^2 3)$. Note that the asymptote at infinite is the line $y=(\log^2 4-\log^2 3)x$. In particular, $y'>0$ and so the curve is strictly increasing and its unique zero is $x_1=2$ (observe that $\log(4)\log(9)=4\log(2)\log(3)=\log(3)\log(16)$, so (96) holds).

4634. *Proposed by George Stoica.*

Let $\sum_{n=1}^{\infty} a_n < \infty$ for $a_n > 0$, $n = 1, 2, \ldots$ Find $\lim_{n\to\infty} n \cdot \sqrt[n]{a_1 \cdots a_n}$.

Solution to problem 4634 Crux Math. 47 (2021), 200 by
Raymond Mortini, Rudolf Rupp

For $a_n > 0$, let $G_n := n(a_1 \cdots a_n)^{1/n}$. Then $\lim_{n\to\infty} G_n = 0$ whenever $\sum_{n=1}^{\infty} a_n$ is convergent. In fact, given $\varepsilon > 0$, choose N so big that $\sum_{n=N}^{\infty} a_n < \varepsilon$. Due to the arithmetic-geometric inequality, for $n > N$,

$$G_n = (a_1 \cdots a_N)^{1/n} \, \frac{n}{n-N} \Big((n-N)(a_{N+1} \cdots a_n)^{1/(n-N} \Big)^{\frac{n-N}{n}} (n-N)^{1-\frac{n-N}{n}}$$

$$\leq \sigma_n \Big(\sum_{j=N+1}^{n} a_j \Big)^{\frac{n-N}{n}},$$

where

$$\sigma_n := (a_1 \cdots a_N)^{1/n} \, \frac{n}{n-N} (n-N)^{N/n}.$$

Since $\lim_n \sigma_n = 1$, we have $\limsup_n G_n \leq \limsup_n \varepsilon^{\frac{n-N}{n}} = \varepsilon$, from which we deduce that $G_n \to 0$.

4615. *Proposed by Anthony Garcia.*

Let f be a twice differentiable function on $[0,1]$ such that $\int_0^1 f(x)dx = \frac{f(1)}{2}$. Prove that

$$\int_0^1 (f''(x))^2 dx \geq 30(f(0))^2.$$

Solution to problem 4615 Crux Math. 47 (2021), 301 by

Raymond Mortini, Rudolf Rupp

If p is a polynomial, we have (due to Cauchy-Schwarz)

$$\left|\int_0^1 f'' p dx\right|^2 \leq \left(\int_0^1 (f'')^2 dx\right)\left(\int_0^1 p^2 dx\right).$$

Now, by using twice integration by parts,

$$\int f'' p dx = (f'+c)p - \left((f+cx+c')p' - \int (f+cx+c')p'' dx\right)$$

Now let $p(x) = x(x-1)$. Evaluation at the end-points and using the hypothesis that $\int_0^1 f dx = f(1)/2$, yields

$$\int_0^1 f'' p dx = -f(0).$$

Since $\int_0^1 p^2 dx = \int_0^1 (x^4 + x^2 - 2x^3) dx = 1/30$, we deduce that

$$\int_0^1 (f'')^2 dx \geq 30 f(0)^2.$$

Equality is given if $f'' = p$ and $f(1) = 2\int_0^1 f dx$; for instance if

$$f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - \frac{1}{30}.$$

Here $f(1) = -7/60$.

7. Elemente der Mathematik

Aufgabe 1441: Sei c die Kurve gegeben durch die Parameterdarstellung

$$x(t) = t - \frac{A}{t}, \quad y(t) = t^2 + \frac{B}{t} \quad \text{für } t \in \mathbb{R} \setminus \{0\}.$$

Für welche ganzzahligen Werte von A und B besitzt c eine Selbstüberschneidung mit senkrechtem Schnittwinkel?

Gregory Dresden, Lexington VA, USA

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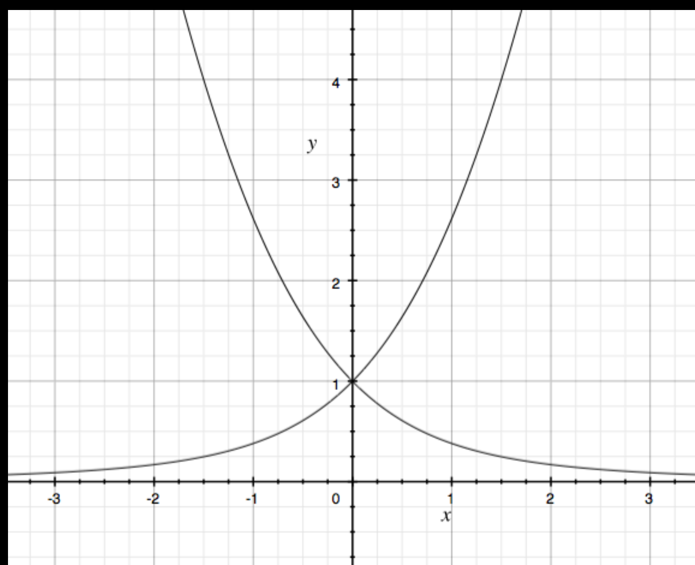

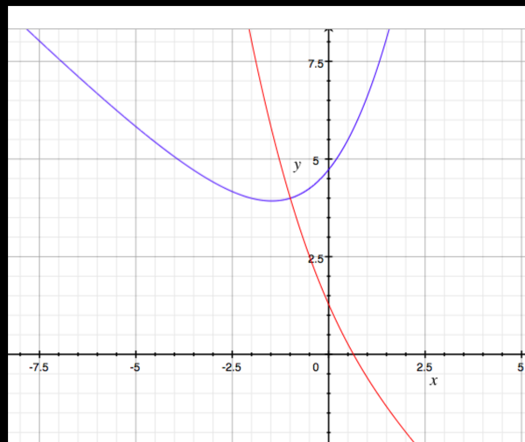

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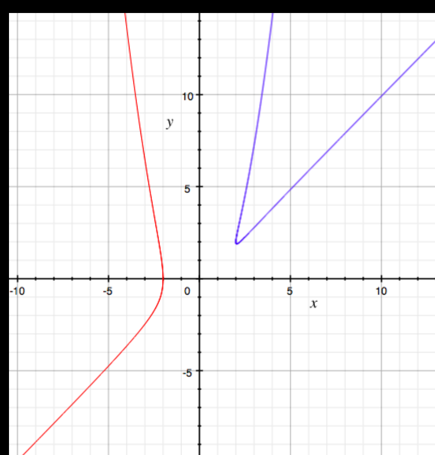

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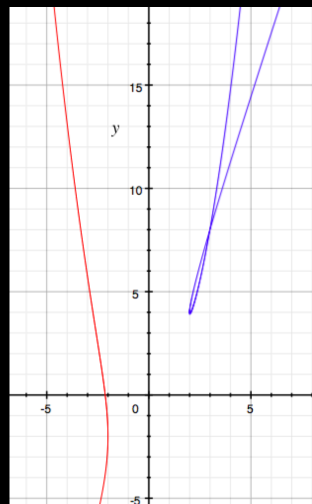

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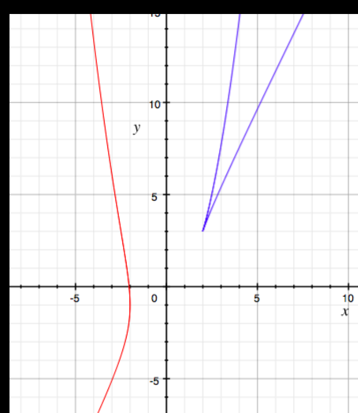

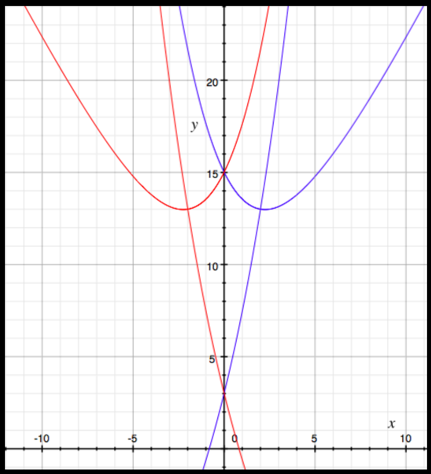

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y
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x
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-5
0
5
10

Aufgabe 1442 (Die einfache dritte Aufgabe): Es sei

$$f(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{2}{z}$$

mit positiven x, y, z in $\mathbb{R}$. Man bestimme

$$\min_{x^2+y^2+z^2=1} f(x, y, z).$$

Frieder Grupp, Bergrheinfeld, D

Aufgabe 1438: Es seien $z_j = r_j e^{it_j}$ zwei Punkte in der komplexen Ebene $\mathbb{C}$ mit $0 < r_j < 1$ und $|t_j| < \pi/2$. Weiterhin sei $\Delta = \langle z_1, z_2, 1 \rangle$ das durch die Punkte z_1, z_2, 1 bestimmte abgeschlossene Dreieck. Man zeige: Für alle $z = re^{it} \in \Delta$ mit $|t| < \pi/2$ gilt

$$\frac{|1-z|}{1-|z|} \leq \max\left\{\frac{|1-z_1|}{1-|z_1|}, \frac{|1-z_2|}{1-|z_2|}\right\} \quad \text{und} \quad \frac{|t|}{1-r} \leq \max\left\{\frac{|t_1|}{1-r_1}, \frac{|t_2|}{1-r_2}\right\}.$$

Raymond Mortini, Metz, F und Rudolf Rupp, Nürnberg, D

Aufgabe 1437: Die Funktion

$$f(x) = \frac{x}{\ln(1-x)}, \quad x \neq 0, \quad f(0) = -1,$$

lässt sich in eine Potenzreihe entwickeln, etwa $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < 1$.

(a) Man beweise $\frac{1}{3k^2} \leq a_k \leq \frac{2}{k}$ für $k \geq 1$.

(b) Konvergieren die Reihen $\sum_{k=0}^{\infty} (-1)^k a_k$ und $\sum_{k=0}^{\infty} a_k$ und falls ja, gegen welchen Grenzwert?

Frieder Grupp, Bergrheinfeld, D

$$-1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{24}x^3+\frac{19}{720}x^4+\frac{3}{160}x^5+\frac{863}{60480}x^6+\frac{275}{24192}x^7+\frac{33953}{3628800}x^8$$

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https://en.wikipedia.org/wiki/Gregory_coefficients 223 224

Aufgabe 1434: Berechne

$$S_1 = \sum_{k=0}^{\infty} \frac{1}{(3k+1)^3}, \quad S_2 = \sum_{k=0}^{\infty} \frac{1}{(3k+2)^3},$$

$$T_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)^3}, \quad T_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)^3}.$$

Michael Vowe, Therwil, CH

101

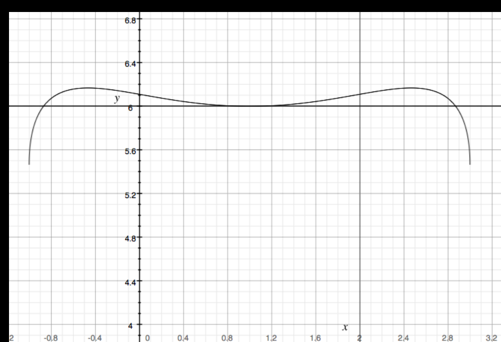
101

Aufgabe 1435: Man beweise für $x, y \geq 0$, $x + y = 2$ die Ungleichung

$$\sqrt{x^2+3} + \sqrt{y^2+3} + \sqrt{xy+3} \geq 6.$$

Šefket Arslanagić, Sarajevo, BIH

[redacted]



[redacted]

Aufgabe 1431: Man bestimme den Wert der Doppelreihe

$$\sum_{k,n=0}^{\infty} \frac{(-1)^{k+n}}{(2n+1)(2k+1)(2n+2k+3)^2} \binom{-1/2}{n}.$$

Raymond Mortini, Metz, F und Rudolf Rupp, Nürnberg, D

Aufgabe 1428: Es seien a, b und c positive reelle Zahlen. Man bestimme die grösste Zahl $k_1 > 0$ und die kleinste Zahl $k_2 > 0$ derart, dass die folgende Ungleichung gilt:

$$k_1 \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{ab+bc+ca}} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le k_2 \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Péter Ivády, Budapest, H

Partial solution to problem 1428 Elem. Math. 77 (2022), 196

Raymond Mortini, Rudolf Rupp

Let

$$f(a,b,c) := \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

Then, by using that $\frac{2xy}{x+y} \le \sqrt{xy} \le \frac{x+y}{2}$,

$$\begin{aligned}
f(a,b,c) &= \frac{1}{a^2+b^2+c^2}\left(a\,\frac{bc}{b+c} + b\,\frac{ca}{c+a} + c\,\frac{ab}{a+b}\right) + 1 \\
&\le \frac{1}{a^2+b^2+c^2}\left(a\,\frac{b+c}{4} + b\,\frac{c+a}{4} + c\,\frac{a+b}{4}\right) + 1 \\
&= \frac{1}{a^2+b^2+c^2}\,\frac{2ab+2bc+2ca}{4} + 1 \\
&= \frac{1}{a^2+b^2+c^2}\,\frac{(a+b+c)^2-(a^2+b^2+c^2)}{4} + 1 \\
&\le \frac{1}{a^2+b^2+c^2}\,\frac{(a^2+b^2+c^2)(1+1+1)-(a^2+b^2+c^2)}{4} + 1 \\
&\le \frac{3}{2}.
\end{aligned}$$

If we let $a=b=c$, then $f(a,a,a)=3/2$ and so $\boxed{k_2 = 3/2}$. To determine k_1, let

$$g(a,b,c) := \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2 \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

Then, by using two of the estimates above, namley $f \ge 1$, and Cauchy-Schwarz,

$$g(a,b,c) = \frac{a^2+b^2+c^2}{ab+bc+ca}\, f(a,b,c)^2 > \frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1.$$

We guess $k_1 = \sqrt{2}$. In fact, we may restrict to triples $(x,1,c)$ (homogeniety). Then it remains to prove that

$$f_c(x) := \left(\frac{1}{x+c} + \frac{x}{c+1} + \frac{c}{1+x}\right)^2 \left(\frac{x+cx+c}{1+x^2+c^2}\right) \ge 2.$$

Now

$$\lim_{c\to 0} f_c(x) = x + \frac{1}{x} = f_0(x) \ge 2.$$

Graphical evidence seems to indicate that $m_c := \min_{x>0} f_c(x) \ge 2$ and $\lim_{c\to 0} m_c = 2$.

As it is customn with this type of questions, the infimum of the two-variable function $f(x,c) := f_c(x)$ is taken on the boundary of the first quadrant; that is when $c=0$. We have no proof though of this last claim.

Aufgabe 1383:

a) Man zeige, dass für $\alpha \in (0,1]$ die in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ holomorphe Funktion $f(z) = (1-z)^\alpha$ Hölder–Lipschitz stetig ist zur Ordnung α, d.h.

$$\sigma_\alpha = \sup_{\substack{z,w\in\mathbb{D}\\ z\neq w}} \frac{|(1-z)^\alpha - (1-w)^\alpha|}{|z-w|^\alpha} < \infty.$$

Hierbei ist, wie üblich, $(1-z)^\alpha = \exp(\alpha \log(1-z))$, wobei der Hauptzweig des Logarithmus in der rechten Halbebene genommen wird.

b) Man bestimme σ_α explizit.

Raymond Mortini, Metz, F und Rudolf Rupp, Nürnberg, D

Solution to problem 1383 in Elem. Math 74 (2019), 38, by
Raymond Mortini and Rudolf Rupp

Theorem 17. *Let* $0 < \alpha \leq 1$. *Then*

$$\begin{aligned}
\sigma(\alpha) &:= \sup\left\{\frac{|(1-z)^\alpha - (1-w)^\alpha|}{|z-w|^\alpha} : |z|, |w| \leq 1, z \neq w\right\} \\
&= \max\{1, 2^{1-\alpha}\sin(\alpha\pi/2)\} \\
&= \begin{cases} 1 & \text{if } 0 < \alpha \leq 1/2 \\ 2^{1-\alpha}\sin(\alpha\pi/2) & \text{if } 1/2 \leq \alpha \leq 1. \end{cases}
\end{aligned}$$

Moreover,

$$\max_{0<\alpha\leq 1} \log\sigma(\alpha) = \left(1 - \frac{2}{\pi}\arctan\left(\frac{\pi}{2\log 2}\right)\right)\log 2 + \log\left(\frac{\pi}{\sqrt{\pi^2 + 4(\log 2)^2}}\right).$$

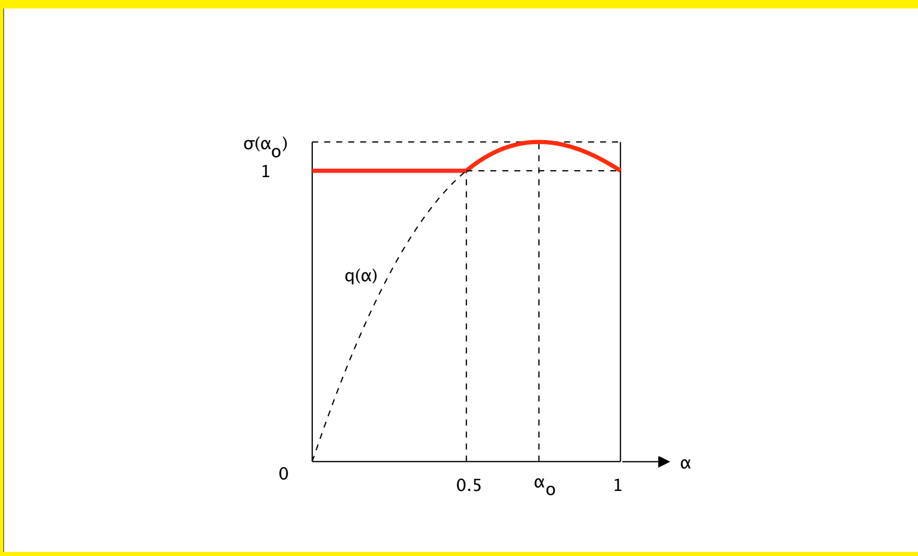


FIGURE 16. The Hölder-Lipschitz constant $\sigma(\alpha)$

See R. Mortini, R. Rupp, The best Hlder-Lipschitz constant associated with the function $(1-z)^\alpha$, Computational Meth. Funct. Theory 20 (2020), 667–676.

Aufgabe 1350: Für $p > 3$ sei

$$f(x) = x^{p-2}(2 - x^p).$$

Man zeige, dass $0 \leq f(x) \leq 1$ gilt, wann immer $0 \leq x \leq \cos(\frac{\pi}{p-1})$ ist.

Raymond Mortini, Metz, F

Solution to problem 1350 in Elem. Math 71 (2016), 84

Dies folgt aus Lemma 2 in [1], welches besagt dass für $0 \leq t \leq \pi/2 - \pi(p-1)$ die Ungleichung $\cos t + (\sin t)^p \leq 1$ gilt, indem man folgende Transformationen benutzt: $x = \sin t$, $0 \leq x \leq \cos(\frac{\pi}{p-1})$,

$$\sqrt{1-x^2} + x^p \leq 1 \iff 1 - x^2 \leq (1-x^p)^2 \iff 0 \leq x^2 + x^{2p} - 2x^p$$
$$\iff 0 \leq 1 + x^{2p-2} - 2x^{p-2} \iff x^{p-2}(2-x^p) \leq 1.$$

Einen davon unabhängigen Beweis würde ich gerne sehen. Ist mir aber unbekannt.

References

[1] Mortini, R; Rhin, G.: Sums of holomorphic selfmaps of the unit disk II, Comp. Meth. Function Theory 11 (2011), 135-142.

Aufgabe 1339: Beweise die Produktdarstellung

$$(1+\sqrt{2})^{\sqrt{2}} = e \cdot \sqrt[2]{e^{1/3}} \cdot \sqrt[4]{e^{1/5}} \cdot \sqrt[8]{e^{1/7}} \cdot \sqrt[16]{e^{1/9}} \cdot \ldots$$

Horst Alzer, Waldbröl, D

solution of problem 1339 Elem. Math. 70 (2015), 82.

Man betrachte die für $|x| < 1$ konvergente Reihe

$$R(x) := \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} = x + \frac{x}{3} + \cdots .$$

Deren Wert ist leicht zu bestimmen durch Ableitung:

$$R'(x) = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}.$$

Somit gilt wegen der gleichmässigen Konvergenz auf $[0,r]$, $0 < r < 1$, und $\int_0^s \sum = \sum \int_0^s$, dass $R(x)$ die Stammfunktion von $(1-x^2)^{-1}$ ist, welche in $x = 0$ verschwindet; also

$$R(x) = \frac{1}{2}\left(\log(1+x) - \log(1-x)\right) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Ersetzt man x durch $1/\sqrt{2}$, so erhält man:

$$R\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{2n+1}\,\frac{1}{2^n}.$$

Desweiteren gilt

$$R\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\log\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) = \log(1+\sqrt{2}).$$

Daher

$$\sum_{n=0}^{\infty} \frac{1}{2n+1}\,\frac{1}{2^n} = \sqrt{2}\log(1+\sqrt{2}).$$

Durch Übergang zur Exponentialfunktion unter Verwendung der für konvergente Reihen geltenden Formel $e^{\sum a_n} = \prod e^{a_n}$, erhält man die gewünschte Gleichung:

$$(1+\sqrt{2})^{\sqrt{2}} = \prod_{n=0}^{\infty} e^{\frac{1}{2^n\,(2n+1)}} = \prod_{n=0}^{\infty} \sqrt[2^n]{e^{1/(2n+1)}}.$$

Aufgabe 1281: Man bestimme den Wert der Reihen

$$S=\sum_{n=2}^{\infty}\left(2+n\log\left(1-\frac{2}{n+1}\right)\right)$$

und

$$S^{*}=\sum_{n=1}^{\infty}\left(1-\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right)\right).$$

Raymond Mortini, Metz, F

Solution to problem 1281 Elem. Math. 65 (2010), 127, by
Raymond Mortini, Jérôme Noël

a) Exponentiating, we have to calculate the value of the infinite product

$$P=\prod_{n=2}^{\infty}\left(e^{2}\left(\frac{n-1}{n+1}\right)^{n}\right).$$

We claim that $P=\frac{4\pi}{e^3}$; so $S=\log(4\pi)-3$.

Let $P_N=\prod_{n=2}^{N}\left(e^2\left(\frac{n-1}{n+1}\right)^n\right)$. Then by Stirlings formula, telling us that $n!\sim e^{-n}n^n\sqrt{2\pi n}$, we obtain

$$P_N=\frac{1}{e^2}e^{2N}\frac{\prod_{n=2}^{N}(n-1)^n}{\prod_{n=2}^{N}(n+1)^n}=$$
$$\frac{1}{e^2}e^{2N}\frac{\prod_{n=2}^{N}(n-1)^n}{\prod_{\mathbf{n=2}}^{\mathbf{N}}\mathbf{n^{n+1}}}\;\frac{\prod_{\mathbf{n=2}}^{\mathbf{N}}\mathbf{n^{n+1}}}{\prod_{n=2}^{N}(n+1)^n}=$$
$$\frac{2}{e^2}\frac{e^{2N}\,(N!)^2}{N^{N+1}(N+1)^N}=\frac{2}{e^2}\left(\frac{e^N N!}{N^N}\right)^2\frac{N^N}{(N+1)^N}\,\frac{1}{N}\sim$$
$$\frac{2}{e^2}\,\frac{(\sqrt{2\pi N})^2}{N}\,\frac{1}{(1+\frac{1}{N})^N}\to\frac{4\pi}{e^3}.$$

b) To determine S^*, we use the same method and calculate the value of

$$P^*=\prod_{n=1}^{\infty}e\left(\frac{n}{n+1}\right)^{n+\frac{1}{2}}$$

We claim that $P^*=\frac{\sqrt{2\pi}}{e}$ and so $S^*=\frac{1}{2}\log(2\pi)-1$.

In fact

$$P_N^*=\prod_{n=1}^{N}e\left(\frac{n}{n+1}\right)^{n+\frac{1}{2}}=\frac{e^N N!}{(N+1)^N}\,\frac{1}{\sqrt{N+1}}.$$

Using Stirling's formula we obtain

$$P_N\sim\frac{N^N}{(N+1)^N}\sqrt{2\pi N}\,\frac{1}{\sqrt{N+1}}=\sqrt{2\pi}\frac{1}{(1+\frac{1}{N})^N}\,\frac{\sqrt{N}}{\sqrt{N+1}}\to\frac{\sqrt{2\pi}}{e}.$$

Aufgabe 901. Die Funktion $f:\{z\in\mathbf{C} \mid |z|<1\}\to\{z\in\mathbf{C} \mid |z|\le 1\}$ sei holomorph und es sei $f(0)=0$. Dann trifft genau eine der beiden folgenden Aussagen zu:

$$\left|\int_{-1}^{1} f(x)\,dx\right| < 2/3. \tag{1}$$

Es gibt eine Konstante $\alpha\in\mathbf{C}$ mit $|\alpha|=1$ derart, dass

$$f(z)=\alpha z^2. \tag{2}$$

Dies ist zu zeigen.

P. von Siebenthal, Zürich

solution of problem 901 Elem. Math. 38 (1983), 128.

El. Math., Vol. 39, 1984 131

Lösung: Es sei $\mathbf{D}$ die offene Einheitskreisscheibe, also $\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$. Zunächst zerlegen wir die Funktion f in den geraden Anteil w und den ungeraden Anteil v, also

$$f(z)=w(z)+v(z), \tag{3}$$

mit

$$w(z)=\frac{1}{2}\big(f(z)+f(-z)\big) \quad \text{und} \quad v(z)=\frac{1}{2}\big(f(z)-f(-z)\big).$$

Aus den Voraussetzungen über f folgt, dass w die Form

$$w(z)=z^2 g(z)$$

hat, wobei die Funktion g holomorph in $\mathbf{D}$ und $|g(z)|\leqslant 1$ für jedes $z\in\mathbf{D}$ ist.

1. Fall. $|g(z)|<1$ für jedes $z\in\mathbf{D}$.

Da die Funktion v ungerade ist, ist das Integral $\int_{-1}^{+1} v(x)\,dx=0$. Daher ergibt sich sofort die gewünschte Ungleichung:

$$\left|\int_{-1}^{+1} f(x)\,dx\right|=\left|\int_{-1}^{+1} w(x)\,dx\right|\leqslant\int_{-1}^{+1}|x^2 g(x)|\,dx<\int_{-1}^{+1}x^2\,dx=\frac{2}{3}.$$

2. Fall. $|g(z_0)|=1$ für ein $z_0\in\mathbf{D}$.

Nach dem Maximumprinzip ist demnach $|g(z)|=1$ für jedes $z\in\mathbf{D}$, also $g(z)\equiv\text{const}=\alpha$ mit $|\alpha|=1$.

Die Funktion f lässt sich daher nach (3) in der Form

$$f(z)=\alpha z^2+v(z)$$

darstellen.

Beachtet man, dass v ungerade ist, so gilt für jedes $z\in\mathbf{D}$.

$$|\alpha z^2+v(z)|=|f(z)|\leqslant 1$$
$$|\alpha z^2-v(z)|=|f(-z)|\leqslant 1.$$

Quadrieren und Addition ergibt wegen $|\alpha|=1$:

$$|z|^4+|v(z)|^2\leqslant 1 \qquad (z\in\mathbf{D}). \tag{4}$$

Die Ungleichung (4) impliziert jedoch, dass $v(z)$ gleichmässig gegen Null geht, falls z gegen den Rand von $\mathbf{D}$ strebt. Nach dem Maximumprinzip ist also v identisch Null. Somit hat f die Gestalt

$$f(z)=\alpha z^2,$$

mit $|\alpha|=1$.

R. Mortini, Karlsruhe, BRD

Q.68 A function f satisfies the equation $f(x+1)+f(x-1)=\sqrt{2}f(x)$ for all real x. Prove that this function is periodic. (*Quantum*)

Aufgabe Q68, EMS Newsletter 25 (1997), 27.

Q.68 *A function f satisfies the equation $f(x+1)+f(x-1)=\sqrt{2}\cdot f(x)$ for all real x. Prove that this function is periodic.*

First Solution (Raymond Mortini, Luxembourg, Université de Metz, Département de Mathématiques)

–Ed. *It is the first time that I received a submission in German language, and astoundingly it came to me from France. The overtures of friendship between Germans and Frenchmen aforementioned appear to work. So, it is fortunate that Mathematics can amplify such advances; hence this solution is presented true to the original.*

Behauptung. Es sei f eine Funktion auf $\mathbb{R}$ welche der Bedeinung genügt:

$$f(x+1)+f(x-1)=\sqrt{2}f(x)$$

Dann hat f die Periode 8.

Beweis. Es sei $x\in\mathbb{R}$ beliebig aber fest gewählt. Dan ergeben sich aus der Voraussetzung die folgenden Gleichungen:

$$f(x+8)=\sqrt{2}f(x+7)-f(x+6)=\sqrt{2}[-f(x+5)+\sqrt{2}f(x+6)]-f(x+6)=-\sqrt{2}f(x+5)+f(x+6)=-f(x+4).$$

Damit ergibt sich sofort die Behauptung $f(x+8)=-f(x+4)=-(-f(x))=f(x)$.

Bemerkung. Alle Lösungen der obigen Funktionalgleichung haben die Form

$$f(x+n)=r(x)\cdot sin(\theta(x)+n\cdot\tfrac{\pi}{4})\text{ für } x\in[0,1[,\quad n\in\mathbb{Z},$$

wobei $r(x)>0$ und $\theta(x)$ beliebige Funktionen sind. ■

Also solved by Dr. J N Lillington.