The ring of real-valued multivariate polynomials

joint work with R. Rupp

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An analyst's perspective

$\boldsymbol{z} := (z_1, \dots, z_n)$ and $\overline{\boldsymbol{z}} := (\overline{z}_1, \dots, \overline{z}_n)$. Associated with $\mathbb{R}[x_1, \dots, x_n]$ is the following algebra of *real-symmetric polynomials*:

$$\mathbb{C}_{\text{sym}}[\boldsymbol{z}] = \left\{ f \in \mathbb{C}[\boldsymbol{z}] : f(\boldsymbol{z}) = \overline{f(\overline{\boldsymbol{z}})} \;\; \forall \boldsymbol{z} \in \mathbb{C}^n \right\}.$$

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$\mathbb{C}_{\text{sym}}[z_1, \dots, z_n]$ is a real algebra of complex-valued polynomials that is real-isomorphic to $\mathbb{R}[x_1, \dots, x_n]$. For $\boldsymbol{a} \in \mathbb{C}^n$ let

$$M_{\boldsymbol{a}} := \{p \in \mathbb{C}[z_1, \dots, z_n] : p(\boldsymbol{a}) = 0\}.$$

By Hilbert's Nullstellensatz an ideal in $\mathbb{C}[z_1, \dots, z_n]$ is maximal if and only if it has the form $M_{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \mathbb{C}^n$.

Theorem

The class of maximal ideals of $\mathbb{C}_{\text{sym}}[z_1, \ldots, z_n]$ *coincides with the class of ideals of the form*

$$S_{\mathbf{a}} := M_{\mathbf{a}} \cap M_{\overline{\mathbf{a}}} \cap \mathbb{C}_{\text{sym}}[z_1, \ldots, z_n],$$

where $\mathbf{a} \in \mathbb{C}^n$*. The set* $\{\mathbf{a}, \overline{\mathbf{a}}\}$ *is uniquely determined for a given maximal ideal.*

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Note that $M_{\mathbf{a}} \cap M_{\overline{\mathbf{a}}} \cap \mathbb{C}_{\text{sym}}[\mathbf{z}] = M_{\mathbf{a}} \cap \mathbb{C}_{\text{sym}}[\mathbf{z}]$, because for every polynomial p in $\mathbb{C}_{\text{sym}}[\mathbf{z}]$ it holds that $p(\mathbf{a}) = 0$ if and only if $p(\overline{\mathbf{a}}) = 0$.

Beweis.

i) $S_{\boldsymbol{a}}$ is maximal:
to see this, suppose that $f \in \mathbb{C}_{\text{sym}}[\boldsymbol{z}]$ does not vanish at $\boldsymbol{a}$. Then

$$\left(f - f(\boldsymbol{a})\right)\left(f - \overline{f(\boldsymbol{a})}\right) = f^2 - \left(2\mathrm{Re}\, f(\boldsymbol{a})\right) f + |f(\boldsymbol{a})|^2 \in S_{\boldsymbol{a}}$$

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and

$$1 = \frac{\big(f - f(\boldsymbol{a})\big)\ \big(f - \overline{f(\boldsymbol{a})}\big)}{|f(\boldsymbol{a})|^2} - f\, \frac{f - \big(f(\boldsymbol{a}) + \overline{f(\boldsymbol{a})}\big)}{|f(\boldsymbol{a})|^2}.$$

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Hence the ideal, $I_{\mathbb{C}_{\text{sym}}[\boldsymbol{z}]}(S_{\boldsymbol{a}}, f)$, generated by $S_{\boldsymbol{a}}$ and f is the whole algebra and so $S_{\boldsymbol{a}}$ is maximal.
ii) if $f(\boldsymbol{a})$ is real, we simply could have argued as follows, since the constant function $\boldsymbol{z} \mapsto f(\boldsymbol{a})$ then belongs to $\mathbb{C}_{\text{sym}}[\boldsymbol{z}]$:

$$1 = -\frac{f - f(\boldsymbol{a})}{f(\boldsymbol{a})} + \frac{f}{f(\boldsymbol{a})} \in I_{\mathbb{C}_{\text{sym}}[\boldsymbol{z}]}(S_{\boldsymbol{a}}, f).$$

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iii) M maximal $\Longrightarrow M = S_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{C}^n$ Suppose, to the contrary, that M is not contained in any ideal of the form $S_{\mathbf{a}}$. Hence, for every $\mathbf{a} \in \mathbb{C}^n$, there is $p_{\mathbf{a}} \in M$ such that $p_{\mathbf{a}}(\mathbf{a}) \neq 0$. By Hilbert's Nullstellensatz, the ideal generated by the set $S = \{p_{\mathbf{a}} : \mathbf{a} \in \mathbb{C}^n\}$ in $\mathbb{C}[\mathbf{z}]$ coincides with $\mathbb{C}[\mathbf{z}]$.

iii) M maximal $\Longrightarrow M = S_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{C}^n$ Suppose, to the contrary, that M is not contained in any ideal of the form $S_{\mathbf{a}}$. Hence, for every $\mathbf{a} \in \mathbb{C}^n$, there is $p_{\mathbf{a}} \in M$ such that $p_{\mathbf{a}}(\mathbf{a}) \neq 0$. By Hilbert's Nullstellensatz, the ideal generated by the set $S = \{p_{\mathbf{a}} : \mathbf{a} \in \mathbb{C}^n\}$ in $\mathbb{C}[\mathbf{z}]$ coincides with $\mathbb{C}[\mathbf{z}]$. Hence there are $q_j \in \mathbb{C}[\mathbf{z}]$ and finitely many $\mathbf{a}_j \in \mathbb{C}^n$, $(j = 1, \dots, N)$, such that

$$\sum_{j=1}^{N} q_j \, p_{\mathbf{a}_j} = 1.$$

$$\Longrightarrow 1 = \overline{\sum_{j=1}^{N} q_j(\overline{\mathbf{z}}) \, p_{\mathbf{a}_j}(\overline{\mathbf{z}})} = \sum_{j=1} \overline{q_j(\overline{\mathbf{z}})} \, p_{\mathbf{a}_j}(\mathbf{z}).$$

Hence , with

$$q_j^*(\mathbf{z}) = \frac{1}{2}\left(\overline{q_j(\overline{\mathbf{z}})} + q_j(\mathbf{z})\right),$$

we conclude that

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Since $q_j^* \in \mathbb{C}_{\rm sym}[\mathbf{z}]$ and $p_{\mathbf{a}_j} \in M$ we obtain the contradiction that $1 \in M$. Thus $M \subseteq S_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{C}^n$. The maximality of M now implies that $M = S_{\mathbf{a}}$.

Generators for the maximal ideals in $\mathbb{R}[x_1, \ldots, x_n]$.

Generators for the maximal ideals in $\mathbb{R}[x_1, \dots, x_n]$. By Hilbert's Nullstellensatz the maximal ideals in $\mathbb{C}[z_1, \dots, z_n]$ are generated by $z_1 - a_1, \dots, z_n - a_n$, where $\boldsymbol{a} := (a_1, \dots, a_n) \in \mathbb{C}^n$.

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Example

(1) Let $\sigma \in \mathbb{C} \setminus \mathbb{R}$ and $r_j \in \mathbb{R}$, $j = 1, 2, \ldots, n-1$. Then the ideal generated by

$$x_n^2 - (2\,\mathrm{Re}\,\sigma)\,x_n + |\sigma|^2 = (x_n - \sigma)(x_n - \overline{\sigma})$$

and $x_j - r_j$, $(j = 1, \ldots, n-1)$, is maximal in $\mathbb{R}[x_1, \ldots, x_n]$. It corresponds to the ideal $S_{(r_1,\ldots,r_{n-1},\sigma)}$.

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$$f = (z^2+1)q_1(z,w) + (w^2+1)q_2(z,w) + r(z,w) \tag{1}$$

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$$r(z,w) = a + bz + cw + d\,zw$$

for some $(a,b,c,d) \in \mathbb{R}^4$. Hence $a + bi + ci - d = 0$ from which we deduce that $a = d$ and $b = -c$.

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We conclude that

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Thus, in view of the maximality of $S_{(i,i)}$, $S_{(i,i)} = I$. Hence I is maximal

examples

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Then the maximal ideal $M = S_{(r_1,\dots,r_k,a_1,\dots,a_m)}$ *of*
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and $2^{n-k} - 2$ *multilinear polynomials* q_j *in* $\mathbb{R}[x_{k+1}, \ldots, x_n]$ *vanshing at* $a_{k+1}, \ldots, a_n$. *More precisely, we have*

$$M = \sum_{j=1}^{n} p_j(x_j)R + \sum_{j=1}^{2^{n-k}-2} q_j(x_{k+1}, \ldots, x_n)\, \mathbb{R}.$$

Lemma

Let $\boldsymbol{i} = (i, \ldots, i) \in \mathbb{C}^m$. *The a (vector-space) basis of*

$$V^* = \left\{ f(\boldsymbol{z}) = \sum_{j_1, \ldots, j_m} c_{\boldsymbol{j}} z_{k+1}^{j_1} \cdots z_{k+m}^{j_m},\ j_\ell \in \{0,1\},\ c_{\boldsymbol{j}} \in \mathbb{R},\ f(\boldsymbol{i}) = 0 \right\}$$

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is given by

$$\begin{array}{rl}
x_1 - x_j & 1 < j \leq m \\
1 + x_{j_1} x_{j_2} & 1 \leq j_1 < j_2 \leq m \\
x_1 + x_{j_1} x_{j_2} x_{j_3} & 1 \leq j_1 < j_2 < j_3 \leq m \\
1 - \prod_{\ell=1}^{4} x_{j_\ell} & 1 \leq j_1 < \cdots < j_4 \leq m \\
x_1 - \prod_{\ell=1}^{5} x_{j_\ell} & 1 \leq j_1 < \cdots < j_5 \leq m
\end{array}$$

The last element has exactly one of the following forms:

$$\begin{cases} x_1 - \prod_{j=1}^{m} x_j & \text{if } m \equiv 1 \mod 4 \\ 1 \quad + \prod_{j=1}^{m} x_j & \text{if } m \equiv 2 \mod 4 \\ x_1 + \prod_{j=1}^{m} x_j & \text{if } m \equiv 3 \mod 4 \\ 1 \quad - \prod_{j=1}^{m} x_j & \text{if } m \equiv 0 \mod 4 \end{cases}$$

Theorem

Let $m + k = n$, $m \geq 2$, *and*
$\boldsymbol{a} := (i, \dots, i, r_{m+1}, \dots, r_{m+k}) \in \mathbb{C}^m \times \mathbb{R}^k$.

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Let $m+k=n$, $m\geq 2$, *and* $\mathbf{a} := (i,\ldots,i,r_{m+1},\ldots,r_{m+k})\in\mathbb{C}^m\times\mathbb{R}^k$. *The maximal ideal* $S_{\mathbf{a}}$ *of* $\mathbb{R}[x_1,\ldots,x_n]$ *is generated by the* 2^m-2 *multilinear polynomials in Lemma 4 and the polynomials*

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The general case of an arbitrary maximal ideal $S_{\mathbf{a}}$ is easily deduced by using the transformation

$$\chi(z_1, \ldots, z_m) = \left(\frac{z_1 - \alpha_1}{\beta_1}, \ \ldots, \ \frac{z_m - \alpha_m}{\beta_m} \right)$$

of $\mathbb{C}^m$ onto $\mathbb{C}^m$, whenever

$$\mathbf{a} = (\alpha_1 + i\beta_1, \ldots, \alpha_m + i\beta_m, r_{m+1}, \ldots, r_{m+k}) \in \mathbb{C}^m \times \mathbb{R}^k \subseteq \mathbb{C}^n,$$

with $\beta_j \neq 0$ for $j = 1, \ldots, m$.

Theorem

Let $p_1, \dots, p_{n+1}$ *be polynomials in* $F[x_1, \dots, x_n]$. *Then there exists a non-zero polynomial* $P \in F[y_1, \dots, y_{n+1}]$ *in* $n+1$ *variables such that*

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Proof Let $k := 1 + \max_{1 \le j \le n+1} \deg p_j$. For big $L \in \mathbb{N}$, to be determined later, we are looking for $P \in F[y_1, \dots, y_{n+1}]$ with $0 \le \deg P \le L$ and $P(p_1, \dots, p_{n+1}) = 0$.

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Let V be the vector space of all polynomials p in $F[x_1, \dots, x_n]$ with $\deg p \leq kL$. Then $\dim V = \binom{kL+n}{n} =: A(L)$.Consider now the following collection $\mathcal{C}$ of (wlog distinct) polynomials:

$$p_1^{j_1} \dots p_{n+1}^{j_{n+1}} : j_i \in \mathbb{N}, \sum_{i=1}^{n+1} j_i \leq L$$

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Note that each of the $B(L) := \binom{L+n+1}{n+1}$ members of $\mathcal{C}$ belongs to V, because for $p \in \mathcal{C}$,

$$\deg p \leq k(j_1 + \dots + j_{n+1}) \leq kL.$$

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Then $\operatorname{card} \mathcal{C} = B(L)$. Note also that $\mathcal{C} \subseteq V$. We claim that $B(L) > A(L)$ for some L (depending on n). In fact, looking upon $B(L)$ and $A(L)$ as polynomials in L, we have that $\deg B = n+1$ and $\deg A = n$. Thus, for large L, we obtain that $B(L) > A(L)$.

Thus the cardinal of set $\mathcal{C}$ is strictly bigger than the dimension of the vector space V it belongs to. Hence $\mathcal{C}$ is a linear dependent set in V. In other words, there is a linear combination of the elements from S that is identically zero. This implies that there is a nonzero polynomial $P \in F[y_1, \dots, y_{n+1}]$ of degree at most L such that $P(p_1, \dots, p_{n+1}) = 0$.

Definition

Let R be a commutative unital ring with identity element 1.

(1) An n-tuple $(f_1, \dots, f_n) \in R^n$ is said to be *invertible* (or *unimodular*), if there exists $(x_1, \dots, x_n) \in R^n$ such that the Bézout equation $\sum_{j=1}^n x_j f_j = 1$ is satisfied.

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(2) The *Bass stable rank* of R, denoted by $\mathrm{bsr}\, R$, is the smallest integer n such that every element in $U_{n+1}(R)$ is reducible. If no such n exists, then $\mathrm{bsr}\, R = \infty$.

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Note that if $\mathrm{bsr}\, R = n$, $n < \infty$, and $m \geq n$, then every invertible $(m+1)$-tuple $(\boldsymbol{f}, g) \in R^{m+1}$ is reducible.

Theorem (Vasershtein)

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Proof $\underline{\operatorname{bsr} \mathbb{R}[x_1, \ldots, x_n] \geq n + 1}$.

Consider the invertible $(n+1)$-tuple $(x_1, \ldots, x_n, 1 - \sum_{j=1}^n x_j^2)$ in $\mathbb{R}[x_1, \ldots, x_n]$.

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Consider the invertible $(n+1)$-tuple $(x_1, \dots, x_n, 1 - \sum_{j=1}^n x_j^2)$ in $\mathbb{R}[x_1, \dots, x_n]$. This tuple cannot be reducible in $\mathbb{R}[x_1, \dots, x_n] \subseteq C(\mathbb{R}^n, \mathbb{R})$, since otherwise the n-tuple $(x_1, \dots, x_n)$, restricted to the unit sphere $\partial\mathscr{B}$ in $\mathbb{R}^n$, would have a zero-free extension $\boldsymbol{e}$ to $\mathscr{B}$, where $\boldsymbol{e}$ is given by

$$\Big(x_1 + u_1 \cdot (1 - \sum_{j=1}^n x_j^2), \dots, x_n + u_n \cdot (1 - \sum_{j=1}^n x_j^2)\Big)$$

for some $u_j \in \mathbb{R}[x_1, \dots, x_n]$.

This contradicts Brouwer's result that the identity map

$$(x_1, \dots, x_n) : \partial\mathscr{B} \to \partial\mathscr{B}$$

defined on the boundary of the closed unit ball $\mathscr{B}$ in $\mathbb{R}^n$ does not admit a zero-free continuous extension to $\mathscr{B}$.

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This follows from a combination of Bass Theorem telling us that the Bass stable rank of a Noetherian ring with Krull dimension n is less than or equal to $n + 1$, and Theorem below, according to which the Krull dimension of $\mathbb{R}[x_1, \ldots, x_n]$ is n.

Definition

Let R be a commutative unital ring, $R \neq \{0\}$.

(1) A chain $\mathfrak{C} = \{I_0, I_1, \dots, I_n\}$ of ideals in R is said to have *length* n if

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the inclusions being strict.

(2) The *Krull dimension* of R is defined to be the supremum of the lengths of all increasing chains of prime ideals

$$P_0 \subset \cdots \subset P_n.$$

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$$a_0^{n_0}\left(a_1^{n_1}\Big(\cdots\big(a_N^{n_N}(1 + a_N x_N) + \cdots\big)\Big) + a_0 x_0\right) = 0, \qquad (2)$$

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in other words

$$L_0 \circ \cdots \circ L_N(1) = 0, \quad (3)$$

where for $a, x \in R$ and $n \in \mathbb{N}$, $L_{a,n,x}(y) = a^n(y + ax)$. If

Proposition (Coquand-Lombardi)

Let F be a field and $R \neq \{0\}$ a commutative unital algebra over F. If any $(n+1)$-tupel $(f_0, \dots, f_n) \in R^{n+1}$ is algebraically dependent over F,

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Let F be a field and $R \neq \{0\}$ a commutative unital algebra over F. If any $(n+1)$-tupel $(f_0, \dots, f_n) \in R^{n+1}$ is algebraically dependent over F, [a] *then the Krull dimension of R is at most n.*

[a] In other words, if there is a non-zero polynomial $Q \in F[y_0, \dots, y_n]$ such that $Q(f_0, \dots, f_n) = 0$.

Proof

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$$a_{i_0,\dots,i_n} f_0^{i_0} f_1^{i_1} \dots f_n^{i_n}$$

be the "first"monomial appearing in the relation above (here the coefficient $a_{i_0,\dots,i_n}$ belongs to F and $(i_0, \dots, i_n) \in \mathbb{N}^n$).

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$$\begin{aligned} Q = f_0^{i_0} \dots f_{n-1}^{i_{n-1}} f_n^{i_n} \quad &+ \quad f_0^{i_0} \dots f_{n-1}^{i_{n-1}} f_n^{1+i_n} R_n + f_0^{i_0} \dots f_{n-1}^{1+i_{n-1}} R_{n-1} + \dots \\ &+ \quad f_0^{i_0} f_1^{1+i_1} R_1 + f_0^{1+i_0} R_0 \end{aligned}$$

where R_j belongs to $F[f_j, f_{j+1}, \dots, f_n]$, $j = 0, 1, \dots, n$.

Hence Q has been written in the form given by equation 2 (with $a_j := f_j$ and $x_j := R_j$), that is

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We conclude from Theorem 10, that the Krull dimension of R is at most n.

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By Perron's Theorem, $R := F[x_1, \ldots, x_n]$ satisfies the assumption of Proposition 11. Hence the Krull dimension of R is less or equal to n. But we have the chain of prime ideals

$$\{0\} \subset I_R(x_1) \subset I_R(x_1, x_2) \subset \cdots \subset I_R(x_1, \ldots, x_n).$$

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Since $\{0\}$ is a prime ideal too, this chain has length n. Thus the Krull dimension of R is n. □