L3 2016 Final

(1) • $\left|n^2 - \frac{1}{n!}\right| \le n^2$, $\sum n^2 z^n$ conv ...

$\Rightarrow R \ge 1$: $z = 1 \Rightarrow n^2 - \frac{1}{n!} \not\to 0 \Rightarrow R = 1$

• $\sum_2 n^2 z^n = z \sum_2 n (z^n)' = z \left(\sum_2 n z^n\right)'$

$\sum_2 n z^n = z \sum_2 (z^n)' = z \left(\sum_2 z^n\right)' = z \left(\frac{z^2}{1-z}\right)'$

$= z\left(\frac{1}{1-z} - 1 - z\right)' = \frac{z}{(1-z)^2} - z$

$\Rightarrow \left(\sum_2 n z^n\right)' = \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} - 1$

$\Rightarrow \sum_2 n^2 z^n = \frac{z}{(1-z)^2} + \frac{2z^2}{(1-z)^3} - z$

$\Rightarrow \sum_2 n^2 z^{n+1} = \frac{z^2}{(1-z)^2} + \frac{2z^3}{(1-z)^3} - z^2$

$\sum_2 \frac{1}{n!} z^{n+1} = z\left(\sum_2 \frac{1}{n!} z^n\right) = z\left(e^z - 1 - z\right)$

$\sum n(n-1) z^n = \sum n^2 z^n - \sum n z^n$

(2) $f(z) = \bar{z} - z\,\mathrm{Im}\,z = x - iy - (x+iy)y = x - xy - i(y + y^2)$

$u(x,y) = x - xy$, $u_x = 1 - y = -1 - 2y$

$v(x,y) = -y - y^2$, $u_y = -x = -0$

$\Rightarrow x = 0$ et $2 + y = 0 \Rightarrow y = -2$

sol : $\{(0,-2)\} = \boxed{\{-2i\}}$

ou : $f(z) = \bar{z} - z\frac{(z - \bar{z})}{2i} \Rightarrow \bar{\partial} f = 1 + \frac{z}{2i} = 0$

$\Leftrightarrow z = -2i$

(3) • $\bar{z} z^2 + 1 = 0 \Rightarrow \bar{z} z^2 = -1 \Rightarrow |z|^3 = 1 \Rightarrow |z| = 1$

$\Rightarrow \bar{z} = \frac{1}{z} \Rightarrow z + 1 = 0 \Rightarrow \boxed{z = -1}$

ou $(x - iy)(x^2 - y^2 + 2ixy) = -1 \Rightarrow$

$\begin{cases} x(x^2 - y^2) + 2y^2 x = -1 \\ -y(x^2 - y^2) + 2x^2 y = 0 \end{cases} \Leftrightarrow \begin{cases} x^3 + xy^2 = -1 & (1) \\ y(x^2 + y^2) = 0 & (2) \end{cases}$

$\Rightarrow y = 0$ ou $x^2 + y^2 = 0 \Rightarrow y = 0$ (car $(x,y) \ne (0,0)$ d'après (1))

$\Rightarrow y = 0 \Rightarrow x = -1$.

$z^2 - z - i + 1 = 0$ $\quad z = -i$ solution

$z^2 - z - i + 1 = (z+i)(z-(1+i))$ $\quad (-i)(1+i) = -i+1$

$S = \{-i, 1+i\}$ $\quad \Delta = 4i-3 \quad \sqrt{\Delta}$

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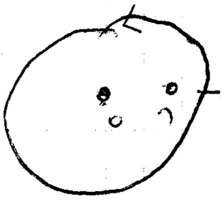
(4) $\sinh z - \sin z = \sum_0^\infty \frac{1}{(2n+1)!} z^{2n+1} - \sum_0^\infty \frac{1}{(2n+1)!} z^{2n+1}(-1)^n$

$= \sum_0^\infty \frac{1}{(2n+1)!}\underbrace{\left(1-(-1)^n\right)}_{\substack{=0 \text{ n pair} \\ =2 \text{ n impair}}} z^{2n+1} = \sum_1^\infty \frac{2}{(4j-1)!} z^{4j-1}$

$= \sum_0 \frac{2}{(4n+3)!} z^{4n+3}$

$f \quad g(z) = z(z-1)(z-2)$

(5)



$\frac{1}{2\pi i}\int_{|z|=r} \frac{e^{f(z)}}{z(z-1)(z-2)} = \begin{cases} \mathrm{Res}(f,0) & 0<r<1 \\ \mathrm{Res}(f,0)+\mathrm{Res}(f,1) & 1<r<\frac{3}{2} \end{cases}$

$= \begin{cases} \frac{e^{f(0)}}{(-1)(-2)} = \frac{1}{2} & 0<r<1 \\ \frac{1}{2} + \frac{e^{f(1)}}{1\cdot(-1)} = \frac{1}{2} - 1 = -\frac{1}{2} & 1<r<\frac{3}{2}. \end{cases}$

$\Rightarrow I = \begin{cases} \pi i & 0<r<1 \\ -\pi i & 1<r<\frac{3}{2}. \end{cases}$

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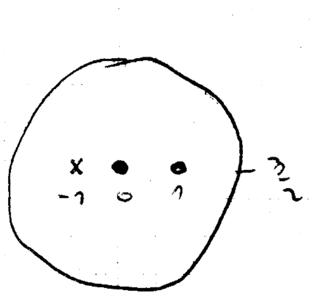
(6)



$\sin \pi z = 0 \iff z = k \quad k \in \mathbb{Z}$

$I = N - P = \underset{2 - 7}{\cancel{\quad}} = -5$

$\frac{\sin \pi z}{z} \to 1 \Rightarrow$ pas de pôle simple en 0.

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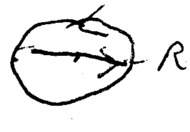
(7) $\int_{D(R)} \frac{z^2+1}{z^4+1}\, dz$



pôles $e^{i\frac{\pi+2k\pi}{4}}$

$= 2\pi i\left(\mathrm{Res}(f, e^{i\frac{\pi}{4}}) + \mathrm{Res}(f, e^{3i\frac{\pi}{4}})\right)$ $\quad -i\frac{k\pi}{4}$ $\quad = -i \frac{}{2 z_1}$

$= 2\pi i\left[\frac{z_0^2+1}{4z_0^3} + \frac{z_1^2+1}{4z_1^3}\right] =$

$z_0 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$, $z_0^3 = e^{i\frac{3\pi}{4}} = z_1$

$z_1 = e^{i\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}(-1+i)$, $z_1^3 = e^{i\frac{9\pi}{4}} = e^{i\frac{\pi}{4}} = z_0$

$\frac{2\pi i}{4}\left[\frac{1}{z_0} + \frac{1}{z_0^3} + \frac{1}{z_1} + \frac{1}{z_1^3}\right]$

$= \frac{\pi i}{2}\left[\frac{1}{z_0} + \frac{1}{z_1} + \frac{1}{z_1} + \frac{1}{z_0}\right]$

$= \frac{\pi i}{2}[2\bar{z}_0 + 2\bar{z}_1] = i\pi(\bar{z}_0 + \bar{z}_1)$

$= \frac{i\pi}{\sqrt{2}}(1-i+(-1-i)] = -2i\frac{i\pi}{\sqrt{2}} = \boxed{\sqrt{2}\pi}$

Γ_R

$\int_{-R}^{R} + \int_{\Gamma_R} = I_1(R) + I_2(R)$

$|I_2(R)| \le \frac{R^2+1}{R^4-1}\pi R \sim \frac{\pi}{R} \to 0$

$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \sqrt{2}\pi$

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$|z|=2$: $\frac{1}{2}z^6 = 2^5 = 16\cdot 2 = 32$

$9\cdot 2^3 = 72$

$g(z) = 9z^3$

$\Rightarrow |p(z) - g(z)| = |\frac{1}{2}z^6 + 13z - 1| \le 32 + 26 + 1 =$

$= 59 < 72 = \underset{|z|=2}{|g|} \le |f| + |g|$

$\Rightarrow Z_{2D}(p) = 3$

$|z|=1$ $h(z) = 13z$

$\Rightarrow |p - h| = |\frac{1}{2}z^6 + 9z^3 - 1| \le \frac{1}{2} + 9 + 1 < 11 < 13 = |h|$ $(|z|=1)$

$= |h(z)| \le |p| + |h|$

$\Rightarrow Z_D(p) = 1 \Rightarrow Z_C(p) = \boxed{2}$

poly degré 6 → 6 racines ⟹ 6 − 3 = $\boxed{3}$ racines
pour $|z| > 2$.

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$\frac{z}{\sin(\frac{1}{z})}$ → $D = \mathbb{C} \setminus [\{0\} \cup \{\frac{1}{n\pi} : n \in \mathbb{Z}^*\}]$

pôles simples $\frac{1}{n\pi}$, $n \in \mathbb{Z}^*$
0 pt. sing. non isolé

$Res(f, 1) = \frac{g(1)}{h'(1)} = \frac{1}{\pi\cos(\frac{\pi}{z})(-\frac{\pi}{z^2})}\Big|_{z=1} = \boxed{1/\pi}$

$Res(f, 3) = \boxed{0}$ car f hol. en 3.

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$f(n) = f(\frac{2}{n}) = 0$ ⟹ $f \equiv 0$
$f(z) = \sin \pi z \sin \frac{2\pi}{z}$, $f \in H(\mathbb{C} \setminus \{0\})$.

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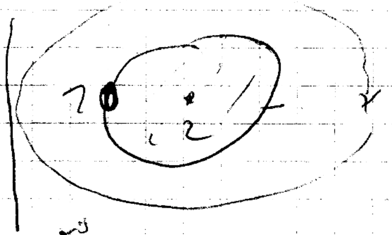
$\frac{1}{z-1}$



pôle 1

$\frac{1}{z-1} = \frac{1}{(z-2)+1} = \sum_0^\infty (z-2)^n (-1)^n$, $R = 1$

Laurent = Mc Laurin
pour déf. en B' (ni l. ni rc).

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