Higher order hulls in H^∞

Pamela Gorkin (*) and Raymond Mortini

summer 1996

Abstract

Using higher order hulls we give a complete description of the closed ideals in H^∞ with hull contained in the set of nontrivial Gleason parts .

AMS-classification number: 46 J 15

Let H^∞ be the Banach algebra of all bounded analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. One way to study a Banach algebra is to understand the ideals in the algebra. In the case of H^∞, the maximal ideals have been well-studied. One can also look at closed ideals with certain special properties. For example, the closed prime ideals in H^∞ have been the object of much recent study. The goal, of course, is to understand the general closed ideals in the algebra and the hope is to use this to learn more about the algebra itself. In this paper, we will give a complete description of certain closed ideals. To do so, we first need to discuss our notation and a few definitions.
It is well-known that the kernel of each nonzero multiplicative linear functional on H^∞ is a maximal ideal, and that every maximal ideal can be thought of as the kernel of a nonzero complex multplicative linear functional. For this reason, we call the space of nonzero complex multiplicative linear functionals on H^∞ the maximal ideal space of H^∞ and denote it by $M(H^\infty)$. When endowed with the weak-star topology, $M(H^\infty)$ becomes a compact Hausdorff space. Because H^∞ is a uniform algebra, we may identify a function $f \in H^\infty$ with its Gelfand transform, $\hat{f}$, defined by $\hat{f}(m) = m(f)$ for $m \in M(H^\infty)$. As usual, we identify $\mathbb{D}$ with a subset of $M(H^\infty)$. The Shilov boundary of H^∞ will be denoted by ∂H^∞.

(*) The first author would like to thank the Universität Bern and the Universit de Metz for their support.

For two points x, m in $M(H^\infty)$, we define the pseudohyperbolic distance of x to m by

$$\rho(x,m) = \sup\{|f(m)| : f \in H^\infty, ||f||_\infty \leq 1, f(x) = 0\}.$$

It is well-know that the relation defined on $M(H^\infty)$ by

$$x \sim m \quad \Leftrightarrow \quad \rho(x,m) < 1$$

defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point m is called the Gleason part of m and is denoted by $P(m)$. If the part, $P(m)$, consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. Hoffman's theory [9] shows that for every Gleason part $P(m)$ there is a continuous map L_m of $\mathbb{D}$ onto $P(m)$ with $L_m(0) = m$ such that $f \circ L_m$ is analytic on $\mathbb{D}$ for all $f \in H^\infty$. When the Gleason part of m is trivial, L_m is just a constant map. When $P(m)$ is nontrivial, the map L_m is a bijection. The set of all nontrivial points in $M(H^\infty)$ is denoted by G, and the set of all trivial points is denoted by Γ. Since $f \circ L_m \in H^\infty$, when $f(m) = 0$ it makes sense to talk about the order of the zero of f at m. For $m \in M(H^\infty)$ and $f \in H^\infty$ with $f(m) = 0$ we let

$$\text{ord}(f,m) = \sup\{n \in \mathbb{N} : f = f_1 \cdots f_n, f_j \in H^\infty, f_j(m) = 0 \text{ for } j = 1, 2, \cdots, n\}.$$

If $f(m) \neq 0$, we say $\text{ord}(f,m) = 0$. If I is an ideal in H^∞, we let $\text{ord}(I,m) = \min\{\text{ord}(f,m) : f \in I\}$.

Hoffman showed that $\text{ord}(f,m) = n$ if and only if $f \circ L_m$ has a zero of order n at f. Moreover, if ord $(f,m) = \infty$ for some $m \in M(H^\infty)$, then f vanishes identically on the part $P(m)$ (see [9], p. 79 and 101).
We are now able to define the objects of primary interest in this paper, the higher order zero sets. Let $n \in \mathbb{N} \cup \{\infty\}$ and define

$$E_n(f) = \{m \in M(H^\infty) : \text{ ord } (f,m) \geq n\}.$$

Thus $E_1(f)$ is the zero set of f which we will also denote using the more common notation $Z(f)$. The zero set of f in $\mathbb{D}$ is denoted by $Z_{\mathbb{D}}(f)$. Given an ideal I and an integer n, one can also define the higher order zero sets or hulls of ideals by $E_n(I) = \bigcap_{f \in I} E_n(f)$; that is, we let

$$E_n(I) = \{x \in M(H^\infty) : \text{ord}(f,x) \geq n \text{ for every } f \in I\}.$$

Once again, $E_1(I)$ is the zero set (or hull) of I, which is also denoted by $Z(I)$. Finally, given an ideal I in H^∞, we let

$$I\Big(E_1(I), \cdots, E_N(I)\Big) =$$

$$= \{f \in H^\infty : \text{ord}(f,x) \geq n \text{ for every } x \in E_n(I), n = 1, 2, \cdots, N\}.$$

While it is relatively easy to determine whether or not a function f belongs to the ideal $I(E_1(I), \cdots, E_N(I))$ (just check whether $E_n(I) \subseteq E_n(f)$ for $n = 1, \cdots, N$), it is, in general, very difficult to decide whether or not a function belongs to a general closed ideal. Our main result in this paper paper shows that if I is a closed ideal such that $Z(I) \subseteq G$, then there exists a positive integer N such that

$$I = I(E_1(I), E_2(I), \cdots, E_N(I)).$$

While the assumption that $Z(I) \subseteq G$ may look as though it can be omitted, an example due to Bourgain shows that it is indeed necessary. In fact, Bourgain showed that there exist two Blaschke products B and C such that BC does not belong to the closure of the ideal I generated by B^2 and C^2. Since $E_n(I) \subseteq E_n(BC)$ for every $n \in \mathbb{N}$, one cannot simply generalize our result to arbitrary closed ideals.

In Section 2, we give a detailed study of higher order hulls of ideals in H^∞. Using an idea of Tolokonnikov we show the relationship between the higher order hulls and higher order pseudohyperbolic derivatives. In Section 3, we study ideals in H^∞ generated by finite products of interpolating Blaschke products and present several results about the localization of the zeros of such generators. This section also contains our main theorem, mentioned earlier, that any closed ideal in H^∞ whose hull does not meet the set of trivial points is uniquely determined by its higher order zero sets.

We conclude this paper in Section 4, where we apply these results to finitely generated ideals in H^∞ and extend a result of Bourgain on closures of finitely generated ideals. We also study finite products of ideals.

1. Notation and Preliminaries

We begin by recalling useful definitions and theorems that we will need in this paper. In this paper, one of the most useful kinds of functions are interpolating Blaschke products. Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ and the associated Blaschke product $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z}$ are called interpolating if for every bounded sequence $(w_n)_{n\in\mathbb{N}}$ there exists a function $f \in H^\infty$ such that $f(a_n) = w_n$ for all $n \in \mathbb{N}$. We will also consider finite Blaschke products with distinct zeros as interpolating Blaschke products. By Carleson's theorem on interpolation (see [5], p. 287) we know that (a_n) is interpolating if and only if

$$(C) \qquad \delta(B) := \inf_{k\in\mathbb{N}} \prod_{j:j\neq k} \left| \frac{a_j - a_k}{1 - \overline{a}_j a_k} \right| \geq \delta > 0$$

The constant $\delta(B)$ is called the uniform separating constant of (a_n) or B. The associated interpolation constant K is defined by

$$K = \sup_{||w_n||_\infty \le 1} \inf\{||f||_\infty : f(a_n) = w_n, \text{for all } n \in N, f \in H^\infty\}.$$

A well-known result (see [5], p. 287) shows that $\frac{1}{\delta(B)} \le K \le \frac{c}{\delta}(1 + \log \frac{1}{\delta})$ for some universal constant c. The connection between interpolating sequences and our theorem will become apparent later. A result of Hoffman [9] stating that a point m in the maximal ideal space is nontrivial if and only if m lies in the closure of an interpolating sequence in $\mathbb{D}$ is one key fact that we will use.
A Blaschke product B that equals a finite product of interpolating Blaschke products is also called a Carleson-Newman Blaschke product. Such a Blaschke product is said to have order p if it can be written as a product of p interpolating Blaschke products, but not written as a product of $p-1$ or fewer interpolating Blaschke products. We recall here a useful result due to K. Hoffman about the constants associated with interpolating Blaschke products.

Hoffman's Lemma ([9], p. 86, 106 and [5], p. 404) . *Let $0 < \delta < 1$, $0 < \eta < (1 - \sqrt{1-\delta^2})/\delta$, (that is, $0 < \eta < \rho(\delta, \eta)$) and let*

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

Furthermore, if B is any interpolating Blaschke product with zeros $\{z_n\}$ such that

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2)|B'(z_n)| \ge \delta,$$

then

1. *$Z(B)$ is the closure of the zero set of B in $\mathbb{D}$,*
2. *$\rho(x, y) \ge \delta$ for any $x, y \in Z(B), x \ne y$, and*
3. $$\{m \in M(H^\infty) : |b(m)| < \varepsilon\} \subseteq \{m \in M(H^\infty) : \rho(m, Z(b)) < \eta\} \subseteq$$ $$\subseteq \{m \in M(H^\infty) : |b(m)| < \eta\}.$$

Moreover, the collection of closures of the pseudohyperbolic disks

$$D(m, \eta) = \{x \in M(H^\infty) : \rho(m, x) < \eta\}$$

for $m \in Z(B)$ are pairwise disjoint.

We note also that $(1 - \sqrt{1-\delta^2})/\delta$ is a monotonically increasing function of $\delta \in (0, 1)$, that $\varepsilon < \eta < \delta$ and that $0 < (1 - \sqrt{1-\delta^2})/\delta < \delta$. Moreover, it is easy to see that $\eta < 2\eta/(1 + \eta^2) < \delta$ is equivalent to $0 < \eta < \rho(\delta, \eta)$.

With the exception of (2), these results are stated in Hoffman's paper. Although (2) is not explicitly stated, one can prove it as follows.

Let $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ be disjoint subsets of the zeros of B in $\mathbb{D}$ such that $x \in \overline{\{u_n\}}$ and $y \in \overline{\{v_n\}}$. Let B_1 denote the subproduct of B associated with $\{u_n\}$, and let B_2 denote the subproduct of B associated with $\{v_n\}$. Then, for each n, we see that

$$|B_2(u_n)| \geq \prod_{k:z_k \neq u_n} \rho(z_k, u_n) \geq \delta.$$

Since B_2 is continuous on the maximal ideal space, $|B_2(x)| \geq \delta$ and $B_2(y) = 0$. Thus

$$\rho(x, y) = \sup\{|f(x)| : f \in H^\infty, ||f||_\infty \leq 1, f(y) = 0\} \geq |B_2(x)| \geq \delta.$$

◇

As in [17], we will also consider the pseudohyperbolic dervatives $D^j f$ of f defined by

$$D^j f(z) = \frac{d^j}{d\zeta^j} f\left(\frac{z+\zeta}{1+\overline{z}\zeta}\right) |_{\zeta=0}$$

for any positive integer j. Thus $D^1 f(z) = (1 - |z|^2) f'(z)$. We let $D^0(f) = f$. Using the Cauchy integral formula for functions in the unit ball of H^∞, it is easy to see that $|D^j f| \leq j!$ for all $j \in \mathbb{N}$. Moreover, we have ([9], p. 94)

$$D^j f(z) = \sum_{k=1}^{j} a_k \overline{z}^{j-k} (1 - |z|^2)^k f^{(k)}(z)$$

for some constants a_k. By Hoffman's theory (see [9], p. 93), for each j the functions $D^j f$ extend continuously to the whole spectrum $M(H^\infty)$ of H^∞. These extensions will also be denoted $D^j f$.

2. Pseudohyperbolic derivatives and higher order zero sets

In this section we give relations between pseudohyperbolic derivatives and higher order zero sets. First we note that for every $n \in \mathbb{N} \cup \{\infty\}$ the sets $E_n(f)$ are closed, or, which is equivalent, the function ord $(f, \cdot)$ is upper semicontinuous (see [6], p. 152). To understand the following, it is useful to remember that a Carleson-Newman Blaschke product B has order N if and only if $E_p(B) = \emptyset$ for $p > N$, but $E_N(B) \neq \emptyset$.

Using pseudohyperbolic derivatives, we obtain the following characterization of higher order zero sets.

Lemma 1.1 *Let $f \in H^\infty$. Then for $n \in \mathbb{N} \cup \{\infty\}$ we have*

$$E_n(f) = \{m \in M(H^\infty) : \sum_{k=0}^{n-1} \frac{1}{k!^2}|D^k f(m)| = 0\}.$$

Proof. Let k be a fixed positive integer. First we show that $D^k f(m) = (f \circ L_m)^{(k)}(0)$. In fact, let $L_z(\xi) = \dfrac{\xi + z}{1 + \bar{z}\xi}$ and let (z_α) be a net in $\mathbb{D}$ converging to m. Then by [9] $(f \circ L_{z_\alpha})^{(k)}$ converges locally uniformly on $\mathbb{D}$ to $(f \circ L_m)^{(k)}$ for every $k \in \mathbb{N} \cup \{0\}$. Evaluating at the origin, we obtain $D^k f(z_\alpha) \to (f \circ L_m)^{(k)}(0)$. The continuity of the pseudohyperbolic derivatives on $M(H^\infty)$ now yields that $D^k f(m) = (f \circ L_m)^{(k)}(0)$.

If $m \in \Gamma$, then $f \circ L_m$ is constant. Hence $(f \circ L_m)^{(k)} \equiv 0$ for $k \in \mathbb{N}$. Therefore $D^k f(m) = 0$ for every $k \in \mathbb{N}$. Now let $m \in E_n(f)$ with $n = \infty$. Then by ([5],p. 403), f vanishes identically on the part $P(m)$. Hence $f \circ L_m \equiv 0$. Thus by the previous argument we may conclude that $D^k f(m) = 0$ for every $k = 0, 1, 2, \cdots$.

Now suppose ord $(f, m) = n$ with n finite. Then by Hoffman ([9], p. 100) there exist n interpolating Blaschke products b_j with $b_j(m) = 0$ and a function $g \in H^\infty$ such that $g(m) \neq 0$ and $f = b_1 \cdots b_n g$. Note that ord $(b, m) = 1$ for every interpolating Blaschke product b with $b(m) = 0$. Because $b_j \circ L_m(0) = 0$ and $(g \circ L_m)(0) \neq 0$, we see that $D^k f(m) = (f \circ L_m)^{(k)}(0) = 0$ for every $0 \leq k \leq n-1$, but $D^n f(m) \neq 0$. Thus $E_n(f) \subseteq \{m \in M(H^\infty) : \sum_{k=0}^{n-1} \frac{1}{k!^2}|D^k f(m)| = 0\}$.

Conversely, let $D^k f(m) = 0$ for $0 \leq k \leq n - 1$. Assume that $p =$ ord $(f, m) < n$. From the lines above we see that $D^p f(m) \neq 0$, a contradiction. Thus ord $(f, m) \geq n$. This shows that $\{m \in M(H^\infty) : \sum_{k=0}^{n-1} \frac{1}{k!^2}|D^k(f)(m)| = 0\} \subseteq E_n(f)$, which gives the conclusion of the lemma. ○

Remark. The factor $\frac{1}{k!^2}$ has been added in order to guarantee the convergence of the infinite sum $\sum_{k=0}^{\infty} \frac{1}{k!^2}|D^k f(m)|$. This function plays an important role in studying analytic structure in $M(H^\infty)$.

Proposition 1.2 ([17]) *Let $f \in H^\infty$ and $n \in \mathbb{N}$. Then $\sum_{k=0}^{n} |D^k f|$ is bounded away from zero on $\mathbb{D}$ if and only if $f = BF$, where B is a Carleson-Newman Blaschke product of order $p \leq n$ and F is invertible in H^∞.*

Proof. Assume that $\sum_{k=0}^{n} |D^k f| \geq \delta > 0$ on $\mathbb{D}$. By Hoffman's theory and the Corona Theorem, this holds on $M(H^\infty)$, too. Since the derivatives $D^j f$ vanish identically for $j = 1, 2, \cdots$ on the set of trivial points (see the proof of Lemma 1.1 or compare with [1]), our hypothesis implies that $|f| \geq \delta > 0$ on Γ. By [8] $f = BF$ for some Carleson-Newman Blaschke product B and some F invertible in H^∞. Assume that for some m we have ord $(B, m) \geq n + 1$. Then ord $(f, m) \geq n + 1$ and by Lemma 1.1 we deduce that $D^j f(m) = 0$ for all $j = 0, 1, \cdots, n$. This is a contradiction.

To prove the converse, let $f = BF$ for some Carleson-Newman Blaschke product B of order $p \leq n$ and some F invertible in H^∞. If $\sum_{k=0}^{n} |D^k f|$ were not bounded away from zero on $\mathbb{D}$, then there would exist an interpolating sequence (z_n) in $\mathbb{D}$ such that

$$\sum_{k=0}^{n} |D^k f(z_n)| \longrightarrow 0 \quad \text{as} \quad n \text{ tends to } \quad \infty.$$

Hence for every $j \in \{0, 1, \cdots, n\}$ and every cluster point m of $\{z_n : n \in \mathbb{N}\}$ we have $D^j f(m) = 0$. But this implies by Lemma 1.1 that ord $(f, m) \geq n+1$. Since $F(m) \neq 0$, we have ord $(B, m) =$ ord $(f, m) \geq n+1$. This is a contradiction. ○

Lemma 1.3 *If $(f_n)_{n\in\mathbb{N}}$ is a sequence in H^∞ converging uniformly on $\mathbb{D}$ to some $f \in H^\infty$, then $(D^j f_n)_{n\in\mathbb{N}}$ converges on $M(H^\infty)$ uniformly to $D^j f$ for every $j \in \mathbb{N}$.*

Proof. It suffices to show that if $f_n \in H^\infty$ tends uniformly to zero, then $D^j f_n$ tends uniformly to zero. Note that

$$D^j f(z) = \sum_{k=1}^{j} a_k \overline{z}^{j-k} (1 - |z|^2)^k f^{(k)}(z)$$

for some constants a_k. Hence it is enough to show that $(1 - |z|^2)^k f_n^{(k)}(z)$ tends uniformly to zero on $\mathbb{D}$ as $n \to \infty$. But for $0 < r < 1$ and $\|f_n\|_\infty \leq \varepsilon$, we have

$$\frac{1}{k!} r^k (1 - |z|^2)^k |f_n^{(k)}(rz)| = \left| \frac{(1 - |z|^2)^k}{2\pi i} \int_{|\eta|=1} \frac{f_n(\eta r)}{(\eta - z)^{k+1}}\, d\eta \right| \leq$$

$$\leq \frac{\varepsilon}{2\pi} \int_{|\eta|=1} \frac{(1 - |z|^2)^k}{|\eta - z|^{k+1}} |d\eta| \leq \frac{\varepsilon}{2\pi} \int_0^{2\pi} \frac{(1 - |z|^2)}{|e^{i\theta} - z|^2} \frac{(1 - |z|^2)^{k-1}}{(1 - |z|)^{k-1}}\, d\theta \leq$$

$$\leq 2^{k-1} \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, d\theta = 2^{k-1} \varepsilon.$$

This yields the assertion. ○

Proposition 1.4 *Let U_j be open sets in $M(H^\infty)$. Suppose that for some interpolating Blaschke products b_k the Blaschke product $B = \prod_{k=1}^n b_k$ satisfies*

$$E_j(B) \subseteq U_j \quad (j = 1, \cdots, n). \tag{2}$$

Then there exists $\lambda_0 > 0$ such that for every $f \in H^\infty$, $\|f\|_\infty \leq 1$ and $0 < \lambda \leq \lambda_0$ the functions $f_\lambda = B + \lambda f$ have the property that

$$E_j(f_\lambda) \subseteq U_j \quad (j = 1, \cdots, n). \tag{3}$$

Moreover, the inner factor of f_λ is a Carleson-Newman Blaschke product of order $p \leq n$ and the outer factor of f_λ is invertible.

Proof. Let j be a positive integer and let D^j be the j-th pseudohyperbolic derivative of $f \in H^\infty$. Because $|D^j f| \leq j!$ on $M(H^\infty)$ whenever $\|f\|_\infty \leq 1$, we obtain from

$$\|D^j f_\lambda - D^j B\|_\infty = \lambda \|D^j f\|_\infty \leq \lambda j!$$

that $D^j f_\lambda$ converges uniformly to $D^j B$ on $M(H^\infty)$ as $\lambda \to 0$. The continuity of the $D^j f$ and the fact that

$$E_j(f_\lambda) = \{m \in M(H^\infty) : \sum_{k=0}^{j-1} |D^k f_\lambda(m)| = 0\} \quad (j = 1, 2, \cdots)$$

(see Lemma 1.1) yields assertion (3).

To prove the remaining assertions, we note that B is a Carleson-Newman Blaschke product of order less than or equal to n. Hence Proposition 1.2 yields that $\sum_{k=0}^{n} |D^k B| \geq \delta$ on $\mathbb{D}$. By the uniform convergence of the $D^k f_\lambda$, we see that there exists a positive λ_0 such that

$$\sum_{k=0}^{n} |D^k f_\lambda| \geq \frac{\delta}{2}$$

on $\mathbb{D}$ for $0 < \lambda \leq \lambda_0$. Note that λ_0 is independent of f. By Proposition 1.2 we obtain that the inner factors of f_λ $(0 < \lambda \leq \lambda_0)$ are Carleson-Newman Blaschke products of order less or equal to n and that the outer factors are invertible. ○

2 Generators of ideals in H^∞

For the sake of better reference, we begin with three easy Lemmas.

Lemma 2.0 *Let I be an ideal in H^∞ and suppose that $f = gh \in I$ for some $g, h \in H^\infty$ with $Z(g) \cap Z(I) = \emptyset$. Then $h \in I$.*

Proof. Because g and I are not contained in any common maximal ideal, the ideal generated by them coincides with H^∞. Hence $1 = gk + q$ for some $k \in H^\infty$ and $q \in I$. Thus $h = (gh)k + hq \in I$. ○

Lemma 2.1 *Let I be an ideal in H^∞. Then the function $\omega : M(H^\infty) \to \mathbb{N}_0 \cup \{\infty\}$, defined by $\omega(x) = \text{ord}\,(I, x)$, is upper semicontinuous.*

Proof. Using the upper semicontinuity of the function $x \to \text{ord}\,(f, x)$, (see [6],

p. 152), and the fact that for any real number α the equality

$$\{x : \text{ord}\,(I,x) \geq \alpha\} = \bigcap_{f\in I} \{x : \text{ord}\,(f,x) \geq \alpha\}$$

holds, we immediately obtain that these sets are closed, which yields the assertion. ◯

Lemma 2.2 *If I is an ideal in H^∞ satisfying $Z(I) \subseteq G$, then*

$$N = \sup_x \text{ord}\,(I,x) < \infty.$$

Proof. Assuming the contrary, there exists a sequence (x_n) in G such that $\text{ord}\,(I,x_n) \to \infty$. Let x be a cluster point of the x_n. Then $x \in Z(I)$. By Lemma 2.1, the upper semicontinuity of $\text{ord}\,(I,x)$ yields that $\text{ord}\,(I,x) = \infty$. Thus, either $x \in \Gamma$, or every function $f \in I$ vanishes identically on the closure of the part $P(x)$ (see [5], p. 403). By Budde ([3], p. 370), $\overline{P(x)}$ contains a trivial point m. Hence in both cases, x or m is in $\Gamma \cap Z(I)$, a contradiction. ◯

Lemma 2.3 *Let $S \subseteq \mathbb{D}$ and let $0 < \eta < 1$. Then*

$$\overline{\{m \in M(H^\infty) : \rho(m,S) \leq \eta\}} = \{m \in M(H^\infty) : \rho(m,\overline{S}) \leq \eta\},$$

where the closure is taken in $M(H^\infty)$.

Proof. Since the pseudohyperbolic distance is lower semicontinuous, we see that $\{m \in M(H^\infty) : \rho(m,\overline{S}) \leq \eta\}$ is closed. Hence

$$\overline{\{m \in M(H^\infty) : \rho(m,S) \leq \eta\}} \subseteq \{m \in M(H^\infty) : \rho(m,\overline{S}) \leq \eta\}.$$

Let $x \in \{m \in M(H^\infty) : \rho(m,\overline{S}) \leq \eta\}$. Choose $s \in \overline{S}$ so that $\rho(x,s) \leq \eta$. Let $x = L_s(\zeta)$ for some $\zeta \in \mathbb{D}$, where L_s is the Hoffman map associated with s. If $s_\alpha \in S$ converges to s, then by Hoffman ([9], p. 92) $L_{s_\alpha}(\zeta)$ converges to $L_s(\zeta) = x$. Let $z_\alpha = L_{s_\alpha}(\zeta)$. Then by ([9], p. 105)

$$\rho(z_\alpha, s_\alpha) = \rho\left(L_{s_\alpha}(\zeta), L_{s_\alpha}(0)\right) = \rho(\zeta,0) = \rho\left(L_s(\zeta), L_s(0)\right) = \rho(x,s) \leq \eta.$$

Therefore $\rho(z_\alpha, S) \leq \eta$ and hence $x \in \overline{\{z \in \mathbb{D} : \rho(z,S) \leq \eta\}}$. Thus

$$\{m \in M(H^\infty) : \rho(m,\overline{S}) \leq \eta\} \subseteq \overline{\{m \in M(H^\infty) : \rho(m,S) \leq \eta\}}.$$

◯

We do not know whether this lemma holds for all $S \subseteq M(H^\infty)$.

For a point $z_0 \in \mathbb{D}$ and $\eta \in (0,1)$ let

$$\overline{D}(z_0,\eta) = \{z \in \mathbb{D} : \rho(z,z_0) \leq \eta\}$$

be a closed pseudohyperbolic disk in $\mathbb{D}$ and let $\mathcal{D}$ be a family of closed pseudohyperbolic disks in $\mathbb{D}$. A union of ℓ disks $\overline{D}(z_j,\eta)$, $(j=1,\cdots,\ell)$, is called an *η-chain within $\mathcal{D}$ of length ℓ* if the union $\bigcup_{j=1}^{\ell}\overline{D}(z_j,\eta)$ is a connected subset of $\mathbb{D}$ which is disjoint from all the other disks in $\mathcal{D}$.

We point out that every η-chain $\bigcup_{j=1}^{\ell}\overline{D}(z_j,\eta)$ of length ℓ is contained in the disks $D(z_{j_0},2\ell\eta)$ for every $j_0\in\{1,2,\cdots,\ell\}$.

Finally, the union of all η-chains within $\mathcal{D}$ of length ℓ is denoted by $H_\eta(\ell)$.

To proceed, we need the following technical Lemma.

Lemma 2.4 *Let C be a product of N interpolating Blaschke products C_j with zero sets $\{z_n^j : n\in\mathbb{N}\}$ $(j=1,\cdots,N)$ in $\mathbb{D}$.*
Let $\delta=\min_{1\le j\le N}\delta(C_j)$ and $0<\eta<\frac{1-\sqrt{1-\delta^2}}{\delta}\cdot\frac{1}{6N}$. Put $\eta^{(0)}=\eta$ and let $\eta^{(j)}$ satisfy

$$0<\eta^{(j)}<\eta^{(j-1)}\left(\frac{\delta-\eta^{(j-1)}}{1-\delta\eta^{(j-1)}}\right),\quad (j=1,\cdots,N).$$

Let

$$\mathcal{D}=\{\overline{D}(z_n^j,\nu):n\in\mathbb{N},1\le j\le N,0<\nu<1\}.$$

For $0<\mu\le\eta$ denote by $H_\mu(\ell)$ the union of all μ-chains within $\mathcal{D}$ of length ℓ. (*)
Finally, for $0<\mu'<\mu\frac{\delta-\mu}{1-\delta\mu}$ with $\mu\le\eta$ let

$$S_\ell=\overline{H_\mu(\ell)}\cap\{m\in M(H^\infty):\rho(m,Z(C))\le\mu'\}\ \ (1\le\ell\le N)$$

and

$$T_\ell=H_{\eta^{(\ell)}}(N-\ell+1)\setminus\bigcup_{n=1}^{\ell-1}H_{\eta^{(n)}}(N-n+1).$$

Then, for $\ell\ne k$ and $1\le\ell,k\le N$, the following assertions hold:

(I) $S_\ell\cap S_k=\emptyset$ *(II)* $\overline{T_\ell}\cap\overline{T_k}=\emptyset$.

Proof. Without loss of generality let $\mu=\eta$. Trivially, $H_\eta(\ell)\cap H_\eta(k)=\emptyset$ for $\ell\ne k$. Moreover, it follows from Hoffman's Lemma, the choice of η and Lemma 2.3 that if M_j and M_j' are disjoint subsets of $Z(C_j)\cap\mathbb{D}$, then

$$\overline{\{z\in\mathbb{D}:\rho(z,M_j)\le 2\eta N\}}\bigcap\overline{\{z\in\mathbb{D}:\rho(z,M_j')\le 2\eta N\}}=\emptyset. \qquad \text{(III)}$$

To prove (I), suppose that for some $\ell\ne k$ there exists $x\in S_\ell\cap S_k$. Let ε satisfy $0<\mu'<\varepsilon<\eta\frac{\delta-\eta}{1-\eta\delta}$. Since $x\in\overline{H_\eta(\ell)}$, there exists a fixed ℓ-tuple $(j_1,\cdots,j_\ell)$

(*) If $z_n^j=z_m^{j'}$ for $j\ne j'$, then the union $\overline{D}(z_n^j,\eta)\cup\overline{D}(z_m^{j'},\eta)$ is considered as a 2-chain, whenever it does not meet any other disk of $\mathcal{D}$ with radius η.

with $1 \leq j_1 < j_2 < \cdots < j_\ell \leq N$ such that x lies in the closure of a sequence of η-chains of length ℓ with building blocks

$$\overline{D}(z^{j_1}, \eta), \cdots, \overline{D}(z^{j_\ell}, \eta),$$

where the $z^{j_n} \in Z_{\mathbb{D}}(C_{j_n})$ $(1 \leq n \leq \ell)$. This implies by Hoffman's Lemma, that for $j \notin \{j_1, \cdots, j_\ell\}$ we have $|C_j(x)| \geq \varepsilon$ and hence, $\rho(x, Z(C_j)) \geq \varepsilon$.

Similarly, $x \in \overline{H_\eta(k)}$ implies that there exists a fixed k-tuple $(j_1^*, \cdots, j_k^*)$ with $1 \leq j_1^* < j_2^* < \cdots < j_k^* \leq N$ such that x lies in the closure of a sequence of η-chains of length k with building blocks

$$\overline{D}(z^{j_1^*}, \eta), \cdots, \overline{D}(z^{j_\ell^*}, \eta).$$

Therefore, as above, we have $\rho(x, Z(C_j)) \geq \varepsilon$ if $j \notin \{j_1^*, \cdots, j_k^*\}$.

On the other hand, $\rho(x, Z(C)) \leq \mu'$ implies that there exists $m \in \bigcup_{j=1}^N Z(C_j)$, say $m \in Z(C_{j_0})$, such that $\rho(x, m) \leq \mu' < \varepsilon$. Hence, by the paragraphs above, $j_0 \in \{j_1, \cdots, j_\ell\} \bigcap \{j_1^*, \cdots, j_k^*\}$. Let $M_{j_0} = H_\eta(\ell) \cap Z_{\mathbb{D}}(C_{j_0})$, $M'_{j_0} = H_\eta(k) \cap Z_{\mathbb{D}}(C_{j_0})$. Obviously $M_{j_0} \cap M'_{j_0} = \emptyset$. Since for suitably chosen z^{j_0} and $z^{j_0^*}$ in $Z_{\mathbb{D}}(C_{j_0})$, $z^{j_0} \neq z^{j_0^*}$, we have

$$D(z^{j_1}, \eta) \cup \cdots \cup D(z^{j_\ell}, \eta) \subseteq D(z^{j_0}, 2\eta\ell)$$

and

$$D(z^{j_1^*}, \eta) \cup \cdots \cup D(z^{j_k^*}, \eta) \subseteq D(z^{j_0^*}, 2\eta k),$$

we conclude from Lemma 2.3 that

$$x \in \overline{\{z \in \mathbb{D} : \rho(z, M_{j_0}) \leq 2\eta\ell\}} \bigcap \overline{\{z \in \mathbb{D} : \rho(z, M'_{j_0}) \leq 2\eta k\}}$$

a contradiction to (III). Thus $S_\ell \cap S_k = \emptyset$.

To prove (II), it suffices to show that

$$\overline{T_\ell} \subseteq \{m \in M(H^\infty) : \rho(m, Z(C)) \leq \eta^{(\ell)}\}, \tag{IV}$$

and

$$T_\ell \subseteq \bigcup_{s=N-\ell+1}^{N-k} H_{\eta^{(k)}}(s) \quad (1 \leq k < \ell \leq N). \tag{V}$$

In fact, suppose that this is true. Recall that by the definition of T_k we have

$$T_k \subseteq H_{\eta^{(k)}}(N - k + 1).$$

Let $\mathcal{N}_{\eta^\ell} = \{m \in M(H^\infty) : \rho(m, Z(C)) \leq \eta^\ell\}$.

Applying (IV) and (V) yields:

$$\overline{T_k} \cap \overline{T_\ell} \subseteq \bigcup_{s=N-\ell+1}^{N-k} \left(\left(\overline{H_{\eta^{(k)}}(N-k+1)} \cap \mathcal{N}_{\eta^{(\ell)}} \right) \cap \left(\overline{H_{\eta^{(k)}}(s)} \cap \mathcal{N}_{\eta^{(\ell)}} \right) \right).$$

Since $\eta^{(\ell)} < \eta^{(k)} \dfrac{\delta - \eta^{(k)}}{1-\delta\eta^{(k)}}$ for $k < \ell$ and $s \neq N-k+1$, we conclude from (I) that $\overline{T_k} \cap \overline{T_\ell}$ is contained in the empty set. This establishes (II).

Let us now verify (IV) and (V). To prove (IV), we first note that $T_\ell \subseteq \{z \in \mathbb{D} : \rho(z, Z(C)) \leq \eta^{(\ell)}\}$. Hence, by Lemma 2.3

$$\overline{T_\ell} \subseteq \{m \in M(H^\infty) : \rho(m, Z(C)) \leq \eta^{(\ell)}\}.$$

To prove (V), we choose $x \in T_\ell$. Recall that

$$T_\ell = H_{\eta^{(\ell)}}(N-\ell+1) \setminus \bigcup_{n=1}^{\ell-1} H_{\eta^{(n)}}(N-n+1).$$

Then x is contained in an $\eta^{(\ell)}$-chain of length $N-\ell+1$. Since $\eta^{(k)} > \eta^{(\ell)}$, we obviously have that x is contained in a certain $\eta^{(k)}$-chain of length at least $N-\ell+1$. Assume that x is contained in an $\eta^{(k)}$-chain of length strictly bigger than $N-k$, say $x \in H_{\eta^{(k)}}(N-k+j_0)$ for some j_0 with $1 \leq j_0 \leq k$. Noticing that for any $q \in \mathbb{N}$ with $N-k+j_0 \leq q \leq N$ we have $\eta^{(N-q+1)} \geq \eta^{(k-j_0+1)} \geq \eta^{(k)}$, we obtain that

$$x \in H_{\eta^{(k)}}(N-k+j_0) \overset{(!)}{\subseteq} \bigcup_{q=N-k+j_0}^{N} H_{\eta^{(N-q+1)}}(q) = \bigcup_{n=1}^{k-j_0+1} H_{\eta^{(n)}}(N-n+1),$$

which is a contradiction to $x \in T_\ell$, because $k - j_0 + 1 \leq \ell - 1$. Thus

$$x \in \bigcup_{s=N-\ell+1}^{N-k} H_{\eta^{(k)}}(s) \quad (1 \leq k < \ell \leq N)$$

which yields (V). ○

Proposition 2.5 *Let I be an ideal in H^∞ and let C_j be interpolating Blaschke products. Assume that $C_1C_2\cdots C_N \in I\Big(E_1(I), E_2(I), \cdots, E_N(I)\Big)$. Suppose that for every $\varepsilon > 0$ there exist interpolating Blaschke products B_j, ($j = 1, \cdots, N$), with $B_1B_2\cdots B_N \in I$ such that $Z(B_j) \subseteq \{|C_j| < \varepsilon\}$ for every $j \in \{1, \cdots, N\}$. Then there exist interpolating Blaschke products b_j with*

$Z(b_j) \subseteq \{|C_j| < \varepsilon\}$ *such that* $b_1 b_2 \cdots b_N \in I$ *and the uniform separation constants* $\delta(b_j)$ *satisfy*

$$\delta(b_j) \geq \tfrac{1}{2} \min\{\delta(C_k) : 1 \leq k \leq N\}\ , \quad (j = 1, \cdots, N).$$

Proof. *Step 1.* Let $\delta = \min_{j=1,\cdots,N} \delta(C_j)$ and let $\{z_n^j : n \in \mathbb{N}\}$ denote the zero set of C_j in $\mathbb{D}$. Choose ε and η with $0 < \varepsilon < \eta$ so that

$$\frac{\delta - \frac{2\eta}{1+\eta^2}}{1 - \delta \frac{2\eta}{1+\eta^2}} > \frac{\delta}{2},$$

$$0 < \eta < \frac{1 - \sqrt{1-\delta^2}}{\delta} \cdot \frac{1}{6N}$$

and

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

Let $\varepsilon^{(0)} = \varepsilon, \eta^{(0)} = \eta$. Using induction on j, $(j = 1, \cdots, N)$, our assumptions allow us to choose interpolating Blaschke products B_k^j with zero sets $\{w_{n,k}^j : n \in \mathbb{N}\}$ in $\mathbb{D}$, such that $B_j = B_1^j B_2^j \cdots B_N^j \in I$ and

$$Z(B_k^j) \subseteq \{|C_k| < \varepsilon^{(j-1)}\}, \tag{1}$$

where

$$0 < \eta^{(j)} \leq \frac{1}{6N} \min \left\{ \varepsilon^{(j-1)}, \min_{1 \leq k \leq N} \inf_{\substack{n,m \\ n \neq m}} \rho(w_{n,k}^j, w_{m,k}^j) \right\} \tag{2}$$

and $\varepsilon^{(j)}$ is chosen so that

$$0 < \varepsilon^{(j)} < \eta^{(j)} \frac{\delta - \eta^{(j)}}{1 - \delta\eta^{(j)}}.$$

By Hoffman's Lemma

$$\bigcup_{n=1}^{\infty} D(z_n^k, \eta^{(j)}) \subseteq \{|C_k| < \varepsilon^{(j-1)}\} \subseteq \bigcup_{n=1}^{\infty} D(z_n^k, \eta^{(j-1)}) \tag{3}$$

and for $n \neq m$, the disks $D(z_n^k, \eta^{(j)})$ and $D(z_m^k, \eta^{(j)})$ are contained in different components of the set $\{|C_k| < \varepsilon^{(j-1)}\}$ and hence of $\{|C_k| < \varepsilon\}$.

Step 2. For $\mu \in (0,1)$ let $\mathcal{C}_\mu$ denote the set of all closed pseudohyperbolic disks $\overline{D}(z_n^k, \mu)$, $1 \leq k \leq N, n \in \mathbb{N}$. Fix $1 \leq j \leq N$ and let $\ell := N - j + 1$. Consider those $\eta^{(j)}$-chains of length ℓ in $\mathcal{C}_{\eta^{(j)}}$ that are not contained in $\bigcup_{k=1}^{j-1} H_{\eta^{(k)}}(N-k+1)$ (for $j = 1$ this is to be interpreted as the empty set). Let us call the union of

these $\eta^{(j)}$-chains the j-th generation. We point out that by (0) the chains of the j-th generation are contained in $H_{\eta^{(j-1)}}(N-j+1)$. Moreover, each of these chains of length ℓ is contained in a hyperbolic disk of radius $2N\eta^{(j)}$, with centers chosen from the zero set of $\prod_{k=1}^{N} C_k$. Hence, by (2), each function B_k^j has at most one zero in each of the chains of the j-the generation. Using (1), (2) and (3) we conclude that in each of these chains B_j has at most $\ell = N-j+1$ zeros.
Running through all the chains of the j-th generation and collecting the zeros of the interpolating Blaschke products B_k^j, we obtain factors b_k^j of B_k^j with the property that each component in $\mathbb{D}$ of the level set $\{|C_k|<\varepsilon\}$ can have at most one zero of b_k^j (see (1), (3) and the two lines following (3)).

Let $f_j = b_1^j b_2^j \cdots b_N^j$ and let

$$T_j = H_{\eta^{(j)}}(N-j+1) \setminus \bigcup_{k=1}^{j-1} H_{\eta^{(k)}}(N-k+1).$$

By the above construction $Z(f_j) \subseteq \overline{T_j}$.

Step 3. We claim that $Z(I) \subseteq \bigcup_{j=1}^{N} \overline{T}_j$. Since $Z(I) \subseteq \bigcup_{k=1}^{N} Z(C_k)$ and $\overline{A} \cup \overline{B} = \overline{A \cup B}$, it suffices to show that

$$\bigcup_{k=1}^{N} Z_{\mathbb{D}}(C_k) \subseteq \bigcup_{j=1}^{N} H_{\eta^{(j)}}(N-j+1).$$

Let $x \in \bigcup_{k=1}^{N} Z_{\mathbb{D}}(C_k)$. Then for every $n \in \{1,\cdots,N\}$ there exists a unique $p = p(n) \in \{1,\cdots,N\}$ such that $x \in H_{\eta^{(n)}}(p)$. Obviously $\varphi : n \mapsto N - p(n) + 1$ is a monotonically increasing map of $\{1,\cdots,N\}$ into $\{1,\cdots,N\}$. Thus φ has a fixed point j_0; that is $p(j_0) = N - j_0 + 1$. Thus $x \in H_{\eta^{(j_0)}}(N-j_0+1)$, which establishes the inclusion above and finishes the proof of Step 3.

Step 4. We claim that $b_k = b_k^1 b_k^2 \cdots b_k^N$ is an interpolating Blaschke product with $\delta(b_k) \geq \frac{\delta}{2}$. To see this, we recall that by Lemma 2.4 the closures $\overline{T_j}$ of the sets T_j are pairwise disjoint. Since for fixed k we have that

$$Z(b_k^j) \subseteq Z(f_j) \subseteq \overline{T}_j,$$

the zero sets of the b_k^j, $(j = 1,\cdots,N)$ are pairwise disjoint. Since b_k^j is an interpolating Blaschke product, we see ([14], p. 69) that b_k is an interpolating Blaschke product.

To prove the assertion that $\delta(b_k) \geq \frac{\delta}{2}$, we note that for $1 \leq j \leq N$

$$Z(b_k^j) \subseteq Z(B_k^j) \subseteq \{|C_k| < \varepsilon^{(j-1)}\} \subseteq \{|C_k| < \varepsilon\}.$$

Since $Z(b_k^j) \subseteq \overline{T_j}$, the disjointness of the $\overline{T_j}$ and the fact established in Step 1 that each component in $\mathbb{D}$ of $\{|C_k| < \varepsilon\}$ can have at most one zero of b_k^j, shows that in each component in $\mathbb{D}$ of the level set $\{|C_k| < \varepsilon\}$ there is at most one zero of the product $b_k = \prod_{j=1}^N b_k^j$. In particular, $\rho(w, Z(C_k)) < \eta$ for every zero $w \in Z_{\mathbb{D}}(b_k)$.
Hence by ([5], p. 310)

$$\delta(b_k) \geq \frac{\delta(C_k) - \frac{2\eta}{1+\eta^2}}{1 - \delta(C_k)\frac{2\eta}{1+\eta^2}} \geq \frac{1}{2}\delta.$$

Step 5. We claim that $f_1 f_2 \cdots f_N \in I$. Fix j. Let $g_j \in H^\infty$ be chosen so that $B_j = f_j g_j$. By construction

$$Z(g_j) \cap T_j = \emptyset. \tag{4}$$

In addition, we will show that

$$Z(g_j) \cap \overline{T_j} \cap Z(I) = \emptyset.$$

This can be seen in the following way. Assume that there exists $x \in Z(g_j) \bigcap \overline{T_j} \bigcap Z(I)$.
Note that $Z(I) \subseteq \bigcup_{k=1}^N Z(C_k)$.
Let C_k^* be the subproduct of C_k satisfying

$$Z_{\mathbb{D}}(C_k^*) = Z(C_k) \cap T_j^0 \quad (k = 1, \cdots, N),$$

where T_j^0 is the interior of T_j.
Write $C_k = C_k^* C_k^{**}$. Noticing that any closed pseudohyperbolic disk of radius $\eta^{(j)}$ and center from $Z_{\mathbb{D}}(C_k)$ either is disjoint from T_j or is entirely contained in T_j, we obtain from Hoffman's Lemma that C_k^{**} is bounded away from zero on T_j. Thus $x \in \overline{T_j}$ implies that $C_k^{**}(x) \neq 0$. Hence

$$x \in \bigcup_{k=1}^N Z(C_k^*) = \overline{Z_{\mathbb{D}}(\prod_{k=1}^N C_k^*)}.$$

Since $x \in \overline{Z_{\mathbb{D}}(g_j)}$, we obtain by ([9], p. 103) that

$$\rho\left(Z_{\mathbb{D}}(g_j), Z_{\mathbb{D}}(\prod_{k=1}^N C_k^*)\right) = 0;$$

that is there exists a point $w_0 \in Z_{\mathbb{D}}(g_j)$ which is very close to one of the j centers—which come from the zeros of $C_1^* \cdots C_N^*$—of the building blocks of T_j. Thus $w_0 \in T_j$. But this is a contradiction to (4).

We conclude that $Z(g_j)\bigcap \overline{T}_j \cap Z(I) = \emptyset$. By Step 3 $Z(I) \subseteq \bigcup_{j=1}^N \overline{T}_j$, so that $\bigcap_{j=1}^N Z(g_j) \cap Z(I) = \emptyset$. Thus there exist functions $h_j \in H^\infty$ and some $f \in I$, so that $1 = f + \sum_{j=1}^N h_j g_j$. Multiplying by the product of the f_k yields:

$$f_1 f_2 \cdots f_N = f(f_1 \cdots f_N) + \sum_{j=1}^N h_j \left(\prod_{k \neq j} f_k \right) B_j \in I.$$

Step 6. We obviously have

$$f_1 \cdots f_N = (b_1^1 b_2^1 \cdots b_N^1) \cdots (b_1^N b_2^N \cdots b_N^N) = (b_1^1 \cdots b_1^N) \cdots (b_N^1 \cdots b_N^N) = b_1 \cdots b_N.$$

Therefore we have found interpolating Blaschke products b_k with uniform separarting constant bigger than $\frac{\delta}{2}$, satisfying $Z(b_k) \subseteq \{|C_k| < \varepsilon\}$ and such that their product is an element of the ideal I. ○

In the sequel, let δ be a number with $0 < \delta < 1$. Then for every interpolating sequence with uniform separating constant δ we let $K = K(\delta)$ denote the interpolation constant associated with δ. Note that K is asymptotically equal to $\frac{1}{\delta} \log \delta$ if $\delta \to 0$. The following distance estimate is one of the key tools used in the proof of Theorem 2.12.

Proposition 2.6 *Let B_j, C_j, $(j = 1, \cdots, N)$ be interpolating Blaschke products. Let ε and δ satisfy $0 < \varepsilon < 1$ and*

$$0 < \delta \leq \min\{\delta(B_j) : 1 \leq j \leq N\}.$$

Finally let K be the interpolation constant associated with δ. Assume that

$$Z(B_j) \subseteq \{|C_j| < \varepsilon\} \quad (j = 1, \cdots, N).$$

Then

$$\operatorname{dist}(C_1 \cdots C_N, B_1 \cdots B_N H^\infty) \leq M_N (K+1)^N \varepsilon, \tag{5}$$

where M_N is a constant depending only on N.

Proof. We shall prove (5) inductively with $M_N = 2^N + 2^{N-1} - 1$. Let $\{z_n : n \in \mathbb{N}\}$ be the zero sequence of B_1 in $\mathbb{D}$. By Carleson's interpolation theorem (see [5], p. 287) there exists $f_1 \in H^\infty$ with $f_1(z_n) = C_1(z_n)$ for every $n \in \mathbb{N}$ such that

$$\|f_1\|_\infty \leq \sup_n |C_1(z_n)| \cdot K.$$

Hence $f_1 = C_1 + h_1 B_1$ for some $h_1 \in H^\infty$ with $\|C_1 + h_1 B_1\|_\infty \leq \varepsilon K$. On the Shilov boundary of H^∞ we obtain

$$|h_1| = |h_1 B_1| \leq \varepsilon K + 1 \leq K + 1.$$

Therefore $\|h_1\|_\infty \leq K+1$. Again by Carleson's interpolation theorem there exists $f_2 \in H^\infty$ with $f_2 = C_2 h_1$ on $Z_{\mathbb{D}}(B_2)$ such that $\|f_2\| \leq K \sup_{Z_{\mathbb{D}}(B_2)} |C_2 h_1|$. Hence $f_2 = C_2 h_1 - B_2 h_2$ for some $h_2 \in H^\infty$ and so

$$\|C_2 h_1 - h_2 B_2\|_\infty \leq \sup_{Z_{\mathbb{D}}(B_2)} |C_2 h_1| \cdot K \leq \varepsilon \|h_1\|_\infty K$$

$$\leq \varepsilon (K+1) K \leq \varepsilon (K+1)^2.$$

Hence

$$|C_1 C_2 + h_2 B_1 B_2| \leq |(C_2 C_1 + C_2 h_1 B_1) - (C_2 h_1 B_1 - h_2 B_2 B_1)| \leq$$

$$\leq |C_2||C_1 + h_1 B_1| + |B_1||C_2 h_1 - h_2 B_2| \leq \varepsilon K + \varepsilon (K+1)^2 \leq M_2 (K+1)^2 \varepsilon,$$

where M_2 is chosen to be $5 = 2^2 + 2 - 1$. Then

$$\operatorname{dist}(C_1 C_2, B_1 B_2 H^\infty) \leq M_2 (K+1)^2 \varepsilon.$$

Moreover

$$\|h_2\|_\infty \leq 2\varepsilon (K+1)^2 + 1 \leq 3(K+1)^2.$$

Now we proceed from $N-1$ to N. Assume that

$$\|C_1 C_2 \cdots C_{N-1} + h_{N-1} B_1 B_2 \cdots B_{N-1}\|_\infty \leq M_{N-1} (K+1)^{N-1} \varepsilon$$

for some h_{N-1} with

$$\|h_{N-1}\|_\infty \leq (2^{N-2} + 2^{N-1})(K+1)^{N-1} \quad (N \geq 2).$$

Once again, since the zero sequence of B_N is an interpolating sequence, there exists $f_N \in H^\infty$ with $f_N = C_N h_{N-1}$ on $Z(B_N)$ and $\|f_n\| \leq K \sup_{Z_{\mathbb{D}}(B_N)} |C_N h_{N-1}|$. Hence $f_N = C_N h_{N-1} - B_N h_N$ for some $h_N \in H^\infty$ and so

$$\|C_N h_{N-1} - h_N B_N\|_\infty \leq \sup_{Z_{\mathbb{D}}(B_N)} |C_N h_{N-1}| \cdot K \leq$$

$$\leq \varepsilon (2^{N-2} + 2^{N-1})(K+1)^{N-1} K \leq (2^{N-2} + 2^{N-1})(K+1)^N \varepsilon.$$

Hence

$$|C_1 C_2 \cdots C_{N-1} C_N + h_N B_1 B_2 \cdots B_{N-1} B_N| \leq$$

$$\leq |C_N (C_1 \cdots C_{N-1} + h_{N-1} B_1 \cdots B_{N-1})| + |B_1 \cdots B_{N-1} (h_N B_N - h_{N-1} C_N)|$$

$$\leq M_{N-1} (K+1)^{N-1} \varepsilon + (2^{N-2} + 2^{N-1})(K+1)^N \varepsilon \leq$$

$$\leq \big((2^{N-1} + 2^{N-2} - 1) + 2^{N-2} + 2^{N-1}\big)(K+1)^N \varepsilon =$$

$$= (2^N + 2^{N-1} - 1)(K+1)^N \varepsilon = M_N (K+1)^N \varepsilon.$$

Hence

$$\mathrm{dist}(C_1 \cdots C_N, B_1 \cdots B_N H^\infty) \le M_N (K+1)^N \varepsilon$$

and

$$\|h_N\|_\infty \le M_N (K+1)^N \varepsilon + 1 \le (M_N + 1)(K+1)^N = (2^N + 2^{N-1})(K+1)^N.$$

$\bigcirc$

Lemma 2.7 *Let B be a Carleson-Newman Blaschke product and let F and F^* be two disjoint closed subsets of $Z(B)$. Then there exist a factorization $B = CD$ of B such that $Z(C) \cap Z(D) = \emptyset$, $D \neq 0$ on F and $C \neq 0$ on F^*.*

Proof. Choose open neighborhoods U_0 of F and V_0 of F^* such that $\overline{U_0} \cap \overline{V_0} = \emptyset$. Let $U = M(H^\infty) \setminus \overline{V_0}$.
Write $B = B_1 \cdots B_p$ for some interpolating Blaschke products B_j. The factors C and D of B will be constructed inductively.

i) Let B_1^* be the interpolating Blaschke product formed with the zeros of B_1 in $U_0 \cap \mathbb{D}$ and let $B_1^{**} = B_1 / B_1^*$. Then $F \cap Z(B_1) \subseteq Z(B_1^*) \subseteq \overline{U_0}$ and $Z(B_1^{**}) \subseteq U_0^c$, the complement of U_0 in $M(H^\infty)$. Since $Z(B_1^*) \cap Z(B_1^{**}) = \emptyset$, there exists open sets $\mathcal{O}_1, U_1$ and V_1 such that

$$\overline{\mathcal{O}_1} \subseteq U_1, \quad \overline{U_1} \cap \overline{V_1} = \emptyset, \quad \overline{U_1} \subseteq U$$

and

$$Z(B_1^*) \subseteq \mathcal{O}_1, \qquad Z(B_1^{**}) \subseteq V_1.$$

ii) Let B_2^* be the interpolating Blaschke product formed with the zeros of B_2 in U_1 and let $B_2^{**} = B_2 / B_2^*$. Then $F \cap Z(B_2) \subseteq Z(B_2^*) \subseteq \overline{U_1}$ and $Z(B_2^{**}) \subseteq U_1^c$. Since $\overline{U_1} \cap \overline{V_1} = \emptyset$, we obtain that $Z(B_2^*) \cap Z(B_1^{**}) = \emptyset$. Moreover, $\overline{\mathcal{O}_1} \subseteq U_1$ implies that $Z(B_1^*) \cap Z(B_2^{**}) = \emptyset$. Since $Z(B_j^*) \cap Z(B_j^{**}) = \emptyset$ anyway, we get that $Z(B_1^* B_2^*) \cap Z(B_1^{**} B_2^{**}) = \emptyset$. Now choose open sets $\mathcal{O}_2, U_2$ and V_2 such that

$$\overline{\mathcal{O}_2} \subseteq U_2,\ \overline{U_2} \cap \overline{V_2} = \emptyset,\ \overline{U_2} \subseteq U$$

and

$$Z(B_1^* B_2^*) \subseteq \mathcal{O}_2, \qquad Z(B_1^{**} B_2^{**}) \subseteq V_2.$$

iii) Let B_3^* be the interpolating Blaschke product formed with the zeros of B_3 in U_2 and let $B_3^{**} = B_3 / B_3^*$. Then $Z(B_3^*) \subseteq \overline{U_2}$ and $Z(B_3^{**}) \subseteq U_2^c$. As above we conclude that $Z(B_3^*) \cap Z(B_1^{**} B_2^{**}) = \emptyset$ and $Z(B_1^* B_2^*) \cap Z(B_3^{**}) = \emptyset$. Hence

$$Z(B_1^* B_2^* B_3^*) \cap Z(B_1^{**} B_2^{**} B_3^{**}) = \emptyset.$$

Moreover $F \cap Z(B_3) \subseteq Z(B_3^*)$.

Continuing this way yields a factorization $B = (B_1^* \cdots B_p^*)(B_1^{**} \cdots B_p^{**})$ of B such that

$$Z(B_1^* \cdots B_p^*) \cap Z(B_1^{**} \cdots B_p^{**}) = \emptyset$$

and $Z(B_1^* \cdots B_p^*) \subseteq U$. Moreover $F \cap Z(B_j) \subseteq Z(B_j^*)$ for $1 \le j \le p$.

Let $C = B_1^* \cdots B_p^*$, $D = B_1^{**} \cdots B_p^{**}$. Then $B = CD$, where C is a Carleson-Newman Blaschke product of order less than or equal to p with $Z(C) \cap Z(D) = \emptyset$. We claim that $D \neq 0$ on F. Let $x \in F$ and let $q = \text{ord } (B, x)$. Since ord $(b, x) = 1$ for every interpolating Blaschke product b with $b(x) = 0$, there exist $j_k \in \{1, \cdots, p\}$, $k = 1, \cdots, q$, such that

$$B_{j_1}(x) = \cdots = B_{j_q}(x) = 0, \quad B_j(x) \neq 0 \text{ for } j \notin \{j_1, \cdots, j_q\}.$$

The construction yields that $B_{j_1}^*(x) = \cdots = B_{j_q}^*(x) = 0$. Hence ord $(C, x) = q = $ ord (B, x), and so $D(x) \neq 0$. Thus $D \neq 0$ on F.
Since $Z(C) \subseteq U = M(H^\infty) \setminus \overline{V_0}$ and $F^* \subseteq V_0$, we see that $C \neq 0$ on F^*. ○

Lemma 2.8 *Let B be a Carleson-Newman Blaschke product, S a closed subset of $M(H^\infty)$ and let $F \subseteq Z(B) \cap S$. Suppose that F is clopen in S (that is a closed and open subset of S). Then there exists a factor C of B such that $B = CD$, where $Z(C) \cap S = F$ and D does not vanish on F. In particular,* ord $(C, x) =$ ord (B, x) *for every $x \in F$. Moreover, C and D can be chosen so that $Z(C) \cap Z(D) = \emptyset$.*

Proof. Let $B = \prod_{j=1}^n b_j$, where the b_j are interpolating Blaschke products. Since F is clopen in S, there exists an open neighborhood U of F in $M(H^\infty)$ such that $S \cap U = S \cap \overline{U} = F$. Let c_j be the factor of b_j formed with the zeros of b_j in $U \cap \mathbb{D}$. Then $Z(c_j) \subseteq \overline{U}$ and

$$Z(c_j) \cap S \subseteq \overline{U} \cap S = F.$$

Let $C^* = c_1 \cdots c_n$. Choose $x \in F$. Then $F \subseteq Z(B) \cap U$ implies that for some $j \in \{1, \cdots, n\}$ the point x belongs to the closure of $Z(b_j) \cap \mathbb{D} \cap U = Z(c_j) \cap \mathbb{D}$. Hence $c_j(x) = 0$. Thus $Z(C^*) \cap S = F$.

Let D^* be chosen so that $B = C^* D^*$ and let $x \in F$. Since $x \in Z(B) \cap S$ we have that $p =$ ord (B, x) satisfies $0 < p \le n$. Since ord $(b, x) = 1$ for every interpolating Blaschke product b with $b(x) = 0$, there exist $j_k \in \{1, \cdots, n\}$ $\quad (k = 1, \cdots, p)$ such that

$$b_{j_1}(x) = \cdots = b_{j_p}(x) = 0 \text{ and } b_j(x) \neq 0 \text{ for } j \notin \{j_1, \cdots, j_p\}.$$

The construction yields that $c_{j_1}(x) = \cdots = c_{j_p}(x) = 0$. Hence

$$\text{ord } (C^*, x) \ge p = \text{ord } (B, x) \ge \text{ord } (C^*, x).$$

This proves the assertion that D^* does not vanish on F. Now let $F^* = Z(C^*) \cap Z(D^*)$. Then $F \cap F^* = \emptyset$. By Lemma 2.7 we may write $C^* = CD_1$ such that D_1 does not vanish on F, C does not vanish on F^* and $Z(C) \cap Z(D_1) = \emptyset$. Note also that $Z(C) \cap Z(D^*) = \emptyset$. Let $D = D_1 D^*$. Then $B = CD$, $D \neq 0$ on F and $Z(C) \cap Z(D) = \emptyset$. Clearly $Z(C) \cap S = Z(C^*) \cap S = F$. ○

Remark. If B is an interpolating Blaschke product and if $S = Z(B) \setminus \mathbb{D}$, we obtain Izuchi's Theorem ([10], p. 339) as a special case.

Lemma 2.9 *Let S be a closed subset of $M(H^\infty)$, $N \in \mathbb{N}$ and $\mathcal{F}$ be a family of functions in H^∞ vanishing identically on S such that*

$$S \subseteq \{x \in M(H^\infty) : 1 \leq \text{ord}\,(f, x) \leq N \text{ for some } f \in \mathcal{F}\}. \tag{6}$$

Then there exists finitely many functions $f_1, \cdots, f_M$ in $\mathcal{F}$ so that

(i) $f_j = c_j g_j$ for some Carleson-Newman Blaschke product c_j of order less than or equal to N,
(ii) $Z(c_j) \cap Z(g_j) = \emptyset \quad (j = 1, \cdots, M)$,
(iii) $Z(c_j) \cap Z(c_k) = \emptyset$ for $j \neq k$,
(iv) $\bigcap_{j=1}^M Z(g_j) \cap S = \emptyset$,
(v) $S \subseteq \bigcup_{j=1}^M Z(c_j)$.

Proof. Let $x \in S$. Then there exists $f \in \mathcal{F}$ with ord $(f, x) \leq N$. Let $p =$ ord (f, x) Then, by Hoffman ([9], p. 100), $f = Bh$ for some Carleson-Newman Blaschke product B of order p and a function $h \in H^\infty$ with $h(x) \neq 0$. Note that p depends on x. We claim that there exists a factor b_0 of B with ord $(b_0, x) = p$ so that $f = b_0 k_0$ and $Z(b_0) \cap Z(k_0) = \emptyset$.

To see this, we choose an open neighborhood U of x such that $h \neq 0$ on $\overline{U}$. By Lemma 2.7 applied to $F = \{x\}$ and $F^* = Z(h) \cap Z(B)$, we obtain factors C and D with $B = CD$, $Z(C) \cap Z(D) = \emptyset$, $D(x) \neq 0$ and $C \neq 0$ on F^*. Then $b_0 = C$ and $k_0 = Dh$ yields the desired factorization. In particular, $Z(b_0)$ is a clopen subset of $Z(f)$.

A compactness argument now yields functions $f_j = b_j k_j \in \mathcal{F}$ with $Z(b_j) \cap Z(k_j) = \emptyset \ \ (j = 1, \cdots, M)$, where the b_j are Carleson-Newman Blaschke products of order less or equal to N such that $E_j := Z(b_j) \cap S$ are clopen subsets of $Z(f_j) \cap S$ satisfying

$$\bigcup_{j=1}^M E_j = S \quad \text{and} \quad \bigcap_{j=1}^M Z(k_j) \cap S = \emptyset.$$

Note that $Z(f_j) \cap S = S$. Thus the E_j are clopen subsets of S.
Let

$$F_1 = E_1, \quad F_j = E_j \setminus \bigcup_{k=1}^{j-1} E_k, \quad (j = 2, \cdots, M).$$

By Lemma 2.8 and the disjointness of the F_j we can choose factors c_j of b_j such that $f_j = c_j g_j$ with

$$Z(c_j) \cap S = F_j, \quad Z(c_j) \cap Z(c_k) = \emptyset \quad (j \neq k),$$

$Z(c_j) \cap Z(g_j) = \emptyset$ and $g_j \neq 0$ on F_j $(j = 1, \cdots, M)$. Thus

$$\bigcap_{j=1}^{M} Z(g_j) \cap S = \emptyset \text{ and } S \subseteq \bigcup_{j=1}^{M} Z(c_j).$$

○

The following result will be one of the major tools used to prove the characterization theorem of closed ideals in H^∞.

Theorem 2.10 *Let I be an ideal in H^∞ such that*

$$N = \sup\{\text{ord }(I, x) : x \in Z(I)\} < \infty. \tag{7}$$

Let U_j be open sets satisfying $E_j(I) \subseteq U_j$ $(j = 1, \cdots, N)$. Then I is algebraically generated by Carleson-Newman Blaschke products B of order N such that

$$E_j(B) \subseteq U_j \quad (j = 1, \cdots, N).$$

Proof. Using induction on j we prove that I is generated by Carleson-Newman Blaschke products B of order N satisfying $E_j(B) \subseteq U_j$ $(j = 1, \cdots, N)$.

Step 1 By Lemma 2.9, applied to $S = Z(I)$ and $\mathcal{F} = I$, we find $f_1, \cdots, f_M \in I$ with $f_j = c_j g_j$ for some Carleson-Newman Blaschke products c_j of order less than or equal to N such that

$$\bigcap_{j=1}^{M} Z(g_j) \cap Z(I) = \emptyset \quad \text{and} \quad Z(c_j) \cap Z(c_k) = \emptyset \ (j \neq k). \tag{8}$$

Hence for some $g \in I$ and $\alpha_j \in H^\infty$ we have $1 = g + \sum_{j=1}^{M} \alpha_j g_j$. This implies that

$$c_1 \cdots c_M = c_1 \cdots c_M g + \sum_{j=1}^{M} \alpha_j \big(\prod_{\substack{k=1 \\ k\neq j}}^{M} c_k\big) c_j g_j \in I.$$

Next we recall that for each j the function c_j is a Carleson-Newman Blaschke product of order less than or equal to N. This and (8) imply that every zero of the product $c_1 \cdots c_M$ is a zero of order less than or equal to N. Equation (7) and the fact that $c_1 \cdots c_M \in I$ now imply that the function $c = c_1 \cdots c_M$ is a Carleson-Newman Blaschke product of order N.

Let $\mathcal{F} = \{f \in I : \|f\|_\infty = 1\}$. Then it is easy to see that for fixed $\lambda_0 > 0$ the set $\{c + \lambda_0 f : f \in \mathcal{F} \cup \{0\}\}$ is a generating set for the ideal I. Taking λ_0 as in Proposition 1.4, we may conclude that the inner factors of the functions $c + \lambda_0 f$ are Carleson-Newman Blaschke products of order less than or equal to N. On the other hand (7) implies that the order is at least N. Since the outer factors of the functions $c + \lambda_0 f$ are invertible in H^∞, we get the assertion that I is generated by Carleson-Newman Blaschke products of order N.

Now let $\mathcal{F}_0$ be the set of these generators and let $B = \prod_{n=1}^N b_n \in \mathcal{F}_0$. Here the b_n are interpolating Blaschke products. Choose an open set W_1 in $M(H^\infty)$ satisfying

$$E_1(I) \subseteq W_1 \subseteq \overline{W}_1 \subseteq U_1.$$

For each n, let c_n be the interpolating Blaschke product formed with the zeros of b_n in $\mathbb{D} \cap W_1$. Let d_n satisfy $b_n = c_n d_n$. Form the Blaschke products $C = c_1 \cdots c_N$ and $D = d_1 \cdots d_N$. Then $B = CD$. Recall that by definition $E_1(f) = Z(f)$. By construction we have $Z(C) \subseteq U_1$ and $Z(D) \cap E_1(I) = \emptyset$. By Lemma 2.0 we conclude that $C \in I$. It is now clear that the set $\mathcal{F}_1$ of all such C is a generating set for the ideal I.

Recall that the complement of a set W in $M(H^\infty)$ will be denoted by W^c.

Step 2 Now assume that for some $j \in \{1, \cdots, N-1\}$, $\mathcal{F}_j$ is a set of generators of I consisting of Carleson-Newman Blaschke products D of order N satisfying

$$E_\ell(D) \subseteq U_\ell \quad (1 \le \ell \le j).$$

Choose open sets W_ℓ satisfying

$$E_\ell(I) \subseteq W_\ell \subseteq \overline{W}_\ell \subseteq U_\ell \quad (\ell = 1, \cdots, j+1),$$

$$W_{\ell+1} \subseteq W_\ell \quad (\ell = 1, \cdots, j). \tag{9}$$

Let

$$S_{j+1} = W_{j+1}^c \cap E_1(I). \tag{10}$$

Then for every $x \in S_{j+1}$ we have $\operatorname{ord}(I, x) \le j$. By Lemma 2.9 applied for $\mathcal{F} = \mathcal{F}_j$ and $S = S_{j+1}$, we obtain a finite number M of Carleson-Newman Blaschke products C_n of order less than or equal to j such that $C_n g_n = D_n$ for some $D_n \in \mathcal{F}_j$ and some $g_n \in H^\infty$ satisfying

$$Z(C_n) \cap Z(g_n) = \emptyset, \quad Z(C_n) \cap Z(C_k) = \emptyset \quad (n \ne k), \tag{11}$$

and

$$\bigcup_{n=1}^{M} Z(C_n) \cap S_{j+1} = S_{j+1}. \tag{12}$$

We can also arrange it so that $Z(C_n) \cap E_N(I) = \emptyset$ for every $n \in \{1, \cdots, M\}$. Let $C = \prod_{n=1}^M C_n$. Then

$$\bigcap_{n=1}^{M} Z(g_n) \cap Z(C) = \emptyset, \tag{13}$$

and

$$E_\ell(C) \subseteq U_\ell \quad (1 \leq \ell \leq j). \tag{14}$$

Note that (14) is true because the C_n are factors of generators in $\mathcal{F}_j$ with disjoint zero sets. We also note that by (11) C is a Carleson-Newman Blaschke product of order less than or equal to j.

In particular from (12) and (13) we get that

$$\bigcap_{n=1}^{M} Z(g_n) \cap S_{j+1} = \emptyset. \tag{15}$$

Let

$$S = W_{j+1}^c \cup \Gamma \cup Z(C), \tag{16}$$

where Γ denotes the set of trivial points in $M(H^\infty)$. Consider the ideal $J = I + I(g_1, \cdots, g_M)$ generated by I and the functions g_n above. Then

$$Z(J) \subseteq Z(I) \quad \text{and} \quad Z(J) \subseteq \bigcap_{n=1}^{M} Z(g_n). \tag{17}$$

Moreover $E_\ell(J) \subseteq E_\ell(I)$ for every $\ell \in \mathbb{N}$. In particular

$$\sup\{\text{ord } (J, x) : x \in Z(J)\} = N.$$

We claim that $Z(J) \cap S = \emptyset$. Assuming the contrary, we can choose $x \in Z(J) \cap S$. Since $Z(J) \subseteq Z(I) \subseteq G$, we have that $x \notin \Gamma$. By (13) and (17) we see that $x \notin Z(C)$. So $x \in W_{j+1}^c \cap Z(I) = S_{j+1}$. Since $x \in Z(J)$, by (17) $x \in \bigcap_{n=1}^M Z(g_n)$. Thus $x \in \bigcap_{n=1}^M Z(g_n) \cap S_{j+1}$. This contradicts (15).

We note also that by (9), (10) and (16) $S_{j+1} \subseteq S$ and $W_\ell^c \subseteq S \quad (1 \leq \ell \leq j)$.

Applying Step 1 of this proof with the ideal I replaced by the ideal J and $U_1 = S^c$, we may conclude that there exists a Carleson-Newman Blaschke product $b \in J$ of order N such that $Z(b) \subseteq S^c$. Hence $b \neq 0$ on S. Since $b \in J$ we can choose $g_0 \in I$ and $\alpha_n \in H^\infty$ so that

$$b = g_0 + \sum_{n=1}^{M} \alpha_n g_n.$$

Then

$$Cb = C_1 \cdots C_M b = Cg_0 + \sum_{n=1}^{M} (\alpha_n \prod_{\substack{k=1 \\ k \neq n}}^{M} C_k)(C_n g_n) \in I + I \subseteq I.$$

Note that by (16) we have

$$Z(b) \cap Z(C) = \emptyset. \tag{18}$$

Since $b \neq 0$ on W_{j+1}^c, by (18) and the fact that C is a Carleson-Newman Blaschke product of order less than or equal to j, we obtain the inclusion $E_{j+1}(Cb) \subseteq W_{j+1} \subseteq U_{j+1}$.

Because $Z(b) \subseteq W_{j+1}$ we get by (9), (18) and (14) that $E_\ell(Cb) \subseteq U_\ell$, $(1 \leq \ell \leq j)$.

Finally, ord $C \leq j$ and ord $b = N$ imply in view of (18) that $\text{ord}(Cb) = N$.

It remains to show that I is generated by Carleson-Newman Blaschke products B of order N satisfying $E_\ell(B) \subseteq U_\ell$ for every $\ell \in \{1, \cdots, j+1\}$. Recall that by our inductive assumption, $\mathcal{F}_j$ was a set of generators D of I of order N satisfying

$$E_\ell(D) \subseteq U_\ell \quad \text{for every } \ell \in \{1, \cdots, j\}.$$

Now define a set $\mathcal{F}_{j+1}^*$ by $\mathcal{F}_{j+1}^* = \{Cb + \lambda_0 f : f \in \mathcal{F}_j \cup \{0\}\}$, where Cb is the Carleson-Newman Blaschke product of order N constructed above. Taking λ_0 as in Proposition 1.4, we can conclude that the outer factors of the functions in $\mathcal{F}_{j+1}^*$ are invertible and that the inner factors of them are just the Carleson-Newman Blaschke products B of order N satisfying $E_\ell(B) \subseteq U_\ell$ for every $\ell \in \{1, \cdots, j+1\}$ we have been looking for. The collection of these B's is now our generating set $\mathcal{F}_{j+1}$ for the ideal I. ○

Theorem 2.11 *Let I be an ideal in H^∞ such that*

$$N = \sup\{\text{ord } (I, x) : x \in Z(I)\} < \infty.$$

Let $E_j = E_j(I)$ be the higher order hulls of I. Assume that for some interpolating Blaschke products C_j the product $C = \prod_{j=1}^N C_j$ is a Carleson-Newman Blaschke product of order N such that $C \in I(E_1, \cdots, E_N)$. Then for every $\varepsilon > 0$ there exist interpolating Blaschke products B_j with $\prod_{j=1}^N B_j \in I$ so that

$$Z(B_j) \subseteq \{|C_j| < \varepsilon\} \quad (j = 1, \cdots, N).$$

Proof. For any positive integer $p \in \{1, \cdots, N\}$ we let

$$\hat{\mathbb{N}}_N^p = \{(j_1, \cdots, j_p) : 1 \leq j_1 < \ldots < j_p \leq N\}.$$

Define the sets

$$V_j = \{|C_j| < \varepsilon\} \quad (j = 1, \cdots, N)$$

and let

$$U_p = \bigcup_{1 \le j_1 < j_2 \cdots < j_p \le N} (V_{j_1} \cap \cdots \cap V_{j_p}). \tag{19}$$

Note that $U_1 = \bigcup_{1 \le j \le N} V_j$ and $U_N = \bigcap_{1 \le j \le N} V_j$.
Then the U_j are open sets containing E_j $(j = 1, \cdots, N)$ and for any integer p,

$$E_p(C) = \bigcup_{1 \le j_1 < j_2 \cdots < j_p \le N} (Z(C_{j_1}) \cap \cdots \cap Z(C_{j_p})).$$

By Theorem 2.10, there exist interpolating Blaschke products D_j with $\prod_{j=1}^N D_j \in I$ so that the Blaschke product $D = \prod_{j=1}^N D_j$ satisfies

$$E_j(D) \subseteq U_j \quad (j = 1, \cdots, N). \tag{20}$$

Our main step will be the construction of an interpolating Blaschke product B_1 dividing D, say $D = B_1 v$, so that v is a Carleson-Newman Blaschke product of order $N-1$ and

$$Z(B_1) \subseteq V_1 \tag{21}$$

$$E_1(v) \subseteq \bigcup_{2 \le j \le N} V_j, \tag{22}$$

$$E_2(v) \subseteq \bigcup_{2 \le j_1 < j_2 \le N} (V_{j_1} \cap V_{j_2}), \tag{23}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E_p(v) \subseteq \bigcup_{2 \le j_1 < j_2 \cdots < j_p \le N} (V_{j_1} \cap \cdots \cap V_{j_p}), \tag{24}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E_{N-1}(v) \subseteq V_2 \cap \cdots \cap V_N. \tag{25}$$

We then use a "backwards" induction argument to obtain the factorization $D = \prod_{j=1}^N B_j$, where the interpolating Blaschke products B_j satisfy $Z(B_j) \subseteq V_j$ $(j = 1, \cdots, N)$.

Step 1 In this first step we factor $D = A_1 u$ where
A_1 is an interpolating Blaschke product with $Z(A_1) \subseteq V_1$ and
u is a Carleson-Newman Blaschke product of order $N-1$ satisfying
$E_1(u) \subseteq \bigcup_{2 \le k \le N} V_k$.

Let

$$S_1 = Z(D_1) \setminus V_1 \tag{26}$$

and

$$S_j = Z(D_j) \setminus (\bigcup_{2 \le k \le N} V_k) \quad (j = 2, \cdots, N). \tag{27}$$

The S_j obviously are closed. Because $Z(D) \subseteq \bigcup_{1 \le j \le N} V_j$, we have

$$S_1 \subseteq \bigcup_{2 \le k \le N} V_k, \quad S_j \subseteq V_1, \quad S_j \cap E_2(D) = \emptyset \tag{28}$$

and

$$S_j \bigcap S_1 = \emptyset \quad (j = 2, \cdots, N). \tag{29}$$

Since

$$E_{p+1}(D) \quad \setminus \quad \bigcup_{\substack{k \ne 1 \\ k \ne j_1, \cdots, j_p}} V_k \quad \subseteq \quad V_1 \cap V_{j_1} \cap \cdots \cap V_{j_p} \subseteq V_1, \tag{30}$$

we conclude that for a fixed p-tuple $(j_1, \cdots, j_p)$ with $2 \le j_1 < \cdots < j_p \le N$ we have

$$S_1 \cap \left(E_{p+1}(D) \quad \setminus \quad \bigcup_{\substack{k \ne 1 \\ k \ne j_1, \cdots, j_p}} V_k \right) = \emptyset \quad (1 \le p \le N-1) \tag{31}$$

For every $j \in \{1, \cdots, N\}$ one can choose open neighborhoods $\mathcal{O}_j$ in $M(H^\infty)$ containing S_j such that

$$\overline{\mathcal{O}_j} \cap \overline{\mathcal{O}_1} = \emptyset \quad (j = 2, \cdots, N) \quad \text{(by (29))} \tag{32}$$

$$\overline{\mathcal{O}_j} \subseteq V_1 \quad (j = 2, \cdots, N) \quad \text{by (28)} \tag{33}$$

$$\overline{\mathcal{O}_1} \subseteq \bigcup_{2 \le k \le N} V_k \quad \text{by (28)} \tag{34}$$

$$\overline{\mathcal{O}_j} \cap E_2(D) = \emptyset \quad (j = 2, \cdots, N) \quad \text{by (28)} \tag{35}$$

$$\overline{\mathcal{O}_1} \cap E_N(D) = \emptyset \quad \text{by (26)} \tag{36}$$

and more generally (by (31))

$$\overline{\mathcal{O}_1} \cap \left(E_{p+1}(D) \quad \setminus \quad \bigcup_{\substack{k \ne 1 \\ k \ne j_1, \cdots, j_p}} V_k \right) = \emptyset \quad (1 \le p \le N-1) \tag{37}$$

for every fixed p-tuple $(j_1, \cdots, j_p)$ with $2 \le j_1 < \cdots < j_p \le N$.
Let D'_j be the interpolating Blaschke products with zero set $\mathcal{O}_j \cap Z_{\mathbb{D}}(D_j)$. Then $D_j = D'_j D''_j$ for some interpolating Blaschke product D''_j. By (33) and (34)

$$Z(D'_j) \subseteq \overline{\mathcal{O}_j} \subseteq V_1 \quad (j = 2, \cdots, N), \tag{38}$$

and

$$Z(D_1') \subseteq \overline{\mathcal{O}_1} \subseteq \bigcup_{2\leq k\leq N} V_k. \tag{39}$$

Let $A_1 = \left(\prod_{j=2}^N D_j'\right)D_1''$ and $u = \left(\prod_{j=2}^N D_j''\right)D_1'$. Then $D = A_1 u$. We claim that A_1 and u satisfy the required conditions; that is:

$$A_1 \text{ is an interpolating Blaschke product with } Z(A_1) \subseteq V_1, \tag{40}$$

and

$$\begin{aligned}&u \text{ is a Carleson-Newman Blaschke product of order } N-1\\ &\text{with } E_1(u) \subseteq \textstyle\bigcup_{2\leq k\leq N} V_k.\end{aligned} \tag{41}$$

Proof of (40).
(i) First we note that $Z(D_1'') \cap Z(\prod_{j=2}^N D_j') = \emptyset$. If not, then there would exist $x \in Z(D_1) \cap Z(\prod_{j=2}^N D_j)$. In particular $x \in E_2(D)$. Thus, by (35), $x \notin \bigcup_{j=2}^N \overline{\mathcal{O}_j}$. By (38) , $x \notin \bigcup_{j=2}^N Z(D_j') = Z(\prod_{j=2}^N D_j')$. This is a contradiction.

(ii) Next we show that $Z(D_{j_1}') \cap Z(D_{j_2}') = \emptyset$ if $2 \leq j_1 < j_2 \leq N$. If not, then again there would exist $x \in E_2(D)$ with $D_{j_1}'(x) = D_{j_2}'(x) = 0$. By (38) $x \in \overline{\mathcal{O}_{j_1}} \cap \overline{\mathcal{O}_{j_2}}$. However, by (35), $\overline{\mathcal{O}_j} \cap E_2(D) = \emptyset$ for $2 \leq j \leq N$. This is a contradiction.

Since a finite product of interpolating Blaschke products is an interpolating Blaschke product if and only if the zero sets are disjoint, (i) and (ii) show that A_1 is an interpolating Blaschke product. To prove the assertion about the zero set of A_1 being contained in V_1, we use (38) to conclude that $Z(D_j') \subseteq \overline{\mathcal{O}_j} \subseteq V_1$ for $2 \leq j \leq N$. But $Z(D_1'') \subseteq V_1$, because otherwise there would exist an $x \in Z(D_1'') \setminus V_1 \subseteq Z(D_1) \setminus V_1 \overset{(26)}{=} S_1$. Since $\mathcal{O}_1 \supseteq S_1$, by the choice of D_1' we have $D_1'(x) = 0$. So ord $(D_1, x) \geq 2$, which contradicts the fact that D_1 is an interpolating Blaschke product. We conclude that $Z(A_1) \subseteq V_1$.

Proof of (41).
(i) If $u = \left(\prod_{j=2}^N D_j''\right)D_1'$ were to have order strictly bigger than $N-1$, then there would exist $x \in Z(u)$ with $D_j''(x) = 0$ for every $j \in \{2, \cdots, N\}$ and $D_1'(x) = 0$. But then $x \in \bigcap_{j=1}^N Z(D_j) = E_N(D)$. Since $Z(D_1') \subseteq \overline{\mathcal{O}_1}$ and (by (36)) $\overline{\mathcal{O}_1} \cap E_N(D) = \emptyset$, we obtain a contradiction. On the other hand, since for $j = 2, \cdots, N$ (35) and (38) imply $Z(D_j') \cap E_N(D) = \emptyset$, we see that $\bigcap_{j=2}^N Z(D_j'') \neq \emptyset$. Since $E_N(D) \neq \emptyset$, we have ord $u = N-1$.

(ii) Now we shall prove that $E_1(u) \subseteq \bigcup_{2\leq k\leq N} V_k$. To see this, we first note that by (39) we have $Z(D_1') \subseteq \bigcup_{2\leq k\leq N} V_k$. But we also have $Z(D_j'') \subseteq \bigcup_{2\leq k\leq N} V_k$

for every $j \in \{2, \cdots, N\}$, since otherwise there exists

$$x \in Z(D_j'') \setminus \Big(\bigcup_{2 \le k \le N} V_k \Big) \quad \subseteq \quad Z(D_j) \setminus \bigcup_{2 \le k \le N} V_k \quad \overset{(27)}{=} \quad S_j \subseteq \mathcal{O}_j .$$

Hence, by the choice of D_j', we get $D_j'(x) = 0$. This would imply that ord $(D_j, x) \geq 2$, a contradiction. Thus

$$Z(\prod_{k=2}^{N} D_k'') = \bigcup_{2 \le k \le N} Z(D_k'') \subseteq \bigcup_{2 \le k \le N} V_k .$$

This proves our claim (41) and completes the proof of Step 1.

Before proceeding to Step 2 we recall that $D = A_1 u$, where $u = \big(\prod_{j=2}^{N} D_j''\big) D_1'$ satisfies

$$E_1(u) \subseteq \bigcup_{2 \le k \le N} V_k ,$$

and in general

$$E_p(u) \subseteq E_p(D) \subseteq \bigcup_{1 \le j_1 < \cdots < j_p \le N} \big(V_{j_1} \cap \cdots \cap V_{j_p}\big) \quad (1 \le p \le N-1). \tag{42}$$

Although u satisfies (41), (see also (22)), the higher order zero sets $E_p(u)$ do not yet satisfy (24) for $2 \le p \le N-1$. To achieve the desired factorization $D = B_1 v$, (v will be a factor of u), we will now indicate in step 2 the construction of these factors.

Step 2 Let p be an integer with $1 \le p \le N-1$ and let

$$M_j = (V_1 \cap V_j) \setminus \bigcup_{\substack{k \neq 1 \\ k \neq j}} V_k \quad (2 \le j \le N)$$

and in general

$$M_{(j_1, \cdots, j_p)} = (V_1 \cap V_{j_1} \cap \cdots \cap V_{j_p}) \quad \setminus \quad \bigcup_{\substack{k \neq 1 \\ k \neq j_1, \cdots, j_p}} V_k \quad (2 \le j_1 < \cdots < j_p \le N).$$

These sets are pairwise disjoint. In general, $E_{p+1}(\prod_{j=2}^{N} D_j'') \cap M_{(j_1, \cdots, j_p)} \neq \emptyset$. So in order to construct our v, we must throw out enough zeros of u so that the final product

$$v = (\prod_{j=2}^{N} D_j''') D_1',$$

where D_j''' is a subfactor of D_j'', satisfies relations (22) to (25) while $B_1 = D/v$ satisfies (21). The interpolating Blaschke products D_j''' will be chosen so that

$$E_{p+1}(\prod_{j=2}^{N} D_j''') \cap M_{(j_1,\cdots,j_p)} = \emptyset.$$

To this end we have yet to introduce other sets (sic!) .
For $j = 2, \cdots, N$, let $R_j = M_j \cap E_2(\prod_{k=2}^{N} D_k'')$. More generally, let

$$R_{(j_1,\cdots,j_p)} = M_{(j_1,\cdots,j_p)} \cap E_{p+1}(\prod_{k=2}^{N} D_k'') \quad (2 \le j_1 < \cdots < j_p \le N)$$

where $1 \le p \le N-1$. We claim that the $R_{(j_1,\cdots,j_p)}$ are *closed* sets. To see this, note that for $2 \le j_1 < \cdots < j_p \le N$ we have by (30)

$$R_{(j_1,\cdots,j_p)} = E_{p+1}(\prod_{k=2}^{N} D_k'') \quad \setminus \bigcup_{\substack{k\neq 1 \\ k\neq j_1,\cdots,j_p}} V_k \quad . \tag{43}$$

Note also, once again using (30), that

$$R_{(j_1,\cdots,j_p)} \quad \subseteq \quad V_1 \cap V_{j_1} \cap \cdots \cap V_{j_p}. \tag{44}$$

It is now easy to see that

$$R_{(j_1,\cdots,j_p)} \cap Z(D_1'') = \emptyset. \tag{44'}$$

In fact, this holds because otherwise such zeros, being of order $p+2$ for D, would be included in the intersection of $R_{(j_1,\cdots,j_p)}$ with $p+2$ sets V_j, which is impossible in view of (44) and (43).

Moreover, the $R_{(j_1,\cdots,j_p)}$ are pairwise disjoint for all $(j_1, \cdots, j_p) \in \hat{\mathbb{N}}_N^p$ with $j_1 \ge 2$ and $p \in \{1, \cdots, N-1\}$. Using (44) and (44') we may choose neighborhoods $\Omega_{(j_1,\cdots,j_p)} \supseteq R_{(j_1,\cdots,j_p)}$ so that the closures of the chosen neighborhoods are pairwise disjoint, too, and for every $p \in \{1, \cdots, N-1\}$ satisfy :

$$\overline{\Omega_{(j_1,\cdots,j_p)}} \subseteq V_1 \cap V_{j_1} \cap \cdots \cap V_{j_p} \quad (2 \le j_1 < \cdots < j_p \le N) \tag{45}$$

$$\overline{\Omega_{(j_1,\cdots,j_p)}} \cap Z(D_1'') = \emptyset, \tag{46}$$

and

$$\overline{\Omega_{(j_1,\cdots,j_p)}} \cap Z(D_j') = \emptyset \quad \text{for every} \quad j \in \{2, \cdots, N\}. \tag{47}$$

Note that the latter can be done because otherwise there would exist $j \in \mathbb{N}$ and

$$x \in E_{p+1}(\prod_{k=2}^{N} D_j''), \quad x \notin \bigcup_{\substack{k\neq 1 \\ k\neq j_1,\cdots,j_p}} V_k, \quad \text{but} \quad D_j'(x) = 0.$$

Hence ord $(D,x) \geq p+2$. Again in this case x would belong to at least $p+2$ sets V_k, in contradiction to (45) and the fact that $x \notin \bigcup_{\substack{k\neq 1\\ k\neq j_1,\cdots,j_p}} V_k$.

For the sequel we note that for each $p \in \{1,\cdots,N-1\}$ and every p-tuple $(j_1,\cdots,j_p)$ with $2 \leq j_1 < \cdots < j_p \leq N$, we have

$$R_{(j_1,\cdots,j_p)} = \bigcup_{2\leq \ell_1<\cdots<\ell_{p+1}\leq N} \Big(Z(D''_{\ell_1}) \cap \cdots \cap Z(D''_{\ell_{p+1}}) \Big) \quad \setminus \bigcup_{\substack{k\neq 1\\ k\neq j_1,\cdots,j_p}} V_k \tag{48}$$

Note also that

$$R_{(j_1,\cdots,j_p)} \subseteq E_{p+1}(D) \setminus E_{p+2}(D). \tag{49}$$

Hence, within each $R_{(j_1,\cdots,j_p)}$ the sets $Z(D''_{\ell_1}) \cap \cdots \cap Z(D''_{\ell_{p+1}})$ (even without the double primes) are pairwise disjoint; that is for every $(\ell_1,\cdots,\ell_{p+1}) \neq (k_1,\cdots,k_{p+1})$ with $2 \leq \ell_1 < \cdots < \ell_{p+1} \leq N$ and $2 \leq k_1 < \cdots < k_{p+1} \leq N$, we have

$$R_{(j_1,\cdots,j_p)} \cap \Big(Z(D_{\ell_1}) \cap \cdots \cap Z(D_{\ell_{p+1}}) \Big) \bigcap \Big(Z(D_{k_1}) \cap \cdots \cap Z(D_{k_{p+1}}) \Big) = \emptyset.$$

Thus for each $(\ell_1,\cdots,\ell_{p+1})$ the sets

$$R_{(j_1,\cdots,j_p)} \cap \Big(Z(D''_{\ell_1}) \cap \cdots \cap Z(D''_{\ell_{p+1}}) \Big)$$

can be separated by pairwise disjoint open sets $U^*_{(\ell_1,\cdots,\ell_{p+1})}$ contained in $\Omega_{(j_1,\cdots,j_p)}$. For each $(j_1,\cdots,j_p)$ with $2 \leq j_1 < \cdots < j_p \leq N$, every $p \in \{1,\cdots,N-1\}$ and each $(\ell_1,\cdots,\ell_{p+1})$ with $2 \leq \ell_1 < \cdots < \ell_{p+1} \leq N$, let d''_{ℓ_1} be the interpolating Blaschke product formed with the zeros of D''_{ℓ_1} in $\mathbb{D} \cap U^*_{(\ell_1,\cdots,\ell_{p+1})}$. Since all the sets $U^*_{(\ell_1,\cdots,\ell_{p+1})}$'s and $\overline{\Omega_{(j_1,\cdots,j_p)}}$ are pairwise disjoint, the product of all such d''_{ℓ_1} is an interpolating Blaschke product d dividing $\prod_{k=2}^N D''_k$.

Now let

$$B_1 = A_1 d, \quad v = u/d = (\prod_{k=2}^N D''_k) D'_1/d.$$

Then $B_1 v = A_1 u = D$. Note that v can be represented in the form $v = (\prod_{k=2}^N D'''_k) D'_1$.
We shall now prove in step 3 that relations (21) to (25) are satisfied.

Step 3 By (45) we have

$$Z(d) \subseteq \bigcup_{p=1}^{N-1} \bigcup_{2\leq j_1<\cdots<j_p\leq N} \overline{\Omega_{(j_1,\cdots,j_p)}} \quad \subseteq \quad V_1 .$$

Together with (40) we obtain $Z(B_1) \subseteq V_1$, so (21) holds. Now (41) yields that (22) is satisfied; that is $E_1(v) \subseteq \bigcup_{2\le j\le N} V_j$.

To prove that B_1 is an interpolating Blaschke product, we have to show that

$$Z(A_1) \cap Z(d) = \emptyset. \tag{50}$$

To see this, let $y \in Z(d)$. Recall that $A_1 = (\prod_{j=2}^{N} D'_j) D''_1$. Because $Z(d) \subseteq \bigcup_{p=1}^{N-1} \bigcup_{2\le j_1<\cdots<j_p\le N} \overline{\Omega_{(j_1,\cdots,j_p)}}$ and $\overline{\Omega_{(j_1,\cdots,j_p)}} \bigcap Z(D''_1) \overset{(46)}{=} \emptyset$, we see that $D''_1(y) \neq 0$. On the other hand, $\overline{\Omega_{(j_1,\cdots,j_p)}} \bigcap Z(D'_j) \overset{(47)}{=} \emptyset$ for $2 \le j \le N$. Therefore $D'_j(y) \neq 0$. We conclude that $A_1(y) \neq 0$. Hence (50) is satisfied.

We shall now prove that (24) (hence (23)-(25)) is true for $1 \le p \le N-1$; that is

$$E_p(v) \subseteq \bigcup_{2\le j_1<j_2\cdots<j_p\le N} (V_{j_1} \cap \cdots \cap V_{j_p}). \tag{51}$$

Note that the case $p = 1$ was proved in the paragraph preceding equation (50). Let $p \ge 2$ and let

$$W_p = \bigcup_{2\le j_1<j_2\cdots<j_p\le N} (V_{j_1} \cap \cdots \cap V_{j_p}).$$

Choose $x \in E_p(v)$ and assume that $x \notin W_p$. Since v is a factor of u, by (42) we have $E_p(v) \subseteq \bigcup_{1\le j_1<\cdots<j_p\le N} \left(V_{j_1} \cap \cdots \cap V_{j_p}\right)$. Therefore

$$x \in \bigcup_{2\le j_2<\cdots<j_p\le N} (V_1 \cap V_{j_2} \cap \cdots \cap V_{j_p}).$$

Suppose, without loss of generality, that

$$x \in V_1 \cap V^*_{j_2} \cap \cdots \cap V^*_{j_p}$$

for some $(j^*_2, \cdots, j^*_p)$ with $2 \le j^*_2 < \cdots < j^*_p \le N$.

We claim that

$$x \notin \bigcup_{\substack{k\neq 1 \\ k\neq j^*_2,\cdots,j^*_p}} V_k \quad .$$

In fact, if this does not hold, there would exist $k_0 \in \{1, \cdots, N\} \setminus \{1, j^*_2, \cdots, j^*_p\}$ such that $x \in V_{k_0}$. Hence $x \in V_{j^*_2} \cap \cdots \cap V_{j^*_p} \cap V_{k_0} \subseteq W_p$. This contradicts the choice of x.

We conclude that

$$x \in E_p(v) \quad \setminus \bigcup_{\substack{k \neq 1 \\ k \neq j_2^*, \cdots, j_p^*}} V_k \,. \tag{52}$$

However, by (37) and (39) we know that

$$Z(D_1') \cap \Big(E_p(D) \setminus \bigcup_{\substack{k \neq 1 \\ k \neq j_2^*, \cdots, j_p^*}} V_k\Big) = \emptyset.$$

Thus $D_1'(x) \neq 0$.
Noticing that $v = D_1' \prod_{j=2}^N D_j'''$, (52) and $p \geq 2$ now yield that

$$x \in E_p(\prod_{j=2}^N D_j''') \setminus \bigcup_{\substack{k \neq 1 \\ k \neq j_2^*, \cdots, j_p^*}} V_k.$$

Thus, by (43), $x \in R_{(j_2^*, \cdots, j_p^*)}$ from which we conclude that $d(x) = 0$. Since $D = B_1 v = B_1 \Big(\prod_{j=2}^N D_j'''\Big) D_1' = A_1 d \Big(\prod_{j=2}^N D_j'''\Big) D_1'$, we have

$$\text{ord } (D, x) \geq \text{ord } \Big((\prod_{j=2}^N D_j''') \cdot d, x\Big) \geq p + 1,$$

which is a contradiction to the fact that by (49)

$$R_{(j_2^*, \cdots, j_p^*)} \cap E_{p+1}(D) = \emptyset.$$

This implies that our assumption $x \notin W_p$ was wrong; so $x \in W_p$. Thus (51) is satisfied.

○

We are now ready to prove our main result.

Theorem 2.12 *Let I be a closed ideal in H^∞ satisfying $Z(I) \subseteq G$. Let*

$$E_n(I) = \{x \in M(H^\infty) : \text{ord } (I, x) \geq n\} \quad (n \in \mathbb{N})$$

be the higher order zero sets (or hulls) of I. Then there exists $N \in \mathbb{N}$ such that $E_n(I) = \emptyset$ for $n \geq N + 1$ and

$$I = I(E_1(I), \cdots, E_N(I)) =$$

$$= \{f \in H^\infty : \text{ord } (f, x) \geq n \quad \text{for every} \quad x \in E_n(I) \quad \text{and} \quad n = 1, 2, \cdots, N\}.$$

In particular, I is uniquely determined by its higher order zero sets.

Proof. *Step 1* Because $Z(I) \subseteq G$, Lemma 2.2 yields that

$$N := \sup\{\text{ord } (I, x) : x \in Z(I)\} < \infty.$$

Hence $E_n(I) = \emptyset$ for every $n > N$. Therefore $I \subseteq I(E_1(I), \cdots, E_N(I)) =: J$ Note that $E_1(I) \supseteq E_2(I) \supseteq \cdots \supseteq E_N(I)$ and $Z(I) = E_1(I)$ and also that $E_n(J) = E_n(I)$ for every $n \in \mathbb{N}$. Moreover one can show that J is a closed ideal. In fact, let $f_n \in J$ converge uniformly to $f \in H^\infty$. Let $x \in E_j(J)$. Then by Lemma 1.1 $D^k f_n(x) = 0$ for $0 \leq k \leq j-1$. By Lemma 1.3, $D^k f_n$converges uniformly on $M(H^\infty)$ to $D^k f$; hence $D^k f(x) = 0$ for $0 \leq k \leq j-1$. Thus by Lemma 1.1 ord $(f, x) \geq j$. This yields that $f \in J$.

Step 2. By Theorem 2.10, I and J are generated by Carleson-Newman Blaschke products of order N. Let $\prod_{k=1}^N C_k$ be such a generator of J. We shall show that $C_1 \cdots C_N \in I$. Let $\varepsilon > 0$ and let $U_j = \{|C_j| < \varepsilon\}$. By Theorem 2.11 I contains a Carleson-Newman Blaschke product $B_1 \cdots B_N$ of order N so that $Z(B_j) \subseteq U_j$. Moreover, by Proposition 2.5, $\delta(B_j)$ can be chosen to be bigger than $\delta = \frac{1}{2} \min\{\delta(C_k) : k = 1, \cdots, N\}$. By Proposition 2.6 there exists a constant $\kappa > 0$ that depends only on δ and N such that

$$\text{dist}(C_1 \cdots C_N, B_1 \cdots B_N \, H^\infty) \leq \varepsilon\kappa.$$

Hence $\text{dist}(C_1 \cdots C_N, I) \leq \varepsilon\kappa$. Since $\varepsilon > 0$ was arbitrary, and I is closed, we have $C_1 \cdots C_N \in I$. Thus every generator of J belongs to I. So $J \subseteq I \subseteq J$; that is $I = J$. ○

3 Applications to finitely generated ideals and finite products of ideals.

Let $f_1, \cdots, f_n \in H^\infty$ and let

$$I = I(f_1, \cdots, f_n) = \Big\{ \sum_{j=1}^n g_j f_j : g_j \in H^\infty \Big\}$$

be the ideal generated by the functions f_j. Let

$$J = J(f_1, \cdots, f_n) = \{ f \in H^\infty : |f| \leq C \sum_{j=1}^n |f_j| \ \text{ on } \mathbb{D} \ \text{ for some } C = C(f) \}.$$

It is well-known that $I \subseteq J$ and that, in general, the inclusion is strict (see [5], p. 369). T. Wolff showed that $f \in J$ implies that $f^3 \in I$ (see [5], p. 329). It is not known whether $f \in J$ implies $f^2 \in I$. In [7], however, it is shown that if the hull of I is contained in the set G of nontrivial points , then $f \in J$ implies that $f^2 \in I$. J. Bourgain [2] proved that $f \in J$ implies $f^2 \in \overline{I}$, the closure of I, for any finitely generated ideal I in H^∞. As a corollary of Theorem 2.12 we now obtain:

Corollary 3.1 *Let $I = I(f_1, \cdots, f_n)$ be a finitely generated ideal in H^∞ such that $Z(I) \subseteq G$. Then $f \in J$ implies that $f \in \overline{I}$.*

Proof. Just note that $E_k(I) = E_k(J)$ for every $k \in \mathbb{N}$ and use Theorem 2.12. ○

Remarks. (1) Bourgain gave an example of two Blaschke products B and C such that $BC \notin \overline{I(B^2, C^2)}$, although $|BC| \le |B^2| + |C^2|$. Thus the condition "$Z(I) \subseteq G$" cannot be removed to conclude that "$f \in J$ implies $f \in \overline{I}$".

(2) Theorem 2.2 in [15], which is a special case of Theorem 3.1, was proven by entirely different methods and applies only to powers of two interpolating Blaschke products. In addition, we obtain the following result mentioned in [15] as Remark 2:

Let B and C be two interpolating Blaschke products. Then

$$J(B^N, C^N) \subseteq \overline{I(B^N, C^N)} = \{f \in H^\infty : \text{ord }(f, m) \ge N \text{ for all } m \in Z(B) \cap Z(C)\}.$$

In the remainder of this section we consider products of ideals. The setting is the following: For $j = 1, \cdots, n$ let I_j be ideals in H^∞. The n-fold ideal product $\bigotimes_{j=1}^{n} I_j = I_1 \otimes \cdots \otimes I_n$ is defined to be the set of all finite sums $\sum_{k=1}^{K} f_{k,1} \cdots f_{k,n}$, where the $f_{k,j} \in I_j$ for $j = 1, \cdots, n$. These products play an important role in describing the ideals $J(f_1, \cdots, f_n)$. For example we have that $J(B^N, C^N) = \bigotimes_{j=1}^{N} I(B, C)$ whenever B and C are interpolating Blaschke products (see [15]).

In order to prove our next result, we begin with a result of Lingenberg. First recall that a set $E \subseteq M(H^\infty)$ is ρ-separated if $\rho(x, y) \ge \delta > 0$ for every $x, y \in E$, $x \ne y$.

Lemma 3.2 ([13], p. 59-60) *Let B be a Carleson-Newman Blaschke product and let E be a closed ρ-separated subset of $Z(B)$. Then there exists an interpolating factor b of B, such that $E \subseteq Z(b)$.*

Theorem 3.3 *Let I be a closed ideal in H^∞ satisfying*

$$(1) \quad Z(I) \subseteq G \text{ and } \qquad (2) \quad Z(I) \text{ is } \rho\text{-separated.}$$

Let

$$N = \sup\{\text{ord }(I, x) : x \in Z(I)\} < \infty.$$

Then

$$I = I(E_1(I), \cdots, E_N(I)) = \bigotimes_{j=1}^{N} I(E_j(I)).$$

Moreover the ideals $I(E_j(I))$ are generated by interpolating Blaschke products.

Proof. By Theorem 2.12 $I = I(E_1(I), \cdots, E_N(I))$. Let $f \in I$ be a Carleson-Newman Blaschke product. Since $E_1(I) = Z(I)$ is ρ-separated, by Lemma 3.2 there exists an interpolating Blaschke product b_1 with $Z(b_1) \supseteq E_1(I)$ such that $f = b_1 g_1$ for some Carleson-Newman Blaschke product g_1. Note that ord $(b_1, x) = 1$ for every $x \in Z(b_1)$. Since $f \in I$, we see that $E_2(f) \supseteq E_2(I)$. Hence $g_1(x) = 0$ for every $x \in E_2(I)$. Moreover $E_2(I)$ is a ρ-separated subset of $Z(g_1)$. Again, by Lemma 3.2, there exists an interpolating Blaschke product b_2 such that $g_1 = b_2 g_2$ with $Z(b_2) \supseteq E_2(I)$ for some Carleson-Newman Blaschke product b_2. After N-steps, we obtain N interpolating Blaschke products b_j so that $f = b_1 b_2 \cdots b_N g$ for some $g \in H^\infty$. Hence $f \in \prod_{j=1}^N I(E_j(I))$. By Theorem 2.9 I^* is generated by Carleson-Newman Blaschke products. Hence I is contained in the ideal generated by the set $\prod_{j=1}^N I(E_j(I))$. Thus $I \subseteq \bigotimes_{j=1}^{N} I(E_j(I))$.

Conversely, if $f \in \bigotimes_{j=1}^{N} I(E_j(I))$, then $f = \sum_{k=1}^{p} \prod_{j=1}^{N} f_{k,j}$ for some $f_{k,j} \in I(E_j(I))$. Since $E_1(f) \supseteq E_2(f) \supseteq \cdots \supseteq E_N(f)$, it immediately follows that $x \in E_\ell(I)$ we have
ord $(f, x) \geq \ell$. Hence $f \in I(E_1(I), \cdots, E_N(I))$.
We conclude that

$$I(E_1(I), \cdots, E_N(I)) = \bigotimes_{j=1}^{N} I(E_j(I)).$$

To prove the assertion about the generators, we note that $E_j(I)$ is a closed ρ-separated set contained in G. Moreover, $Z\big(I(E_j(I))\big) \subseteq Z(I) \subseteq G$. Hence, by ([13], Corollary 3.5 and Theorem 3.3) or ([11], p. 552), we have $E_j(I) \subseteq Z(b)$ for some interpolating Blaschke product b. By [14], $I(E_j(I))$ then is generated by interpolating Blaschke products. ○

Remark. In order to obtain the product representation of Theorem 3.3 we note that the assumption that $Z(I)$ be ρ-separated is necessary. To see this, let B and C be two interpolating Blaschke products such that $Z(B) \cup Z(C)$ is not ρ-separated. Let $I = BCH^\infty$. Then, trivially, I is closed with $E_1(I) = Z(B) \cup Z(C)$ and $E_2(I) = Z(B) \cap Z(C)$. Moreover $E_3(I) = \emptyset$. But as we show below, the ideal $I^* = I(E_1(I)) \bigotimes I(E_2(I))$ satisfies $E_3(I^*) \neq \emptyset$.

In fact, let $x_n \in Z(B), y_n \in Z(C)$ so that $\rho(x_n, y_n) \to 0, x_n \neq y_k$ for all $n, k \in \mathbb{N}$. Let $f \in I(E_1(I))$. Then $f(x_n) = f(y_n) = 0$. Hence, by ([12], p. 442), we have ord $(f, x) \geq 2$ for every cluster point x of $\{x_n : n \in \mathbb{N}\}$. By the lower semicontinuity of the pseudohyperbolic distance ρ, the point x also is a cluster point of the y_n. Therefore $x \in E_2(I)$ and so $x \in E_3(I^*)$.

We conclude this paper with the following open problems:

(1) Let I be an ideal in H^∞. Is $\prod_{j=1}^N I(E_j(I))$ an ideal? In other words, do we have

$$\prod_{j=1}^{N} I(E_j(I)) = \bigotimes_{j=1}^{N} I(E_j(I))?$$

By Hoffman's theory, we know that this is true if the $E_j(I)$ are singletons.

(2) Under the assumptions of Theorem 3.3 the ideal $\bigotimes_{j=1}^N I(E_j(I))$ is closed. Now let E_j be closed ρ-separated subsets of G, $(j = 1, \cdots, n)$. Is the product $\bigotimes_{j=1}^n I(E_j)$ then a closed ideal? If this were true, then

$$\bigotimes_{j=1}^{n} I(E_j) = I(S_1, \cdots, S_n),$$

where

$$S_1 = \bigcup_{1\le j\le n} E_j,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_p = \bigcup_{1\le j_1<\cdots<j_p\le n} (E_{j_1} \cap \cdots \cap E_{j_p}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_n = \bigcap_{1\le j\le n} E_j,$$

are the higher order hulls.

In the special case that $E_j = Z(B_j) \cap Z(C_j) \quad (j = 1, 2)$ for interpolating Blaschke products B_j and C_j, the question has an affirmative answer. In fact there we have :

$$\overline{I(B_1B_2, B_1C_2, C_1B_2, C_1C_2)} = \overline{I(B_1, C_1) \otimes I(B_2, C_2)} =$$
$$= I(E_1) \otimes I(E_2) = I(E_1 \cup E_2, E_1 \cap E_2).$$

References

[1] Axler, S; Zhu, K.: *Boundary behaviour of derivatives of analytic functions*, Mich. Math. J. 39 (1992), 129-143.

[2] Bourgain, J.: *On finitely generated closed ideals in $H^\infty(\mathbb{D})$*, Ann. Inst. Fourier (Grenoble) 35 (1985), 163-174.

[3] Budde, P.: *Support sets and Gleason parts*, Mich. Math. J. 37 (1990), 367-383.
[4] Gamelin, T.: *Uniform Algebras*, 2nd ed., Chelsea, New York, 1984.
[5] Garnett, J.B.: *Bounded Analytic Functions* , Academic Press, New York 1981.
[6] Gorkin, P.; Mortini, R.: *Interpolating Blaschke products and factorization in Douglas algebras*, Mich. J. Math. 38 (1991), 147-160.
[7] Gorkin, P.; Mortini, R.; Nicolau, A.: *The generalized corona theorem*, Math. Annalen 301 (1995), 135-154.
[8] Guillory, C.; Izuchi, K.; Sarason, D.: *Interpolating Blaschke products and division in Douglas algebras*, Proc. Roy. Irish Acad. 84 A (1984), 1-7.
[9] Hoffman, K.: *Bounded analytic functions and Gleason parts*, Ann. Math. 86 (1967), 74-111.
[10] Izuchi, K.: *Zero sets of interpolating Blaschke products*, Pacific J. Math. 119 (1985), 337-342.
[11] Izuchi, K.: *Interpolating Blaschke products and factorization theorems* J. London Math. Soc. 50 (1994), 547-567.
[12] Izuchi, K.; Izuchi, Y.: *Inner functions and division in Douglas algebras*, Mich. Math. J. 33 (1986), 435-443.
[13] Lingenberg, H.-M.: *Interpolating sets in the maximal ideal space of* H^∞, Mich. Math. J. 39 (1992), 53-63.
[14] Mortini, R.: *Ideals generated by interpolating Blaschke products*, Analysis 14 (1994), 67-74.
[15] Mortini, R.: *On an example of J. Bourgain on closures of finitely generated ideals in* H^∞, to appear in Math. Z. .
[16] Tolokonnikov, V.: *Interpolating Blaschke products and ideals of the algebra* H^∞, Sov. J. Math. 27 (1984), 2549-2553.
[17] Tolokonnikov, V.: *Blaschke products satisfying the Carleson-Newman condition and ideals of the algebra* H^∞, Sov. J. Math. 42 (1988), 1603-1610.

Pamela Gorkin
Department of Mathematics
Bucknell University
Lewisburg, Pennsylvania

PA 17837
USA

pgorkin@bucknell.edu

Raymond Mortini
Dpartement de Mathmatiques
Universit de Metz
Ile du Saulcy

F-57045 Metz
France

mortini@poncelet.univ-metz.fr