SOLUTION TO PROBLEM 4816 CRUX MATH. 49 (1) 2023, 101

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4816. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $a, b, k \geq 0$. Calculate

$$\lim_{n\to\infty} \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2 x^{2n}}\, \mathrm{d}x.$$

We show that for $a, b, k \geq 0$ (k not necessary an integer)

$$I_n := \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2 x^{2n}}\, dx \overset{n\to\infty}{\longrightarrow} \sqrt{b} + \frac{\sqrt{a}}{k+1/2}.$$

Write

$$f_n(x) = x^{k-1/2}\sqrt{a + bn^2 x^{2n+1}}.$$

If $a = 0$, then

$$I_n = \int_0^1 \sqrt{b} n x^{n+k} dx = \frac{n\sqrt{b}}{n+k+1} \to \sqrt{b}.$$

For $a > 0$, let

$$d_n(x) := x^{k-1/2}\Big(\sqrt{a + bn^2 x^{2n+1}} - \sqrt{bn^2 x^{2n+1}}\Big).$$

Then

$$0 \leq d_n(x) = x^{k-1/2}\,\frac{a}{\sqrt{a + bn^2 x^{2n+1}} + \sqrt{b} n x^{n+1/2}} \leq \frac{a}{\sqrt{a}} x^{k-1/2}.$$

Hence d_n is dominated by an $L^1[0,1]$ function and so, by using that $nx^n \to 0$ for $0 < x < 1$,

$$\lim_n \int_0^1 d_n(x)dx = \int_0^1 \lim_n d_n(x)dx = \int_0^1 \sqrt{a} x^{k-1/2} = \frac{\sqrt{a}}{k+1/2}.$$

Consequently,

$$\begin{aligned}
\int_0^1 f_n(x)dx &= \int_0^1 d_n(x)dx + \sqrt{b}\int_0^1 n x^{k-1/2} x^{n+1/2} dx \\
&= \int_0^1 d_n(x)dx + \sqrt{b}\,\frac{n}{k+n+1} \\
&\underset{n\to\infty}{\longrightarrow} \frac{\sqrt{a}}{k+1/2} + \sqrt{b}.
\end{aligned}$$

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