Real and Complex Analysis

Solutions to Problems in

Amer. Math. Monthly
Math. Magazine
College Math. J.
Elemente der Math
Crux Math.
EMS Newsletter
Math. Gazette

Edited by

Raymond Mortini

14.9.2025

In this arxiv-post I present my solutions (published or not) to Problems that appeared in Amer. Math. Monthly, Math. Magazine, Elemente der Mathematik and CRUX, that were mostly done in collaboration with Rudolf Rupp. Some of them (including a few own proposals which were published) were also done in cooperation with Rainer Brück, Bikash Chakraborty, Pamela Gorkin, Gerd Herzog, Jérôme Noël, Peter Pflug, Amol Sasane and Robrto Tauraso.

A few of these contributions to "Recreational Mathematics" actually were the base for interesting generalizations that led to some of my publications (partially co-authored) in research journals ([4, 5, 6, 7, 8, 9, 10, 11]).

The content will surely be attractive to all undergraduate/graduate students in Mathematics who want to solve challenging problems in Analysis by calculating explicit values of funny looking integrals, sums and products, by deriving astonishing inequalities and by solving functional equations. It is also a valuable source for teachers in mathematics in preparing exercise sheets for their students. Moreover, I think that it is worth to see in most cases quite different solutions than those already published in the above listed journals.

My main reason to post this collection of (mainly unpublished solutions), is to keep also for future generations an archive for historians in Mathematics, interested in the work of one of the very few mathematicians with Luxembourgish Nationality. Without this digital archiving, these contributions to education and science would for ever disappear in a few years.

Some technical remarks: The first items for each journals are still hidden, as the submission deadline has not yet occurred. Regular updates are planned. Own proposals are presented with a yellow background.

We obtained permission from the MAA, the EMS, Math. Gazette and CRUX to post a scan of the original statements and to reproduce our published (and non-published) solutions.

14.9.2025

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Aufgabe 901, Elemente Math. **38** (1983), 128.	solution: Elemente Math. **39** (1984), 130-132.
Problem E 3325, Amer. Math. Monthly **96** (1989), 445.	solution: Amer. Math. Monthly **97** (1990), 855-856.
Aufgabe Q68, EMS Newsletter **23** (1997), 21.	solution: EMS Newsletter **25** (1997), 27.
Problem 11684 Amer. Math. Monthly **120** (2013), 76.	solution: Amer. Math. Monthly **122** (2015), 80-81 (with R. Rupp).
Problem 4816 Crux **49** (2023), 101.	solution: Crux **49** (2023), 392-393 (with R. Rupp).
Problem 12290 Amer. Math. Monthly **128** (2021), 946.	solution: Amer. Math. Monthly **130** (2023), 774 (with R. Rupp).
Problem 4835, Crux Mathematicorum 49 (2023), 213.	solution: Crux Mathematicorum 49 (2023), 502-503 (with R. Rupp)
Problem 12308, Amer. Math. Monthly 129 (2022), 285.	solution: Amer. Math. Monthly 131 (2024), 77-78
Problem 4825, Crux Mathematicorum 49 (2023), 157.	solution: Crux Mathematicorum 49 (2023), 449-451 (with R. Rupp).
Problem 4896, Crux Mathematicorum 49 (2023), 540.	solution: Crux Mathematicorum 50 (2024), 269-270 (with R. Rupp)
Problem 4909, Crux Mathematicorum 50 (2024), 38.	solution: Crux Mathematicorum 50 (2024), 330-333 (with R. Rupp)
Problem 4910, Crux Mathematicorum 50 (2024), 38.	solution: Crux Mathematicorum 50 (2024), 333-337 (with R. Rupp)
Problem 4914, Crux Mathematicorum 50 (2024), 82.	solution: Crux Mathematicorum 50 (2024), 378-379 (with R. Rupp)
Problem 4918, Crux Mathematicorum 50 (2024), 83.	solution: Crux Mathematicorum 50 (2024), 384 (with R. Rupp)
Problem 2193, Math. Mag. 97 (2024), 223. (with R. Rupp)	solution: Math. Mag 98 (2025), 151-153.

published proposals

Problem 10890, Amer. Math. Monthly **108** (2001), 668.	solution: Amer. Math. Monthly **110** (2003), 62-63.
Problem 10991, Amer. Math. Monthly **110** (2003), 155.	solution: Amer. Math. Monthly **111** (2004), 826-827.
Problem 11136, Amer. Math. Monthly **112** (2005), 181.	solution: Amer. Math. Monthly **113** (2006), 763-764.
Problem 11147, Amer. Math. Monthly **112** (2005), 366. (with P. Gorkin)	solution: Amer. Math. Monthly **113** (2006), 854.
Problem 11185, Amer. Math. Monthly **112** (2005), 840. (with R. Brueck)	solution: Amer. Math. Monthly **114** (2007), 552-554.
Problem 11456, Amer. Math. Monthly **116** (2009), 747.	solution: Amer. Math. Monthly **118** (2011), 185.
Aufgabe 1281, Elemente der Mathematik **65** (2010), 127.	solution: Elemente der Mathematik **66** (2011), 133-134.
Problem 11584, Amer. Math. Monthly **118** (2011), 558. (with J. Noel)	solution: Amer. Math. Monthly **120** (2013), 661-662.
Problem 11684, Amer. Math. Monthly **120** (2013), 76. (with R. Rupp)	solution: Amer. Math. Monthly **122** (2015), 80-81.
Problem 1947, Math. Magazine **87** (2014), 230. (with. J. Noel)	solution: Math. Magazine **88** (2015), 288.
Aufgabe 1350, Elemente der Mathematik **71** (2016), 84	solution: Elemente der Mathematik **72** (2017), 84-85.
Quicky 1075 Math. Magazine **90** (2017), 384	solution: Math. Magazine **90** (2017), 393
Aufgabe 1383 Elemente der Mathematik **74** (2019), 38 (with R. Rupp)	solution: Elemente der Mathematik **75** (2020), 38-41.
Aufgabe 1431 Elemente der Mathematik **78** (2023), 44 (with R. Rupp)	solution: Elemente der Mathematik **79** (2024),39-42
Problem 12406 Amer. Math. Monthly **130** (2023), 679 (with R. Rupp)	solution: Amer. Math. Monthly
Aufgabe 1438 Elemente der Mathematik **78** (2023), 135 (with R. Rupp)	solution: Elemente der Mathematik 79 (2024), 134-135.
Problem 2181 Math. Magazine **96** (2023), 566 (with P. Pflug and R. Rupp)	solution: Math. Mag. 97 (2024), 576-577.
Aufgabe 1443 Elemente der Mathematik **79** (2024), 38	solution: Elemente der Mathematik 80 (2025), 39-40.
Quicky 1140 Math. Magazine **97** (2024), 224 (with R. Rupp)	solution: Math. Magazine **97** (2024), 233
Aufgabe 1449 Elemente der Mathematik **79** (2024), 131	solution: Elemente der Mathematik
Problem 12487 Amer. Math. Monthly**131** (2024), 723	solution: Amer. Math. Monthly
Aufgabe 1455 Elemente der Mathematik **80** (2024), 38. (with R. Rupp)	solution: Elemente der Mathematik

See [12] for the whole bibliographic list.

1. American Math. Monthly

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12550. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*
(a) Evaluate $\lim_{n\to\infty} \int_0^{\pi/2} \left|\sin^n(x) - \cos^n(x)\right|^{1/n} \, dx$.
(b) Evaluate

$$\lim_{n\to\infty} n^2 \left(L - \int_0^{\pi/2} \left|\sin^n(x) - \cos^n(x)\right|^{1/n} \, dx \right),$$

where L is the limit in **(a)**.

Solution to problem 12550 in Amer. Math. Monthly 132 (2025), p. ?

Raymond Mortini and Rudolf Rupp

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12540. *Proposed by Robert Dragomirescu, Stanford University, Stanford, CA, and Cezar Lupu, Tsinghua University, Beijing, China.* What is the maximum value of

$$\int_0^1 x^3 f(x)^2\,dx - \int_0^1 x^2 f(x)^3\,dx$$

over all continuous functions $f : [0,1] \to [0,\infty)$?

Solution to problem 12540 in Amer. Math. Monthly 132 (2025), p. 592

Raymond Mortini and Rudolf Rupp

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12537. *Proposed by Jinhai Yan, Fudan University, Shanghai, China.* Let $f(x)$ be a continuous real-valued function on $[0, \infty)$, differentiable on $(0, \infty)$, and satisfying $f(0) \geq 0$ and $f'(x) \geq (f(x))^3$ for $x > 0$. Prove that $f(x) = 0$ for all $x \geq 0$.

Solution to problem 12537 in Amer. Math. Monthly 132 (2025), p. ?

Gerd Herzog, Raymond Mortini and Rudolf Rupp

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12535. *Proposed by Necdet Batır, Nevşehir Haci Bektaş Veli University, Nevşehir, Turkey.*
Let x be a real number and n a nonnegative integer. Prove

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+kx}{n}=x^n.$$

Solution to problem 12535 in Amer. Math. Monthly 132 (2025), p. ?

Raymond Mortini and Rudolf Rupp

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12534. *Proposed by Marián Štofka, Bratislava, Slovakia.* Prove

$$\int_0^1 \frac{1}{x}\Big(6(\ln(1+x)\ln(1-x))^2 + (\ln(1+x))^4\Big)\,dx = \frac{21}{4}\zeta(5),$$

where ζ is the Riemann zeta function.

Solution to problem 12534 in Amer. Math. Monthly 132 (2025), p. ?

Raymond Mortini and Rudolf Rupp

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12527. *Proposed by Said Attaoui, University of Science and Technology of Oran—Mohamed Boudiaf, Oran, Algeria.* Prove

$$\int_0^{\pi/2} \frac{\tanh\left(\tan^2\theta\right)}{\sin(2\theta)\left(1+\cosh\left(2\tan^2\theta\right)\right)}\,d\theta = \frac{7\zeta(3)}{8\pi^2},$$

where $\zeta(3)$ is Apéry's constant.

Solution to problem 12527 in Amer. Math. Monthly 132 (2025), p. ?

Raymond Mortini and Rudof Rupp

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12521. *Proposed by Marián Štofka, Bratislava, Slovakia.* Let a be a real number greater than 1, and let m be a nonnegative integer. Prove

$$\int_0^\infty \frac{(\ln x)^m}{1+x^a}\,dx = \left(\frac{\pi}{a}\right)^{m+1}\left[\frac{d^m}{dx^m}\csc(x)\right]_{x=\pi/a}.$$

Solution to problem 12521 in Amer. Math. Monthly 132 (2025), p. ?

Raymond Mortini and Rudof Rupp

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12518. *Proposed by Paolo Perfetti, Tor Vergata University of Rome, Rome, Italy.* Evaluate

$$\lim_{n\to\infty} n \sin\left(4\sum_{k=1}^{6n} \arctan\left(\frac{\left(\sqrt{3}+1\right)k^2 - 2\left(\sqrt{3}+1\right)k + 3\sqrt{3} - 1}{\left(\sqrt{3}-1\right)k^2 - 2\left(\sqrt{3}-1\right)k - 3 - \sqrt{3}}\right)\right),$$

where the limit is over integer values of n.

Solution to problem 12518 in Amer. Math. Monthly 132 (2025), p. ?

Raymond Mortini and Rudof Rupp

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12515. *Proposed by Robert Smyth, Cooper Union, New York, NY.* Determine the set of accumulation points of the sequence $\{(\sin n)^n\}_{n=1}^{\infty}$.

Solution to problem 12515 in Amer. Math. Monthly 132 (2025), p. 181

Raymond Mortini and Myriam Ounaies

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12510. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $c > 0$ and $n \geq 1$, let

$$R_n(c) = \sqrt{c + \sqrt{c + \sqrt{c + \cdots + \sqrt{c}}}}$$

with n radicals. Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \prod_{j=k}^{n} \frac{1}{R_j(c)}.$$

Solution to problem 12510 in Amer. Math. Monthly 132 (2025), p. 180

Raymond Mortini and Rudolf Rupp

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12509. *Proposed by Lawrence Glasser, Clarkson University, Potsdam, NY.* Let n be a nonnegative integer and a a positive real number. Prove

$$\int_0^\infty \frac{\ln(ax)}{\prod_{k=0}^n \left(x^2+(2k+1)^2\right)}\,dx = \frac{\pi}{2^{2n+1}}\sum_{k=0}^n \frac{(-1)^k}{(n+k+1)!(n-k)!}\ln\bigl((2k+1)a\bigr).$$

Solution to problem 12509 in Amer. Math. Monthly 132 (2025), p. 90

Raymond Mortini and Rudolf Rupp

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12501. *Proposed by Hervé Grandmontagne, Paris, France.* Prove

$$\int_0^\infty \frac{\left(\ln(x/(1+x))\right)^4 \ln\left(x^3(1+x)^{17}\right)}{1+x}\,dx = -240\,\zeta(3)^2,$$

where $\zeta(3)$ is Apéry's constant $\sum_{n=1}^\infty 1/n^3$.

Solution to problem 12501 in Amer. Math. Monthly 131 (2024), p. 906

Raymond Mortini and Rudolf Rupp

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Substituting $x/(1+x) =: t$, equivalently $x = t/(1-t)$ with $t \in]0,1[$ yields

$$\begin{aligned} I := \int_0^\infty \left(\log\frac{x}{1+x}\right)^4 \log\left(x^3(1+x)^{17}\right)\,dx &= \int_0^1 \frac{\log^4 t\left(3\log\left(\frac{t}{1-t}\right)+17\log\left(\frac{1}{1-t}\right)\right)}{\frac{1}{1-t}}\,\frac{dt}{(1-t)^2}\\ &= \int_0^1 \frac{\log^4 t\Big(3\log t - 20\log(1-t)\Big)}{1-t}\,dt\\ &= 3\int_0^1 \frac{\log^5 t}{1-t}\,dt - 20\int_0^1 \frac{\log^4 t\,\log(1-t)}{1-t}\,dt\\ &=: 3I_1 - 20 I_2. \end{aligned}$$

Now, as the integrands have constant sign, $\int\sum = \sum\int$, and so, using 5-times integration by parts,

$$\begin{aligned} I_1 &= \int_0^1 \sum_{n=0}^\infty t^n \log^5 t\,dt = \sum_{n=0}^\infty \int_0^1 t^n\,\log^5 t\,dt\\ &= -\sum_{n=0}^\infty \frac{5!}{(n+1)^6} = -120\zeta(6). \end{aligned}$$

Let $H_k = 1+\frac12+\frac13+\cdots+\frac1k$ denote the k-th harmonic number. Using that for $0\le x<1$,

$$\frac{\log(1-x)}{1-x} = -\sum_{k=1}^\infty H_k\,x^k$$

(Cauchy-product), we obtain

$$\begin{aligned} I_2 &= -\int_0^1 \log^4 t\sum_{k=1}^\infty H_k t^k\,dt = -\sum_{k=1}^\infty \int_0^1 t^k \log^4 t\,dt\\ &= -4!\sum_{k=1}^\infty \frac{H_k}{(k+1)^5}. \end{aligned}$$

Due to Euler's formula (see [23, p. 416], [24]),

$$2\sum_{k=1}^\infty \frac{H_k}{(k+1)^n} = n\zeta(n+1) - \sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1), \quad (n\ge 2),$$

we conclude that

$$\begin{aligned} \sum_{k=1}^\infty \frac{H_k}{(k+1)^5} &= \frac12\left(5\,\zeta(6) - \zeta(4)\zeta(2) - \zeta(3)^2 - \zeta(2)\zeta(4)\right)\\ &= \frac52\zeta(6) - \zeta(4)\zeta(2) - \frac12\zeta(3)^2. \end{aligned}$$

Since $\zeta(6) = \frac{\pi^6}{945}$ and $\zeta(4)\zeta(2) = \frac{\pi^4}{90}\,\frac{\pi^2}{6} = \frac{\pi^6}{540} = \frac{945}{540}\zeta(6) = \frac74\zeta(6)$, we deduce that

$$\begin{aligned} I = 3I_1 - 20I_2 &= -360\zeta(6) + 20 \cdot 24\Big(\frac{5}{2}\zeta(6) - \zeta(4)\zeta(2) - \frac{1}{2}\zeta(3)^2\Big) \\ &= \zeta(6)\Big(- 360 + 1200 - 840\Big) - 240\zeta(3)^2 \\ &= -240\zeta(3)^2. \end{aligned}$$

12494. *Proposed by Joseph Santmyer, Las Cruces, NM.* Let ζ be the Riemann zeta function, H_r the harmonic number $\sum_{k=1}^{r} 1/k$, and $H_r^{(2)} = \sum_{k=1}^{r} 1/k^2$. Prove

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2n+mn^2+rmn} = \begin{cases} 2\zeta(3), & \text{if } r=0;\\ \frac{1}{r}\left((H_r)^2+H_r^{(2)}\right), & \text{if } r \text{ is a positive integer.}\end{cases}$$

Solution to problem 12494 in Amer. Math. Monthly 131 (2024), p. 815

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Let

$$S := S(r) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2n+mn^2+rmn}.$$

We additionally show that for $r \in \mathbb{N} = \{0,1,2,3,\dots\}$

$$S(r) = \int_0^1 x^{r-1}\log^2(1-x)dx = \begin{cases} 2\zeta(3) & \text{if } r=0\\ \displaystyle\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\frac{2}{(j+1)^3} & \text{if } r\neq 0.\end{cases}$$

If, more generally, $r \geq 0$ is a real number, then this formula also holds in the form

$$S(r) = \sum_{j=0}^{\infty}\binom{r-1}{j}(-1)^j\frac{2}{(j+1)^3}.$$

1.1. **The calculations.** Let $m,n \in \mathbb{N}^* := \mathbb{N}\setminus\{0\}$ and $r \in \mathbb{R}, r \geq 0$. Then

$$\begin{aligned}\frac{1}{m^2n+mn^2+rmn} &= \frac{1}{mn}\,\frac{1}{m+n+r} = \frac{1}{m(m+r)}\left(\frac{1}{n}-\frac{1}{n+(m+r)}\right)\\ &= \frac{1}{m(m+r)}\int_0^1 (x^{n-1}-x^{n+m+r-1})dx\\ &= \frac{1}{m(m+r)}\int_0^1 x^{n-1}(1-x^{m+r})dx.\end{aligned}$$

Since all terms considered are positive, $\int\sum = \sum\int$. Hence, by partial integration,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{m^2n+mn^2+rmn} &= \frac{1}{m(m+r)}\int_0^1 (1-x^{m+r})\sum_{n=1}^{\infty}x^{n-1}dx \qquad (1)\\ &= \frac{1}{m(m+r)}\int_0^1\frac{1-x^{m+r}}{1-x}dx\\ &= -\frac{1}{m}\int_0^1 x^{m+r-1}\log(1-x)dx\end{aligned}$$

Consequently, if $r \geq 0$, $r \in \mathbb{R}$,

$$\begin{aligned}S(r) &= -\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 x^{m+r-1}\log(1-x)dx\\ &= -\int_0^1 x^{r-1}\left(\sum_{m=1}^{\infty}\frac{x^m}{m}\right)\log(1-x)dx = \int_0^1 x^{r-1}\log^2(1-x)dx \qquad (2)\\ &= \int_0^1 (1-x)^{r-1}\log^2 x\,dx. \qquad (3)\end{aligned}$$

Hence, if $r \in \mathbb{N}^*$,

$$\begin{aligned} S(r) &= \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \int_0^1 x^j \log^2 x \, dx \\ &= \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \frac{2}{(j+1)^3}. \end{aligned}$$

where the latter identity is shown using twice partial integration.

If $r = 0$, we have the convergent integral (two singularities),

$$\begin{aligned} S &= \int_0^1 (1-x)^{-1} \log^2 x \, dx = \int_0^1 \sum_{j=0}^{\infty} x^j \log^2 x \, dx = \sum_{j=0}^{\infty} \int_0^1 x^j \log^2 x \, dx \\ &= \sum_{j=0}^{\infty} \frac{2}{(j+1)^3} = 2\zeta(3). \end{aligned}$$

If $r \in \mathbb{R}$ and $r > 0$,

$$\begin{aligned} S(r) &= \sum_{j=0}^{\infty} \binom{r-1}{j} (-1)^j \int_0^1 x^j \log^2 x \, dx \\ &= \sum_{j=0}^{\infty} \binom{r-1}{j} (-1)^j \frac{2}{(j+1)^3}. \end{aligned}$$

Next, if $r \in \mathbb{N}^*$, we derive the desired equality from 2. We do this using the following trick: consider the function

$$f(a) := (1-x)^{a-1}, a > 0.$$

Then

$$\frac{d^2}{da^2}(1-x)^{a-1} = \frac{d}{da}(1-x)^{a-1} \log(1-x) = (1-x)^{a-1} \log^2(1-x).$$

In particular,

$$\log^2(1-x) = \frac{d^2}{da^2}(1-x)^{a-1}\Big|_{a=1}.$$

For $a > 0$, let

$$G(a) := \int_0^1 x^{r-1} \frac{d^2}{da^2}(1-x)^{a-1} dx.$$

The continuity of G on $]0,\infty[$ and $(x,a) \mapsto x^{r-1}\frac{d^2}{da^2}(1-x)^{a-1}$ on $]0,1[\times]0,\infty[$ yields that

$$\lim_a \int = \int \lim_a .$$

Hence

$$\begin{aligned} G(1) &= \lim_{a\to 1} \int_0^1 x^{r-1} \frac{d^2}{da^2}(1-x)^{a-1} dx = \int_0^1 x^{r-1} \lim_{a\to 1} \frac{d^2}{da^2}(1-x)^{a-1} dx \\ &= \int_0^1 x^{r-1} \log^2(1-x) dx. \end{aligned}$$

Now, in view of the definition of the Eulerian beta-function, $B(r,a)$, and the fact that $\frac{d^2}{da^2}\int = \int \frac{d^2}{da^2}$

$$\begin{aligned} G(a) &= \frac{d^2}{da^2}\int_0^1 x^{r-1}(1-x)^{a-1}dx \\ &= \frac{d^2}{da^2}B(r,a) = \frac{d^2}{da^2}\frac{\Gamma(r)\Gamma(a)}{\Gamma(a+r)} \\ &= \frac{d^2}{da^2}\left(\frac{(r-1)!}{a(a+1)\cdots(a+r-1)}\right). \end{aligned}$$

For $a > 0$, put

$$L(a) := \log \frac{1}{\prod_{j=0}^{r-1}(a+j)} = -\sum_{j=0}^{r-1}\log(a+j).$$

Then

$$F(a) := \frac{B(r,a)}{(r-1)!} = e^{L(a)}.$$

Hence

$$G(1) = (r-1)!F''(1) = (r-1)!F(1)\big((L'(1))^2 + L''(1)\big) = \frac{(r-1)!}{r!}\big((H_r)^2 + H_r^{(2)}\big).$$

Remark. Using that for $r \in \mathbb{N}^*$

$$\begin{aligned} \frac{1}{m(m+r)}\int_0^1 \frac{1-x^{m+r}}{1-x}dx &= \frac{1}{m(m+r)}\int_0^1 \sum_{k=0}^{m+r-1} x^k dx \\ &= \frac{H_{m+r}}{m(m+r)}, \end{aligned}$$

we also have that, in view of (1), $S(r) = \sum_{m=1}^{\infty}\frac{H_{m+r}}{m(m+r)}$.

12490. *Proposed by Cezar Lupu, Tsinghua University, Beijing, China, and Tudorel Lupu, Decebal High School, Constanța, Romania.*
(a) Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function that is nondecreasing. Prove

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} f(x)\, dx \leq 0.$$

(b) Let $g : [0, 1] \to \mathbb{R}$ be a twice continuously differentiable function that is convex. Prove

$$\sum_{n=1}^{\infty} \int_0^1 \cos(2\pi n x) g(x)\, dx \geq 0.$$

Solution to problem 12490 in Amer. Math. Monthly 131 (2024), p. 814

Raymond Mortini and Rudolf Rupp

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a1) We prove that $\sum \int$ is well defined and that

$$\sum_{n=1}^{\infty} \int_0^1 \frac{\sin(2\pi nx)}{n}\, f(x)\, dx \leq 0.$$

As $f' \geq 0$, partial integration yields

$$\begin{aligned} I_n := \int_0^1 \frac{\sin(2\pi nx)}{n}\, f(x)\, dx &= -\frac{\cos(2\pi nx)}{2\pi n^2} f(x)\Big|_0^1 + \int_0^1 \frac{\cos(2\pi nx)}{2\pi n^2} f'(x)dx \\ &\leq \frac{f(0) - f(1)}{2\pi n^2} + \frac{1}{2\pi n^2} \int_0^1 f'(x)dx = 0. \end{aligned}$$

Since $|I_n| \leq \frac{c}{n^2}$ the series converges $S := \sum I_n$ and so $S \leq 0$.

a2) We prove that $\int \sum$ is well defined and that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n}\, f(x) \right) dx \leq 0.$$

A standard exercise in Fourier analysis tells us

$$S(x) := \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} = \frac{\pi - 2\pi x}{2} \text{ for } 0 < x < 1.$$

Since $f' \geq 0$,

$$\begin{aligned} \int_0^1 S(x) f(x) dx &= \frac{\pi}{2} \left(\int_0^1 f(x)dx - 2\int_0^1 x f(x) dx \right) \\ &= \frac{\pi}{2} \left(x f(x)\Big|_0^1 - \int_0^1 x f'(x)dx - x^2 f(x)\Big|_0^1 + \int_0^1 x^2 f'(x)dx \right) \\ &= \frac{\pi}{2} \left((x - x^2) f(x)\Big|_0^1 + \int_0^1 (x^2 - x) f'(x)dx \right) \\ &= \frac{\pi}{2} \int_0^1 (x^2 - x) f'(x)dx \leq 0. \end{aligned}$$

b) Since $g'' \geq 0$,

$$\begin{aligned} J_n &= \int_0^1 \cos(2\pi nx) g(x)dx = -\frac{1}{2\pi n}\int_0^1 \sin(2\pi nx) g'(x)dx \\ &= \frac{1}{4\pi^2 n^2}\cos(2\pi nx)g'(x)\Big|_0^1 - \frac{1}{4\pi^2 n^2}\int_0^1 \underbrace{\cos(2\pi nx)}_{\leq 1} g''(x)dx \\ &\geq \frac{g'(1)-g'(0)}{4\pi^2 n^2} - \frac{g'(1)-g'(0)}{4\pi^2 n^2} = 0. \end{aligned}$$

Since $|J_n| \leq \frac{c}{n^2}$, the series $S^* := \sum J_n$ converges and so $S^* \geq 0$.

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Second proof of b)

Extend g 1-periodically to $\mathbb{R}$. The Fourier series for g is given by

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(2\pi nx) + b_n \sin(2\pi nx)\right)$$

with $a_n = 2\int_0^1 g(x)\cos(2\pi nx)dx$ and $b_n = 2\int_0^1 g(x)\sin(2\pi nx)dx$. Since g is smooth, Dirichlet's rule yields

$$\frac{g(0)+g(1)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = \int_0^1 g(x)dx + 2\sum_{n=1}^{\infty}\int_0^1 \cos(2\pi nx)g(x)dx. \tag{4}$$

Now $\frac{g(0)+g(1)}{2} \geq \int_0^1 g(x)dx$ since the convexity of g implies $g(x) \leq g(0) + x(g(1)-g(0))$ and so

$$\int_0^1 g(x)dx \leq g(0)\cdot 1 + \frac{1}{2}(g(1)-g(0)) = \frac{g(0)+g(1)}{2}.$$

Hence, in view of (4), $\displaystyle\sum_{n=1}^{\infty}\int_0^1 \cos(2\pi nx)g(x)dx \geq 0$.

Second proof of a)

This follows from b): Given f in a), put $g(x) := \int_0^x f(t)dt$. Then $g'' \geq 0$; hence g is convex. Now, by partial integration,

$$\begin{aligned} \int_0^1 g(x)\cos(2\pi nx)dx &= g(x)\frac{\sin(2\pi nx)}{2\pi n}\Big|_0^1 - \int_0^1 \frac{\sin(2\pi nx)}{2\pi n}g'(x)dx \\ &= -\int_0^1 \frac{\sin(2\pi nx)}{2\pi n}f(x)dx. \end{aligned}$$

Thus b) implies a).

12487. *Proposed by Raymond Mortini, University of Luxembourg, Esch-sur-Alzette, Luxembourg.* Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$ and assume that either $f(0) = f(1)$ or $f(1)$ is not an extremal value for f.
(a) Prove that for any real λ except $\lambda = 1$, there are distinct $c_1, c_2 \in (0, 1)$ such that $f'(c_1) = \lambda f'(c_2)$.
(b) Prove that, whenever $0 \le s_1 < s_2 < s_3 < s_4$, there are distinct $c_1, c_2, c_3, c_4 \in (0, 1)$ such that $s_1 f'(c_1) + s_2 f'(c_2) + s_3 f'(c_3) + s_4 f'(c_4) = 0$.

Solution to problem 12487 in Amer. Math. Monthly 131 (2024), p. 723

Raymond Mortini

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(a) Suppose that f is not constant. Due to the hypotheses, f takes at least one of its extrema inside $]0, 1[$. By Rolle's theorem, there is $x_0 \in]0, 1[$ for which $f'(x_0) = 0$. Now f' must also take negative as well as positive values, since otherwise f is decreasing, respectively increasing, on $[a, b]$, contradicting the assumption that $f(0) = f(1)$ or that $f(1)$ is not an extremal value. Say $f'(x_1) < 0$ and $f'(x_2) > 0$. By the intermediate value property for the derivative (or Darboux property, see [25, Theorem 5.12, p.108])), f' takes every value η with $f'(x_1) < \eta < f'(x_2)$. In particular, every value η with small modulus is taken, say $|\eta| \le \varepsilon$. Now let $r_1 \neq r_2$ be taken so that $|r_j| \le \varepsilon$ and $r_1/r_2 = \lambda$ (this is possible: for instance, if $|\lambda| > 1$, choose $r_1 = \varepsilon$ and $r_2 = \varepsilon/\lambda$, and if $|\lambda| < 1$, choose $r_1 = \lambda\varepsilon$ and $r_2 = \varepsilon$). Now let $c_j \in]0, 1[$ be such that $f'(c_j) = r_j$. Then $c_1 \neq c_2$ and

$$f'(c_1)/f'(c_2) = r_1/r_2 = \lambda.$$

(b) Let $0 < s_1 < s_2 < s_3 < s_4$. By i) there are $c_1, c_2 \in]0, 1[$ with

$$f'(c_1)/f'(c_2) = \lambda := -s_2/s_1 = r_1/r_2$$

and $c_3, c_4 \in]a, b[$ with

$$f'(c_3)/f'(c_4) = \lambda := -s_4/s_3 = r_3/r_4.$$

Since the modulus of r_j can be taken to be arbitrarily small, the proof above guarantees that all the c_j are distinct. We conclude that $\sum_{j=1}^4 s_j f'(c_j) = 0$.

If $s_1 = 0$, we choose c_3, c_4 as above, then $c_2 := x_0$ and c_1 arbitrary, but different from c_2, c_3, c_4.

Remark This was motivated by problem 4779 Crux Math. 48 (8) 2022, 484.

12481. *Proposed by Bernhard Elsner, Université de Versailles Saint-Quentin-en-Yvelines, Versailles, France, and Eric Müller, Villingen-Schwenningen, Germany.* Let $f_1, \dots, f_n$ be holomorphic functions on U, where U is an open, connected subset of $\mathbb{C}$. Suppose that the function $g : U \to \mathbb{R}$ given by $g(z) = |f_1(z)| + \cdots + |f_n(z)|$ takes a maximum value in U. Must each function f_k be constant on U?

Solution to problem 12481 in Amer. Math. Monthly 131 (2024), p. 630

Raymond Mortini, Peter Pflug and Rudolf Rupp

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The answer is yes, and this is an easy classical exercise in a complex analysis course; see [27, p. 168], [28, p. 353], [26, p. 161], [30, p. 164, Ex. 300/302], [32, p. 41], [33]. Since the modulus of a holomorphic function is subharmonic, it satisfies the maximum principle. Thus, if the maximum value of $f \in H(U)$ is taken inside the domain U, then f is constant. This establishes the problem for $n = 1$. Now if $n \geq 2$, we use that the finite sum u of subharmonic functions is subharmonic again, a fact best seen by using the definition via the mean-value inequality

$$u(x_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + re^{i\theta}) d\theta$$

for all closed disks $D(x_0, r) \subseteq U$. Thus $g = \sum_{j=1}^n |f_j|$ is subharmonic, and so is constant c under the assumption of the problem. Next, fix $j_0 \in \{1, \dots, n\}$. Then

$$|f_{j_0}| = c - \underbrace{\sum_{\substack{j=1 \\ j \neq j_0}}^n |f_j|}_{:=v}.$$

The subharmonicity of v implies that $-v$ is superharmonic. Hence, by adding the constant c, $c - v$ is superharmonic, too. Consequently, $|f_{j_0}|$ is superharmonic, as well as subharmonic. In other words, $|f_{j_0}|$ is harmonic. Any holomorphic function f in a domain U whose modulus is harmonic, is constant though. In fact, let D be a closed disk in U such that f has no zeros in D. Then f admits a holomorphic square-root q in D; that is $q^2 = f$ ([16, p. 816]). Now $|q|^2$ is harmonic by assumption. Since $\Delta|q|^2 = 4|q'|^2$ (where Δ is the Laplacian [16, p. 222]), we get that $q' \equiv 0$ in D, and so q is constant in D. Consequently, $f = q^2$ is constant in D. By the uniqueness theorem for holomorphic functions [28, p. 347], f is constant in the connected open set U.

Remark 1. A more direct, but not so elegant way to prove that the harmonicity of $|f|$ implies constancy, is purely computational and uses the $\overline{\partial}$-calculus, that is the Wirtinger derivatives $\partial u = u_z$ and $\overline{\partial} u = u_{\overline{z}}$ (see [16, sect. 4]):

Let $s(z) := |f(z)|$, $f \not\equiv 0$. Then $s^2 = f\overline{f}$ and $\partial s^2 = 2s\partial s = f'\overline{f}$. Hence, outside the discrete zero set of f, $\partial s = \frac{f'\overline{f}}{2s}$, and so

$$\Delta s = 4\overline{\partial}\partial s = \frac{4}{2} \frac{f'\overline{f'}\, s - f'\overline{f}\,\overline{\frac{f'f}{2s}}}{s^2} \overset{!}{=} 0 \iff |f'|^2 \left(2s^2 - |f|^2\right) = 0 \iff |f'|^2|f|^2 = 0 \iff f' = 0.$$

This could of course also be established via the "real"-method by calculating $s_{xx} + s_{yy}$ with $s(z) \sim s(x, y)$, $z = x + iy$ and $s = \sqrt{u^2 + v^2}$, where $f = u + iv$, and applying the Cauchy-Riemann equations in their 'real' form, instead of the shorter form $f_{\overline{z}} = 0$ above.

Remark 2. An analysis of the proof shows that one obtains the same conclusion replacing $\sum_{j=1}^n |f_j|$ by $\sum_{j=1}^n |f_j|^{\alpha_j}$, where $\alpha_j > 0$. Just note the following two points:

1) if $f \not\equiv 0$ is holomorphic, then $u := \alpha \log |f|$ is subharmonic, and so its left composition e^u with the exponential function yields the subharmonicity of $|f|^\alpha$ (see [29, Chap.1, §6] or [31, p.44]).

2) The square root q of f above is replaced by an $\alpha/2$-root of f, so that $q^2 = f^\alpha$.

12479. *Proposed by Marc Chamberland, Grinnell College, Grinnell, IA.* Evaluate $\sum_{k=1}^{\infty} r_k^{-6}$, where $r_1, r_2, \dots$ are the real roots of $2\cos\left(\sqrt{3}x\right) = -e^{-3x}$.

Solution to problem 12479 in Amer. Math. Monthly 131 (2024), p. 630

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For technical reasons, we use the new enumeration $r_0, r_1, r_2, \dots$ to denote the real roots of the equation $2\cos(\sqrt{3}x) = -e^{-3x}$. We show that

$$\boxed{S := \sum_{n=0}^{\infty} r_n^{-6} = \frac{8}{5}}.$$

Recall the Bourbaki notation $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.
Let

$$g(z) := 2\cos(\sqrt{3}z) + e^{-3z},$$

and, by multiplication with e^{3z},

$$f(z) := 2e^{3z}\cos(\sqrt{3}z) + 1.$$

Note that f also writes as

$$f(z) = e^{(3+i\sqrt{3})z} + e^{(3-i\sqrt{3})z} + 1.$$

Depending on which is best adapted to the computations, we shall use in the sequel all these variants. Note that the functions f, g have the same zero sets.

Property 1 Let

$$\alpha := e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Then

$$f(\alpha z) = e^{(-3+i\sqrt{3})z} f(z).$$

In fact, by using that

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(3 + i\sqrt{3}\right) = -3 + \sqrt{3}i$$

and

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(3 - i\sqrt{3}\right) = 2i\sqrt{3},$$

we obtain

$$\begin{aligned} f(\alpha z) &= e^{(-3+\sqrt{3}i)z} + e^{2i\sqrt{3}z} + 1 \\ &= e^{\sqrt{3}iz}\left(e^{-3z} + e^{i\sqrt{3}z}\right) + 1 \\ &= e^{\sqrt{3}iz}\left(e^{-3z} + e^{i\sqrt{3}z} + e^{-i\sqrt{3}z}\right) - 1 + 1 \\ &= e^{\sqrt{3}iz}\left(e^{-3z} + 2\cos(\sqrt{3}z)\right) \\ &= e^{\sqrt{3}iz} e^{-3z} f(z). \end{aligned}$$

As an immediate consequence we have the following useful property:

Property 2 $f(z) = 0$ *if and only* $f(\alpha z) = 0$.

Property 3 *The function f has infinitely many positive zeros and no negative one. More precisely, for every $n \in \mathbb{N}$ each interval*

$$I_n := \left[\frac{2n\pi}{\sqrt{3}}, \frac{2(n+1)\pi}{\sqrt{3}}\right]$$

contains exactly two positive zeros of f, one contained in the left part of the interval $\left[\frac{\frac{\pi}{2}+2n\pi}{\sqrt{3}}, \frac{\frac{3\pi}{2}+2n\pi}{\sqrt{3}}\right]$, and one in the right part.

Proof. • Let us first show that g (hence f) has no negative zeros. In fact if $-\frac{\pi}{2\sqrt{3}} \le x \le 0$, then $\cos(\sqrt{3}x) \ge 0$, and so $g(x) = e^{-3x} + 2\cos(\sqrt{3}x) > 0$. If $x < -\frac{\pi}{2\sqrt{3}}$, then $e^{-3x} > e^{\sqrt{3}\frac{\pi}{2}} \sim 15.1909\cdots > 2$, and again $g(x) > 0$.

• Now we deal with the positive solutions to $g(x) = 0$; or equivalently $f(x) = 2e^{3x}\cos(\sqrt{3}x) + 1 = 0$. Consider the points

$$a_n = \frac{2n\pi}{\sqrt{3}}, \quad b_n = \frac{2n\pi + \frac{\pi}{2}}{\sqrt{3}}, \quad c_n = \frac{2n\pi + \pi}{\sqrt{3}}, \quad d_n = \frac{2n\pi + \frac{3\pi}{2}}{\sqrt{3}}, \quad e_n = \frac{2n\pi + 2\pi}{\sqrt{3}}.$$

To achieve our goal, we study for $x \ge 0$ the variation of $f(x) = 2e^{3x}\cos(\sqrt{3}x) + 1$. Note that

$$f'(x) = 2e^{3x}\big(3\cos(\sqrt{3}x) - \sqrt{3}\sin(\sqrt{3}x)\big).$$

In particular, $f'(x) = 0 \iff \sqrt{3} = \tan\sqrt{3}x \iff x = x_n := \frac{\arctan\sqrt{3}+n\pi}{\sqrt{3}} = \frac{\frac{\pi}{3}+n\pi}{\sqrt{3}}$, $n \in \mathbb{N}$. Thus f is increasing on $[\frac{2n\pi}{\sqrt{3}}, \frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}}]$, decreasing on $[\frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}}, \frac{\frac{4\pi}{3}+2n\pi}{\sqrt{3}}]$ and increasing on $[\frac{\frac{4\pi}{3}+2n\pi}{\sqrt{3}}, \frac{2\pi+2n\pi}{\sqrt{3}}]$.

Now

$$f(a_n) = 2e^{3a_n} + 1, \quad f(b_n) = 1, \quad f(c_n) = -2e^{3c_n} + 1 < 0, \quad f(d_n) = 1, \quad f(e_n) = 2e^{3e_n} + 1.$$

Since $a_n < \frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}} < b_n < \frac{\pi+2n\pi}{\sqrt{3}} = c_n < \frac{\frac{4\pi}{3}+2n\pi}{\sqrt{3}} < \frac{\frac{3\pi}{2}+2n\pi}{\sqrt{3}} = d_n$, we see that there are exactly two zeros on $[a_n, e_n]$, namely one between b_n and c_n and one between c_n and d_n [1].

Let us denote the zeros by r_n, in increasing order ($n \in \mathbb{N}$). Note that r_n is eventually close to the zero

$$\boxed{s_n := \frac{\frac{\pi}{2} + n\pi}{\sqrt{3}}}$$

of $2\cos(\sqrt{3}x)$, since e^{-3x} tends rapidly to 0 as $x \to \infty$, and that for $n \in \mathbb{N}$,

$$s_{2n} < r_{2n} < r_{2n+1} < s_{2n+1} < s_{2n+2} < r_{2n+2} \tag{5}$$

(because the cosinus values at r_n must be negative). □

Here are some numerical examples:

$r_0 \sim 0.924906\ldots$, $s_0 = \frac{\pi}{2\sqrt{3}} \sim 0.9068996\ldots$, $|s_0 - r_0| \sim 0.0180064 < 0.125 = 1/8$,
$r_1 \sim 2.720616677\ldots$, $s_1 = \frac{3\pi}{2\sqrt{3}} \sim 2.7206990\ldots$, $|r_1 - s_1| \sim 0.00008223\cdots$,
$r_2 \sim 4.53449876\ldots$, $s_2 = \frac{5\pi}{2\sqrt{3}} \sim 4.53449841\ldots$, $|r_2 - s_2| \sim 0.00000035\cdots$.

$|r_1 - r_0| \sim 1.7957107\cdots$, $|r_2 - r_1| \sim 1.81388208\cdots$

The items (1)–(4) below are nice additional properties of the real zeros of f. We do not need these, though.

[1]Using only the information $f(a_n) > 0, f(c_n) < 0$ and $f(e_n) > 0$ (so without studying the variation of f), allows us to conclude that f has at least two zeros in $I_n = [a_n, e_n]$. Using Rouchés theorem and the additional information that in the right half-plane the complex zeros of $\cos(\sqrt{3}z)$, all simple, are exactly the real zeros $\frac{\frac{\pi}{2}+2n\pi}{\sqrt{3}}$ and $\frac{\frac{3\pi}{2}+2n\pi}{\sqrt{3}}$, gives us the possibility to conclude that f has no other zeros in the strips $\{z \in \mathbb{C} : \mathrm{Re}\, z \in I_n\}$ (see Property 6).

Property 4

(1) $|r_n - s_n| < \frac{1}{16\sqrt{3}} < \frac{1}{8}$ for all $n \in \mathbb{N}$.
(2) $|r_n - s_n| \to 0$.
(3) $|r_n - r_{n-1}| > 1$ for all $n \in \mathbb{N}^*$.
(4) $|r_n - r_{n-1}| \to \frac{\pi}{\sqrt{3}} \sim 1.813799\cdots$.
(5) $r_n^3 \sim cn^3$, for some constant c.

Noticing that $\{r_{2n}, r_{2n+1}\} \subseteq I_n$, one can easily show that $|r_{2n+1} - r_{2n}|$ increases to $\frac{\pi}{\sqrt{3}}$ and $|r_{2n} - r_{2n-1}|$ decreases to $\frac{\pi}{\sqrt{3}}$.

Proof. Recall that $f(z) = 2e^{3z}\cos(\sqrt{3}z) + 1$.

(1) For $n = 2m$ even, it is sufficient to prove that $s_n < r_n < s_n + \frac{1}{16\sqrt{3}} =: p_n$, where $s_n, r_n \in I_m$. Since $f(s_n) = 1$ and $f(r_n) = 0$, and f is decreasing on $[\frac{\frac{\pi}{3}+2m\pi}{\sqrt{3}}, \frac{\frac{4\pi}{3}+2m\pi}{\sqrt{3}}] \supseteq [s_n, p_n]$, this proof is done by showing that $f(p_n) < 0$. Now, noticing that the cosinus term is negative,

$$\begin{aligned} f(p_n) &= 2e^{\sqrt{3}(\frac{\pi}{2}+n\pi+\frac{1}{16})}\cos\left(\frac{\pi}{2} + \frac{1}{16}\right) + 1 \\ &\leq 2e^{\sqrt{3}(\frac{\pi}{2}+\frac{1}{16})}\cos\left(\frac{\pi}{2} + \frac{1}{16}\right) + 1 \sim -1.114587\ldots. \end{aligned}$$

For $n = 2m+1$ odd, it is proved in the same way that $q_n := s_n - \frac{1}{16\sqrt{3}} < r_n < s_n$, where $s_n, r_n \in I_m$. In fact, f is increasing on $[\frac{\frac{4\pi}{3}+2m\pi}{\sqrt{3}}, \frac{2\pi+2m\pi}{\sqrt{3}}] \supseteq [q_n, s_n]$. So, by noticing that the cosinus term is negative,

$$\begin{aligned} f(q_n) &= 2e^{\sqrt{3}(\frac{3\pi}{2}+(n-1)\pi-\frac{1}{16})}\cos\left(\frac{3\pi}{2} - \frac{1}{16}\right) + 1 \\ &\leq 2e^{\sqrt{3}(\frac{3\pi}{2}-\frac{1}{16})}\cos\left(\frac{3\pi}{2} - \frac{1}{16}\right) + 1 \sim -391.97708\cdots \end{aligned}$$

(2) This works in the same way as above; just replace $1/16$ be an number $\varepsilon > 0$ arbitrary close to 0. Then the cosinus term, still negative, may be very small. But the power n in the exponential factor can be made sufficiently big (depending on ε), so that $f(p_n)$, resp. $f(q_n)$ is strictly negative.

(3) Let n be even. Then

$$|r_n - r_{n-1}| = r_n - r_{n-1} \geq s_n - s_{n-1} = \frac{\pi}{\sqrt{3}} \sim 1.813799\cdots > 1.$$

If n is odd, then by (1)

$$|r_n - r_{n-1}| = r_n - r_{n-1} \geq (s_n - \frac{1}{16\sqrt{3}}) - (s_{n-1} + \frac{1}{16\sqrt{3}}) = \frac{\pi}{\sqrt{3}} - \frac{1}{8\sqrt{3}} \sim 1.7416305\cdots > 1.$$

(4) Due to (2)

$$r_n - r_{n-1} = (r_n - s_n) + (s_n - s_{n-1}) + (s_{n-1} - r_{n-1}) \to 0 + \frac{\pi}{\sqrt{3}} + 0.$$

(5) Using (5), we obtain that

$$\frac{\frac{\pi}{2} + 2n\pi}{\sqrt{3}} < r_{2n} < r_{2n+1} < \frac{\frac{3\pi}{2} + 2n\pi}{\sqrt{3}}. \tag{6}$$

Hence $\frac{r_n}{n} \to \frac{\pi}{\sqrt{3}}$ and so $r_n^3 \sim cn^3$.

□

Let $Z := \{r_n, \alpha r_n, \alpha^2 r_n\}$. Combining the properties (1) and (2), we see that Z is a symmetric zero-set for f, resp. g.

Property 5 *The elements of Z are simple zeros for f (hence for g)* [2].

Proof. Suppose, contrariwise, that for some real z,

$$f(z) = 2e^{3z}\cos(\sqrt{3}z) + 1 = 0, \tag{7}$$

and

$$f'(z) = 3 \cdot 2e^{3z}\cos\sqrt{3}z - 2 \cdot e^{3z}\sqrt{3}\sin\sqrt{3}z = 0. \tag{8}$$

Plugging equality (7) into (8) yields

$$0 = f'(z) = -3 - 2\sqrt{3}e^{3z}\sin\sqrt{3}z. \tag{9}$$

Hence

$$1 = \sin^2(\sqrt{3}z) + \cos^2(\sqrt{3}z) \overset{(9)}{\underset{(7)}{=}} \frac{3}{4}e^{-6z} + \frac{1}{4}e^{-6z} = e^{-6z}.$$

Therefore $z = 0$. But $f(0) = 3$, a contradiction. We conclude that, due to Property 2, all elements in Z are simple zeros for f. ☐

Property 6 *The exact zero-set of f coincides with Z.*

Proof. Numerically it can be shown that for $k \in \mathbb{N}^*$

$$\frac{1}{2\pi i}\int_{|z|=2k\pi/\sqrt{3}} \frac{f'}{f}dz = 6k,$$

which yields the assertion, as we already know that within the annuli

$$\left\{z \in \mathbb{C} : \frac{2n\pi}{\sqrt{3}} \leq |z| \leq \frac{2(n+1)\pi}{\sqrt{3}}\right\}$$

there are 2 real zeros and their rotations by α and α^2.

For our genuine proof, we first restrict the calculations to the case where $\operatorname{Re} z \geq 0$. The other case will be deduced at the very end of the proof.

- We know that $|\cos z|^2 = (\cos x)^2 + (\sinh y)^2$, where $z = x + iy$. In fact,

$$\begin{aligned} 4|\cos z|^2 &= |e^{i(x+iy)} + e^{-i(x+iy)}|^2 = |e^{-y}e^{ix} + e^{y}e^{-ix}|^2 \\ &= e^{-2y} + e^{2y} + 2\mathrm{Re}(e^{2ix}) = 2\cosh(2y) + 2\cos(2x) \\ &= 2(\cosh^2 y + \sinh^2 y) + 2(\cos^2 x - \sin^2 x) \\ &= 2(1 + 2\sinh^2 y) + 2(\cos^2 x - (1 - \cos^2 x)) \\ &= 4(\sinh^2 y + \cos^2 x). \end{aligned}$$

- Recall that $g(z) = e^{-3z} + 2\cos(\sqrt{3}z)$. We now show that $g \neq 0$ on $[0, \infty[\times[1, \infty[$. In fact,

$$|g(z)| \geq 2|\cos(\sqrt{3}z)| - |e^{-3z}| = 2\sqrt{\cos^2(\sqrt{3}x) + \sinh^2(\sqrt{3}y)} - e^{-3x} \geq 2\sinh\sqrt{3} - 1 > 0.$$

The same proof also shows that $g \neq 0$ on $[0, \infty[\times\,]-\infty, -1]$.

Next we use Rouché's theorem for the rectangles $R_k := \left[\frac{2k\pi}{\sqrt{3}}, \frac{2(k+1)\pi}{\sqrt{3}}\right] \times [-1, 1]$, $k \in \mathbb{N}$. If $x = \frac{2k\pi}{\sqrt{3}}$ or $x = \frac{2(k+1)\pi}{\sqrt{3}}$, then

$$\begin{aligned} |g(z) - 2\cos(\sqrt{3}z)| &= e^{-3x} \leq 1 < 2|\cos\sqrt{3}iy| = 2(\sqrt{1 + \sinh^2(\sqrt{3}y)} = 2|\cos(\sqrt{3}z)| \\ &\leq |g(z)| + 2|\cos(\sqrt{3}z)|. \end{aligned}$$

[2] Also note here, that Rouché's theorem (applied as in the proof of Property 6) automatically yields this fact, too. We preferred though a straightforward elementary proof.

If $|y| = 1$, then

$$\begin{aligned} |g(z) - 2\cos(\sqrt{3}z)| &= e^{-3x} \le 1 < 2\sinh\sqrt{3} \le 2\sqrt{\cos^2(\sqrt{3}x) + \sinh^2(\sqrt{3})} \le 2|\cos(\sqrt{3}z)| \\ &\le |g(z)| + 2|\cos(\sqrt{3}z)|. \end{aligned}$$

By Rouché's Theorem [16, p. 852], g and $\cos(\sqrt{3}z)$ have the same number of zeros on R_k, namely 2. As we already know that g has two real zeros in these intervals $\left[\frac{2k\pi}{\sqrt{3}}, \frac{2(k+1)\pi}{\sqrt{3}}\right]$, we are done.

• Having only the real zeros r_k of f (equivalently g) in the right-half plane, the symmetry $f(z) = 0 \iff f(e^{i2\pi/3}z) = 0$ now implies that no other zeros are in the left half-plane excepted the rotations $e^{i2\pi/3}r_k$ and $e^{i4\pi/3}r_k$ as any $\zeta := re^{i\theta}$ with $\pi/2 \le \theta \le 3\pi/2$ would yield that at least one of the points $e^{i4\pi/3}\zeta$ or $e^{i2\pi/3}\zeta$ belongs to the right-half plane. This finishes the proof of Property 6. □

We are finally ready to derive the value for the desired sum S. The main idea is to apply the residue theorem for the function

$$F(z) := \frac{1}{z^6}\frac{f'(z)}{f(z)},$$

and the formula $\operatorname{Res}\left(h\frac{f'}{f}, p\right) = h(p)$, where p is a simple pole of f and h is holomorphic in a neighborhood of p.

Observe that [3]

$$\frac{f'(z)}{f(z)} = 2 - 4z^2 - \frac{24}{5}z^5 - \frac{212}{35}z^8 - \frac{14736}{1925}z^{11} - \frac{1694832}{175175}z^{14} - \cdots$$

Thus,

$$\operatorname{Res}\left(\frac{1}{z^6}\frac{f'}{f}, z = 0\right) = -\frac{24}{5}.$$

Note that the zeros $z_n = \alpha r_n$, resp. $z_n = \alpha^2 r_n$ have the property that $z_n^6 = r_n$.

Recall from (6) that for $n \in \mathbb{N}$ the numbers r_{2n}, r_{2n+1} belong to

$$\left[\frac{2n\pi + \frac{\pi}{2}}{\sqrt{3}}, \frac{2n\pi + \frac{3\pi}{2}}{\sqrt{3}}\right].$$

For $N \in \mathbb{N}$, let C_N be the circle centered at the origin and with radius $R_N := \frac{2\pi(N+1)}{\sqrt{3}}$. Then the associated disk contains $r_0, r_1, \dots, r_{2N+1}$ and

(1) $R_N > N$,
(2) $R_N^3 - r_{2N+1}^3 \ge 1$.

Note that (2) is a consequence to $R_N - r_{2N+1} \ge \pi/2$.

Due to the residue theorem

$$\begin{aligned} L := \lim_{N\to\infty} \frac{1}{2\pi i}\int_{C_N} \frac{1}{z^6}\frac{f'(z)}{f(z)}dz &= \operatorname{Res}\left(\frac{1}{z^6}\frac{f'}{f}, 0\right) + \sum_{\xi\in Z}\operatorname{Res}\left(\frac{1}{z^6}\frac{f'}{f}, \xi\right) \\ &= -\frac{24}{5} + 3\sum_n \frac{1}{r_n^6}. \end{aligned}$$

We claim that $L = 0$, from which we deduce that

$$S = \sum_n \frac{1}{r_n^6} = \frac{8}{5}.$$

To prove this claim, we use that by Property (4), $\sum \frac{1}{r_n^3}$ converges. Hence the infinite product

$$p(z) := \prod_{n=0}^{\infty}\left(1 - \left(\frac{z}{r_n}\right)^3\right)$$

[3] Obtained e.g. with wolframalpha.

converges locally uniformly on $\mathbb{C}$. According to Weierstrass's factorization theorem, $f = pq$, where q is a zero-free entire function. Note that

$$\frac{f'}{f} = \frac{p'(z)}{p(z)} + \frac{q'(z)}{q(z)}$$

and that

$$\int_{C_N} \frac{1}{z^6}\frac{f'(z)}{f(z)}dz = \int_{C_N} \frac{1}{z^6}\frac{p'(z)}{p(z)}dz + 0.$$

Now

$$\frac{p'(z)}{p(z)} = -3\sum_{n=0}^{\infty} \frac{z^2}{r_n^3 - z^3},$$

which is a locally uniformly convergent series on $\mathbb{C}\setminus Z$ (note that $r_n^3 \sim c\cdot n^3$). Hence $\int\sum = \sum\int$ and so, by Cauchy's integral theorem,

$$2\pi i J_N := \int_{C_N} \frac{1}{z^6}\frac{p'(z)}{p(z)}dz = -3\sum_{n=0}^{2N+1}\int_{C_N} \frac{z^{-4}}{r_n^3 - z^3}dz.$$

Now

$$\left|\frac{z^{-4}}{r_n^3 - z^3}\right| \leq \frac{R_N^{-4}}{R_N^3 - r_n^3} \leq R_N^{-4}.$$

Thus

$$0 \leq 2\pi J_N \leq 3(2N+2)R_N^{-4}2\pi R_N \leq C\cdot N^{-2}.$$

We conclude that $L = \lim_{N\to\infty} J_N = 0$. □

Related problems are given in [36, p. 279] and [35] ($\cos x \cosh x + 1 = 0$) and [34] ($\tan x = kx$) [4].

<u>*Second, but more complicated proof of Property 6*</u>

Note that $g(z) = e^{-3z} + 2\cos(\sqrt{3}z) = 0$ implies that

$$4|\cos^2(\sqrt{3}z)| = e^{-6x},$$

and so

$$4\sinh^2(\sqrt{3}y) = e^{-6x} - 4\cos^2(\sqrt{3}x).$$

In particular

$$4\cos^2(\sqrt{3}x) \leq e^{-6x}. \tag{10}$$

If $k \in \mathbb{N}$, equality in (10) holds if $x = r_k$ and if $x = \frac{\frac{\pi}{2}+k\pi}{\sqrt{3}} = s_k$, then trivially $0 = 4\cos^2(\sqrt{3}x) \leq e^{-6x}$.

In the figure below, we display one "branch" of the curves y_j, given by

$$y_2 = \frac{1}{\sqrt{3}}\operatorname{arsinh}\sqrt{\frac{1}{4}e^{-6x} - \cos^2(\sqrt{3}x)} \text{ and } y_1 = 2\cos(\sqrt{3}x) + e^{-3x}. \tag{11}$$

They are very tiny, as the coordinates show. To recapitulate, if z is a zero of g with non-negative real part, then it belongs to these tiny arcs.

• To show that in the right half-plane g has only the zeros r_k, it remains to study the behaviour of $g(z) = 2\cos(\sqrt{3}z) + e^{-3z}$ on the disks $D(r_k, 1/8)$. A major difficulty will be to show that the union of these disks contains the graph of the curve y_2. A tool will be the following result from complex analysis:

Zero-set criterium 1. [16, p. 365] *Let Φ be bounded by 1 and holomorphic in $\mathbb{D}$ and suppose that $\Phi(0) = 0$ as well as $|\Phi'(0)| \geq \delta > 0$. Then Φ has no zeros on $\{0 < |z| < \delta\}$.*

[4] Communicated to us by A. Sasane.

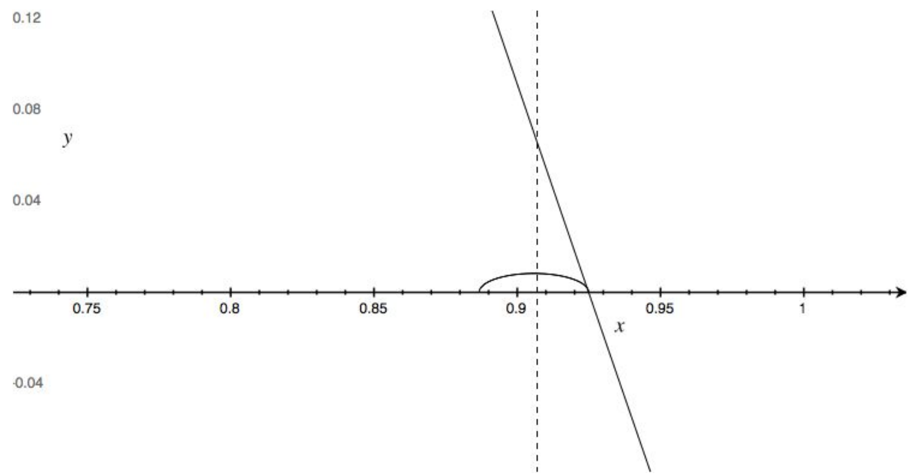

FIGURE 1. The curves $y_2 = \frac{1}{\sqrt{3}}\text{arsinh}\sqrt{\frac{1}{4}e^{-6x} - \cos^2(\sqrt{3}x)}$ and $y_1 = 2\cos(\sqrt{3}x) + e^{-3x}$ and $s_0 = \frac{\pi}{2\sqrt{3}}$

We first note that for $z = x + iy$ with $x \geq 0$ and $|y| \leq 1$, g is bounded by 8 :

$$\begin{aligned} |g(z)| &\leq 2|\cos(\sqrt{3}z)| + e^{-3x} = 2(\sqrt{\cos^2(\sqrt{3}x) + \sinh^2(\sqrt{3}y)}) + 1 \leq 2(\sqrt{1 + \sinh^2\sqrt{3}}) + 1 \\ &= 2\cosh(\sqrt{3}) + 1 \leq e^{\sqrt{3}} + 2 \leq 8. \end{aligned}$$

Moreover $|g'(r_k)| \geq 1$, since

$$g'(z) = -2\sqrt{3}\sin(\sqrt{3}z) - 3e^{-3z},$$

and so

$$|g'(r_k)| = |-2\sqrt{3}\sin(\sqrt{3}r_k) + 6\cos(\sqrt{3}r_k)| \to 2\sqrt{3} \sim 3.4641\cdots.$$

respectively

$$|g'(r_k)| = |(-1)^k\sqrt{3}\sqrt{4 - e^{-6r_k}} - 3e^{-3r_k}|$$

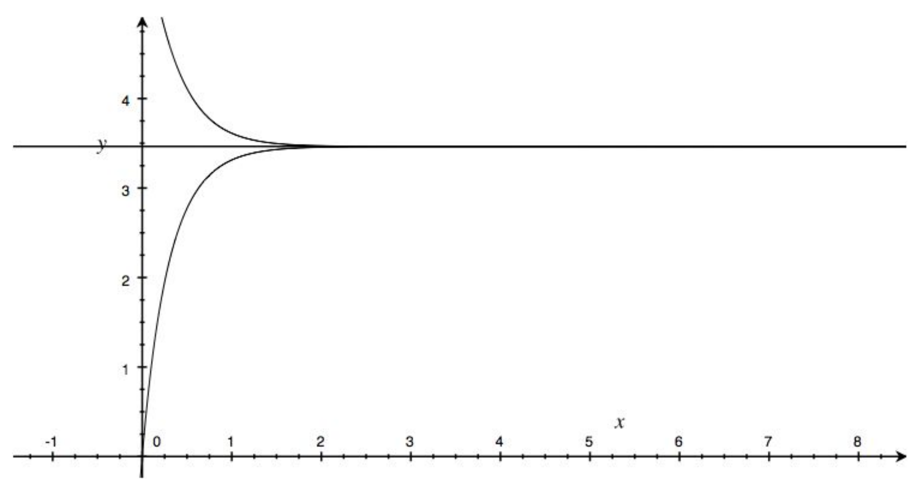

FIGURE 2. The curves $y = |\pm\sqrt{3}\sqrt{4 - e^{-6x}} - 3e^{-3x}|$

The value of the lower branch at $x = 1/6$ is $\sim 1.481371\cdots$ and the branch is increasing. So $|g'(r_k)| \geq 1$ for r even. The upper branch is always bigger than 3. So, $|g'(r_k)| \geq 1$, too if r is odd.

We conclude from the zero set criterium 1 that $g \neq 0$ on $D(r_k, 1/8)$, excepted at r_k.

• The final part which remains is now the proof that the graph of the (different branches) of the curve y_2 is contained in the union of the disks $D(r_k, 1/8)$.

We first determine the positive zeros of y_2. These are given by $e^{-6x} - 4\cos^2(\sqrt{3}x) = 0$. Of course the r_k, the positive zeros of $2\cos(\sqrt{3}x) - e^{-3x}$ are zeros of y_2, too. The other ones are given by the zeros of the function

$$h^*(x) = 2e^{3x}\cos(\sqrt{3}x) - 1 = h(x) - 2.$$

This works as in the proof oh Property 3; note that $(h^*)' = h'$.

$$h^*(a_n) = 2e^{3a_n} - 1 > 0, \quad h(b_n) = -1, \quad h(c_n) = -2e^{3c_n} - 1 < 0, \quad h(d_n) = -1, \quad h(e_n) = 2e^{3e_n} - 1 > 0.$$

Since $a_n < \frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}} < b_n < \frac{\pi+2n\pi}{\sqrt{3}} = c_n < \frac{\frac{4\pi}{3}+2n\pi}{\sqrt{3}} < \frac{\frac{3\pi}{2}+2n\pi}{\sqrt{3}} = d_n$, and h^* is increasing on $[\frac{2n\pi}{\sqrt{3}}, \frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}}]$, decreasing on $[\frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}}, \frac{\frac{4\pi}{3}+2n\pi}{\sqrt{3}}]$ and increasing on $[\frac{\frac{4\pi}{3}+2n\pi}{\sqrt{3}}, \frac{2\pi+2n\pi}{\sqrt{3}}]$, we see that there are exactly two zeros on $[a_n, e_n]$, namely one between $\frac{\frac{\pi}{3}+2n\pi}{\sqrt{3}}$ and $b_n = \frac{\frac{\pi}{2}+2n\pi}{\sqrt{3}}$ and one between $d_n = \frac{\frac{3\pi}{2}+2n\pi}{\sqrt{3}}$ and $e_n = \frac{2\pi+2n\pi}{\sqrt{3}}$. We enumerate these in an increasing order, say r_n^*, and

$$r_{2n}^* < s_{2n} < r_{2n} \text{ as well as } r_{2n+1} < s_{2n+1} < r_{2n+1}^*.$$

As in the proof of Property 5 (1) we see that

$$|r_n^* - s_n| < \frac{1}{16\sqrt{3}}.$$

In fact,

For $n = 2m$ even, it is sufficient to prove that $p_n^* := s_n - \frac{1}{16\sqrt{3}} < r_n^* < s_n$, where $s_n, r_n^* \in I_m = \left[\frac{2\pi m}{\sqrt{3}}, \frac{2\pi m+2\pi}{\sqrt{3}}\right]$. Since $h(s_n) = -1$ and $h(r_n^*) = 0$, and h is decreasing on $[\frac{\frac{\pi}{3}+2m\pi}{\sqrt{3}}, \frac{\frac{4\pi}{3}+2m\pi}{\sqrt{3}}] \supseteq [p_n^*, s_n]$, this is done by showing that $h^*(p_n) > 0$. Now, noticing that the cosinus term is positive,

$$\begin{aligned} h^*(p_n^*) &= 2e^{\sqrt{3}(\frac{\pi}{2}+n\pi-\frac{1}{16})}\cos\left(\frac{\pi}{2} - \frac{1}{16}\right) - 1 \\ &\geq 2e^{\sqrt{3}(\frac{\pi}{2}-\frac{1}{16})}\cos\left(\frac{\pi}{2} - \frac{1}{16}\right) - 1 \sim 0.7029349\cdots > 0. \end{aligned}$$

For $n = 2m+1$ odd, it is proved in the same way that $s_n < r_n^* < q_n^* := s_n + \frac{1}{16\sqrt{3}}$, where $s_n, r_n^* \in I_m$. In fact, h^* is increasing on $[\frac{\frac{4\pi}{3}+2m\pi}{\sqrt{3}}, \frac{2\pi+2m\pi}{\sqrt{3}}] \supseteq [s_n, q_n^*]$. So, by noticing that the cosinus term is positive,

$$\begin{aligned} h^*(q_n^*) &= 2e^{\sqrt{3}(\frac{3\pi}{2}+(n-1)\pi+\frac{1}{16})}\cos\left(\frac{3\pi}{2} + \frac{1}{16}\right) - 1 \\ &\geq 2e^{\sqrt{3}(\frac{3\pi}{2}+\frac{1}{16})}\cos\left(\frac{3\pi}{2} + \frac{1}{16}\right) - 1 \sim 486.971814\cdots \end{aligned}$$

Next we estimate the local maxima of y_2. Since $r_n^* \geq r_0^* > \pi/3$,

$$y_2 \leq \frac{1}{\sqrt{3}}\text{arsinh}\sqrt{\frac{1}{4}e^{-6\pi/3}} = \frac{1}{\sqrt{3}}\text{arsinh}\frac{e^{-\pi}}{2} \sim 0.01247381\cdots < 0.02 := \rho$$

Now the rectangle

$$R := \left[r_0 - \frac{1}{8\sqrt{3}}, r_0 + \frac{1}{8\sqrt{3}}\right] \times [0, \rho] \subseteq D(r_0, \frac{1}{8}),$$

because the vertex $(\frac{1}{8\sqrt{3}}, \rho)$ satisfies

$$\left(\frac{1}{8\sqrt{3}}\right)^2 + \rho^2 \sim 0.00568\cdots < 0.01562 = \frac{1}{64}.$$

It is easily shown that $\{(x, y_2(x)) : r^*_{2n} \leq x \leq r_{2n}\}$ and $\{(x, y_2(x)) : r_{2n+1} \leq x \leq r^*_{2n+1}\}$ are contained in the rectangle R. □

• Having only the real zeros r_k of f in the right-half plane, the symmetry $f(z) = 0 \iff f(e^{i2\pi/3}z) = 0$ now implies that no other zeros are in the left half-plane excepted the rotations $e^{i2\pi/3}r_k$ and $e^{i4\pi/3}r_k$ as any $\zeta := re^{i\theta}$ with $\pi/2 \leq \theta \leq 3\pi/2$ would yield that at least one of the points $e^{i4\pi/3}\zeta$ or $e^{i2\pi/3}\zeta$ belongs to the right-half plane. This finishes the proof of Property 6. □

A detailed analysis of the zeros of the functions $e^{az} + e^{-az} + 1$ appears in [4]. That paper is based on the preceding methods and text.

12470. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh(2^{n-1})}\right).$$

Solution to problem 12470 in Amer. Math. Monthly 131 (2024), p. 536

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We show that

$$\boxed{S := \sum_{n=1}^{\infty} \frac{1}{2^n} \log\left(\frac{\tanh 2^n}{\tanh 2^{n-1}}\right) = \log(e^2+1) - 2 \sim 0.1269280110\cdots}.$$

One has to transform this into a telescoping series.

$$\begin{aligned}\frac{\tanh 2^n}{\tanh 2^{n-1}} &= \frac{\sinh 2^n}{\sinh 2^{n-1}} \frac{\cosh 2^{n-1}}{\cosh 2^n} = \frac{2\ \sinh 2^{n-1} \cosh 2^{n-1}}{\sinh 2^{n-1}} \frac{\cosh 2^{n-1}}{\cosh 2^n} \\ &= 2\frac{(\cosh 2^{n-1})^2}{\cosh 2^n}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2^n} \log\left(\frac{\tanh 2^n}{\tanh 2^{n-1}}\right) &= \frac{1}{2^n}\Big(\log 2 + 2\log(\cosh 2^{n-1}) - \log\cosh 2^n\Big) \\ &= \frac{\log 2}{2^n} + \frac{1}{2^{n-1}} \log(\cosh 2^{n-1}) - \frac{1}{2^n} \log(\cosh 2^n).\end{aligned}$$

Note that $\varepsilon_n := \frac{1}{2^n} \log(\cosh 2^n) \to 1$ since (by using l'Hospital's rule).

$$\lim_{x\to\infty} \frac{\log(e^x + e^{-x})}{x} = 1.$$

Consequently the series below converges and

$$\begin{aligned}S &= \sum_{n=1}^{\infty} \frac{\log 2}{2^n} + \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} \log(\cosh 2^{n-1}) - \frac{1}{2^n} \log(\cosh 2^n)\right) \\ &= 1\cdot \log 2 + \log\cosh 1 - \lim_n \varepsilon_n = \log 2 + \log\left(\frac{e^1 + e^{-1}}{2}\right) - 1 = \log(e^2+1) - 2.\end{aligned}$$

12460. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Suppose that $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous. Prove that the following are equivalent:

(1) Whenever $\langle a_n\rangle$ and $\langle b_n\rangle$ are sequences of rationals such that $\langle a_n+b_n\rangle$ converges, the sequence $\langle f(a_n)+b_n g(b_n)\rangle$ also converges.

(2) There are constants m and c such that $f(x)=mx+c$ and $g(x)=m$.

Solution to problem 12460 in Amer. Math. Monthly 131 (2024), 354

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(2) $\Longrightarrow$ (1): If $f(x)=mx+c$ and $g(x)=m$, then trivially

$$S(n):=f(a_n)+b_ng(b_n)=ma_n+c+b_nm=m(a_n+b_n)+c$$

and so $S(n)$ converges whenever a_n+b_n converges.

(1)$\Longrightarrow$ (2): Let $\mathbb{N}:=\{0,1,2,\dots\}$. Fix $a,q\in\mathbb{Q}$. Consider the sequences

$$(a_n)_{n\in\mathbb{N}}=(a+q,a,a+q,a,a+q,a,\dots)\quad\text{and}\quad(b_n)_{n\in\mathbb{N}}=(-q,0,-q,0,\dots).$$

Then $a_n+b_n=a$ for all n. Moreover

$$s_n:=f(a_n)+b_ng(b_n)=\begin{cases}f(a+q)-qg(-q) & \text{if } n \text{ is even}\\ f(a) & \text{if } n \text{ is odd.}\end{cases}.$$

As by assumption (s_n) converges, we deduce that

$$f(a+q)-qg(-q)=f(a) \tag{12}$$

Next consider the sequences $(a_n)_{n\in\mathbb{N}}=(q,0,q,0,\dots)$ and $(b_n)_{n\in\mathbb{N}}=(-q,0,-q,0,\dots)$. Then $a_n+b_n=0$ and

$$r_n:=f(a_n)+b_ng(b_n)=\begin{cases}f(q)-qg(-q) & \text{if } n \text{ is even}\\ f(0) & \text{if } n \text{ is odd.}\end{cases}$$

Since by assumption also (r_n) converges, we have

$$f(q)-qg(-q)=f(0). \tag{13}$$

Now (12)–(13) yields that for every $q\in\mathbb{Q}$ and $a\in\mathbb{Q}$

$$f(a+q)-f(q)=f(a)-f(0).$$

Since f is assumed to be continuous, $f(a+x)-f(a)=f(x)-f(0)$ for every $x\in\mathbb{R}$. This implies that f is an affine function. In fact, let $h(x)=f(x)-f(0)$. Then $h(a+x)=h(x)+h(a)$, that is, h is a continuous additive function. By a classical result due to Cauchy, h is linear; that is $h(x)=mx$ for some $m\in\mathbb{R}$. Consequently $f(x)=mx+f(0)$. Let $c:=f(0)$. Then the condition on f and g has the form

$$S_n=f(a_n)+b_ng(b_n)=ma_n+c+b_ng(b_n)=c+m(a_n+b_n)+b_n(g(b_n)-m).$$

Let $p,q\in\mathbb{Q}$ and consider the sequences $(b_n)_{n\in\mathbb{N}}=(p,q,p,q,\dots)$ and $(a_n)_{n\in\mathbb{N}}=(-p,-q,-p,-q,\dots)$. Then $a_n+b_n=0$ for all n and, by assumption,

$$S_n=\begin{cases}c+p(g(p)-m) & \text{if } n \text{ is even}\\ c+q(g(q)-m) & \text{if } n \text{ is odd}\end{cases}$$

converges. Hence the function $x(g(x)-m)$ must be constant on $\mathbb{Q}$, hence on $\mathbb{R}$ (due to continuity). Consequently $g(x)=m$ for every $x\in\mathbb{R}$.

12459. *Proposed by Hervé Grandmontagne, Paris, France.* Let α be a real number greater than 1. Evaluate

$$\int_0^\infty \frac{\mathrm{Li}_2\left(-x^\alpha\right)+\mathrm{Li}_2\left(-x^{-\alpha}\right)}{1+x^\alpha}\,dx,$$

where Li_2 is the dilogarithm function, defined by $\mathrm{Li}_2(x)=\sum_{k=1}^\infty x^k/k^2$ when $|x|<1$ and extended by analytic continuation.

Solution to problem 12459 in Amer. Math. Monthly 131 (2024), 354

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We prove that for $a>1$

$$\boxed{I(a):=\int_0^\infty \frac{\mathrm{Li}_2(-x^a)+\mathrm{Li}_2(-x^{-a})}{1+x^a}\,dx=\frac{\pi^3}{3a}\left(\frac{\sin^2(\pi/a)-3}{\sin^3(\pi/a)}\right).}$$

To start with, we use the known formula [37]

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(\frac{1}{z})=-\frac{\pi^2}{6}-\frac{1}{2}\log^2(-z),\quad z\in\mathbb{C}\setminus[0,\infty[,$$

for $z=-x^a$. So the integral to be computed is

$$I(a)=-\frac{\pi^2}{6}\int_0^\infty \frac{1}{1+x^a}dx-\frac{1}{2}\int_0^\infty \frac{\log^2(x^a)}{1+x^a}\,dx.$$

The change of variable $x^a\mapsto e^{-t}$ now yields

$$I(a)=-\frac{\pi^2}{6a}\int_{-\infty}^\infty \frac{e^{-\frac{t}{a}}}{1+e^{-t}}dt-\frac{1}{2a}\int_{-\infty}^\infty \frac{t^2e^{-\frac{t}{a}}}{1+e^{-t}}dt.$$

We solve this with the help of the residue theorem. So, for $m=0$ or $m=2$, let

$$f_m(z):=z^m\frac{e^{-\frac{z}{a}}}{1+e^{-z}}.$$

In order the obtained series converge and the path-integrals tend to 0 when "blowing up" the contours, we consider for $0<r<1$ the auxiliary functions

$$u_r(z):=r^{-iz/2\pi}:=e^{-i\frac{\log r}{2\pi}z}$$

(which converge locally uniformly to 1 as $r\to 1$) and

$$F_{m,r}(z):=f_m(z)u_r(z),$$

and calculate the integral

$$J_r(a):=-\frac{\pi^2}{6a}\int_{-\infty}^\infty \frac{e^{-\frac{t}{a}}r^{-it/2\pi}}{1+e^{-t}}dt-\frac{1}{2a}\int_{-\infty}^\infty \frac{t^2e^{-\frac{t}{a}}r^{-it/2\pi}}{1+e^{-t}}dt. \tag{14}$$

As $|u_r(t)|\leq 1$, we deduce from Lebesgue's dominated convergence theorem that

$$\lim_{r\to 1}J_r(a)=I(a).$$

Note that $F_{m,r}$ is meromorphic in $\mathbb{C}$ with simple poles at $z_n=i\pi(1+2n)$ for $n\in\mathbb{Z}$. We integrate $F_{m,r}$ over the positively oriented boundary $\Gamma_N=\gamma_1+\gamma_2+\gamma_3$ of the rectangles $[-2N\pi,2N\pi]\times[0,2N\pi]$, where $N\in\mathbb{N}^*$. Let $s_N:=2\pi N$. Then

$$\begin{aligned}\gamma_1(t)&=s_N(1+it),&&0\leq t\leq 1,\\ \gamma_2^{[-1]}(t)&=s_N(t+i),&&-1\leq t\leq 1,\\ \gamma_3^{[-1]}(t)&=s_N(-1+it),&&0\leq t\leq 1.\end{aligned}$$

By the residue theorem

$$\int_{\Gamma_N} F_{m,r}(z)dz = 2\pi i \sum_{n=0}^{N-1} \text{Res } (F_{m,r}, z_n).$$

Now $F_{m,r} = g/h$ and so

$$\text{Res } (F_{m,r}, z_n) = \frac{g(z_n)}{h'(z_n)}.$$

Moreover, $\int_{\Gamma_N} F_{m,r}(z)dz \to 0$ as $N \to \infty$. To see this we have to consider three cases:

Since on γ_j for $j = 1, 3$, we have $|u_r(\gamma_j(t))| \le e^{tN \log r} \le 1$, we see that

$$\begin{aligned}\left|\int_{\gamma_1} F_{m,r}(z)dz\right| &\le \int_0^1 \frac{|s_N(1+it)|^m \; |e^{-s_N(1+it)/a}| \, |u_r(s_N(1+it))|}{|1+e^{-s_N(1+it)}|} s_N dt \\ &\le \frac{CN^{m+1}e^{-s_N/a}}{1-e^{-s_N}} \to 0 \text{ as } N \to \infty.\end{aligned}$$

$$\begin{aligned}\left|\int_{\gamma_3} F_{m,r}(z)dz\right| &\le \int_0^1 \frac{|s_N(-1+it)|^m \; |e^{-s_N(-1+it)/a}| \, |u_r(s_N(-1+it))|}{|1+e^{-s_N(-1+it)}|} s_N dt \\ &\le \frac{CN^{m+1}e^{s_N/a}}{e^{s_N}-1} \cdot \frac{e^{-s_N}}{e^{-s_N}} \\ &= \frac{CN^{m+1}e^{-s_N\left(1-\frac{1}{a}\right)}}{1-e^{-s_N}} \to 0 \text{ as } N \to \infty.\end{aligned}$$

Next we observe that on γ_2 we have

$$|u(\gamma_2(t))| = \left|e^{-iN\log r(t+i)}\right| = e^{N \log r},$$

and that

$$N^{m+1}e^{N\log r} = e^{(m+1)\log N + N \log r} \to 0 \text{ as } N \to \infty.$$

Hence

$$\begin{aligned}\left|\int_{\gamma_2} F_{m,r}(z)dz\right| &\le \int_{-1}^1 \frac{|s_N(i+t)|^m |e^{-s_N(t+i)/a}| \, |u_r(s_N(i+t))|}{|1+e^{-s_N(t+i)}|} s_N dt \\ &\le CN^{m+1}e^{N\log r}\int_{-1}^1 \frac{e^{-s_N t/a}}{1+e^{-ts_N}}dt \\ &\le CN^{m+1}e^{N\log r}\left(\int_0^1 \underbrace{e^{-s_N t/a}}_{\le 1}dt + \int_{-1}^0 \underbrace{e^{s_N t\left(1-\frac{1}{a}\right)}}_{\le 1}dt\right) \\ &\to 0 \cdot 0 \text{ as } N \to \infty,\end{aligned}$$

where the property $\lim \int = \int \lim$ is used (Lebesgue's dominated convergence theorem: the integrands are bounded (in moduli) by 1 and converge to 0 on the associated open intervals).

By letting $N \to \infty$, we conclude that

$$J_r(a) = 2\pi i \left(-\frac{\pi^2}{6a}\sum_{n=0}^{\infty}\text{Res } (F_{0,r}, z_n) - \frac{1}{2a}\sum_{n=0}^{\infty}\text{Res } (F_{2,r}, z_n)\right).$$

That the series converge will be clear in a moment. To this end, we need to calculate the residua. Note that

$$u_r(z_n) = e^{-i\frac{\log r}{2\pi}i\pi(1+2n)} = (\sqrt{r})^{1+2n}.$$

Hence

$$\begin{aligned}\text{Res } (F_{m,r}, z_n) &= z_n^m \frac{e^{-z_n/a}}{-e^{-z_n}} u_r(z_n) = -(i\pi(1+2n))^m \frac{e^{-i\pi(1+2n)/a}}{e^{-i\pi(1+2n)}} u_r(z_n) \\ &= (\sqrt{r})^{1+2n} \; (i\pi(1+2n))^m e^{-i\pi(1+2n)/a}.\end{aligned}$$

Let $\zeta := e^{-i\pi/a}$. Then

$$\begin{aligned} J_r(a) &= 2\pi i\left(-\frac{\pi^2}{6a}\sum_{n=0}^{\infty}(\sqrt{r}\zeta)^{2n+1} - \frac{1}{2a}\sum_{n=0}^{\infty}(i\pi(1+2n))^2(\sqrt{r}\zeta)^{2n+1}\right) \\ &= 2\pi i\left(-\frac{\pi^2}{6a}\sum_{n=0}^{\infty}(\sqrt{r}\zeta)^{2n+1} + \frac{\pi^2}{2a}\sum_{n=0}^{\infty}(2n+1)^2(\sqrt{r}\zeta)^{2n+1}\right). \end{aligned}$$

It is straightforward to check the following result:

$$\sum_{n=0}^{\infty}(2n+1)^2 z^{2n+1} = z\frac{z^4+6z^2+1}{(1-z^2)^3} =: S(z). \tag{15}$$

Hence

$$J_r(a) = 2\pi i\left(-\frac{\pi^2}{6a}\frac{\sqrt{r}\zeta}{1-r\zeta^2} + \frac{\pi^2}{2a}S(\sqrt{r}\zeta)\right),$$

and so

$$I = \lim_{r\to 1} J_r(a) = 2\pi i\left(-\frac{\pi^2}{6a}\frac{\zeta}{1-\zeta^2} + \frac{\pi^2}{2a}S(\zeta)\right).$$

Note that $\zeta\,\overline{\zeta} = 1$, but $\zeta \neq 1$. A short calculation yields

$$S(\zeta) = \frac{\zeta^2+6+\overline{\zeta}^2}{(\overline{\zeta}-\zeta)^3}.$$

Consequently, by using that $\zeta^2+6+\overline{\zeta}^2 = (\zeta-\overline{\zeta})^2+8$,

$$\begin{aligned} I &= 2\pi i\left(-\frac{\pi^2}{6a}\frac{\zeta}{1-\zeta^2} + \frac{\pi^2}{2a}\frac{\zeta^2+6+\overline{\zeta}^2}{(\overline{\zeta}-\zeta)^3}\right) \\ &= 2\pi i\left(-\frac{\pi^2}{6a}\frac{1}{\overline{\zeta}-\zeta} + \frac{\pi^2}{2a}\frac{(\zeta-\overline{\zeta})^2+8}{(\overline{\zeta}-\zeta)^3}\right) \\ &= 2\pi i\left(-\frac{\pi^2}{6a}\frac{(\overline{\zeta}-\zeta)^2}{(\overline{\zeta}-\zeta)^3} + \frac{\pi^2}{2a}\frac{(\zeta-\overline{\zeta})^2+8}{(\overline{\zeta}-\zeta)^3}\right) \\ &= -\frac{\pi^3}{3a}\left(i\,\frac{(-4)\sin^2(\pi/a)}{(2i)^3\sin^3(\pi/a)} - i\,\frac{(-12)\sin^2(\pi/a)+24}{(2i)^3\sin^3(\pi/a)}\right) \\ &= -\frac{\pi^3}{3a}\left(\frac{3-\sin^2(\pi/a)}{\sin^3(\pi/a)}\right). \end{aligned}$$

Remark A more classical way is to compute

$$I(a) = -\frac{\pi^2}{6}\int_0^{\infty}\frac{1}{1+x^a}dx - \frac{1}{2}\int_0^{\infty}\frac{\log^2(x^a)}{1+x^a}\,dx$$

"directly" without the change of variable by applying the residue theorem to the functions $\frac{1}{1+z^a}$, $\frac{\log z}{1+z^a}$ and $\frac{\log^2 z}{1+z^a}$ for the standard branches of the power and logarithm and the boundary of the sectors

$$\{z \in \mathbb{C} : |z| < R, 0 \le \arg z \le \frac{2\pi}{a}\},$$

which contains one simple pole.

12451. *Proposed by Adam L. Bruce, Dexter, MI.* Let A and B be complex n-by-n and n-by-m matrices, respectively, let $0_{m,n}$ denote the m-by-n zero matrix, let I_m denote the m-by-m identity matrix, and let exp be the matrix exponential function. Prove

$$\exp\begin{bmatrix} A & B \\ 0_{m,n} & 0_{m,m}\end{bmatrix} = \begin{bmatrix} \exp(A) & \left(\int_0^1 \exp(tA)dt\right)\cdot B \\ 0_{m,n} & I_m \end{bmatrix}.$$

Solution to problem 12451 in Amer. Math. Monthly 131 (2024)

Raymond Mortini and Rudolf Rupp

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As usual, $M^0 = I_s$ where M is a square $s \times s$ matrix. Via induction

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} A^k & A^{k-1}B \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned}\exp\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} &= \sum_{k=0}^\infty \frac{1}{k!}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^k = I_{n+m} + \sum_{k=1}^\infty \frac{1}{k!}\begin{pmatrix} A^k & A^{k-1}B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \exp A & \sum_{k=1}^\infty \frac{1}{k!}A^{k-1}B \\ 0 & I_m \end{pmatrix}.\end{aligned}$$

But

$$\int_0^1 \exp(At)dt = \sum_{j=0}^\infty A^j \int_0^1 \frac{t^j}{j!} = \sum_{j=0}^\infty \frac{1}{(j+1)!}A^j.$$

Hence

$$\exp\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp A & \left(\int_0^1 \exp(At)dt\right) B \\ 0 & I_m \end{pmatrix}.$$

12436. *Proposed by Lorenzo Sauras-Altuzarra, Vienna University of Technology, Vienna, Austria.* For a positive integer n, evaluate

$$\prod_{k=1}^{n}\left(x+\sin^2\left(\frac{k\pi}{2n}\right)\right).$$

Solution to problem 12436 in Amer. Math. Monthly 130 (2023)

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We show that

$$\boxed{P(x):=\prod_{k=1}^{n}\left(x+\sin^2\left(\frac{k\pi}{2n}\right)\right)=2^{-2n+2}(x+1)U_{n-1}(2x+1)},$$

where $U_0=1$ and

$$U_n(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\binom{n-k}{k}(2x)^{n-2k}\overset{n\geq 1}{=}2^n\prod_{k=1}^{n}\left(x-\cos\left(\frac{k\pi}{n+1}\right)\right)$$

is the Chebyshev polynomial of the second kind.

This is very easy, though.

$$\begin{aligned}
P(x) &= \prod_{k=1}^{n}\left(x+\sin^2\left(\frac{k\pi}{2n}\right)\right)=\prod_{k=1}^{n}\left(x+\frac{1}{2}\left(1-\cos\left(\frac{k\pi}{n}\right)\right)\right)\\
&= 2^{-n}\prod_{k=1}^{n}\left(2x+1-\cos\left(\frac{k\pi}{n}\right)\right)=2^{-n}\prod_{k=1}^{n-1}\left(2x+1-\cos\left(\frac{k\pi}{n}\right)\right)\ (2x+1-\cos(\pi))\\
&= 2^{-2n+2}(x+1)U_{n-1}(2x+1).
\end{aligned}$$

12433. *Proposed by Etan Ossip, student, Queen's University, Kingston, ON, Canada.* For $x > 1$, prove

$$\frac{i}{2}\int_{-\infty}^{\infty} \frac{\tanh(\pi t)}{\left(\frac{1}{2}+it\right)^x}\,dt = \zeta(x),$$

where ζ is the Riemann zeta function.

Solution to problem 12433 in Amer. Math. Monthly 130 (2023), ?

Raymond Mortini and Rudolf Rupp

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Let $G := \{z \in \mathbb{C} : \text{Im}z < 0.5\}$ be the shifted lower half-plane, and let $\log z = \log|z| + i\arg z$ with $-\pi < \arg z < \pi$ be the standard holomorphic branch of the logarithm. Since for $z \in G$ we have Re $(0.5 + iz) > 0$, the function

$$(0.5 + iz)^x = e^{x\log(0.5+iz)}$$

is well defined and holomorphic in G. Consequently, the function

$$f(z) := \frac{\tanh(\pi z)}{(0.5 + iz)^x},$$

is meromorphic in G with simple poles at $z_k := -i(0.5 + k) \in G$, where $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. We apply now the residue theorem to f. To this end, we integrate for $N \geq 1$ the function f along the positively oriented boundary Γ_N of the rectangles $R_N := [-N, N] \times [0, -N]$ and conclude that

$$\int_{\Gamma_N} f(z)dz = 2\pi i\sum_{k=0}^{\infty} n(\Gamma_N, z_k)\text{Res}(f, z_k),$$

where $n(\Gamma, z)$ denotes the number of times the point z is surrounded by Γ. Observe that at most a finite number of terms in this sum are not equal to 0 as

$$n(\Gamma_N, z_k) = \begin{cases} 1 & \text{if } k = 0, 1, \dots, N \\ 0 & \text{if } k > N. \end{cases}$$

Let us calculate the residue now. We use the formula Res $(\frac{g}{h}, a) = \frac{g(a)}{h'(a)}$, whenever a is a simple zero of h. That is, when we choose $g(z) = \dfrac{\sinh(\pi z)}{(0.5 + iz)^x}$ and $h(z) = \cosh(\pi z)$,

$$\text{Res}(f, z_k) = \frac{\sinh(\pi z)}{(1 + k)^x}\frac{1}{\pi\sinh(\pi z)}\Big|_{z=-i(0.5+k)} = \frac{1}{\pi}\,\frac{1}{(1 + k)^x}.$$

It remains to show that the integral along the three parts Γ_N^j of Γ_N that are contained in the lower half plane Im $z < 0$ tends to zero. First note that

$$\tanh(\pi z) = \frac{e^{2\pi z} - 1}{e^{2\pi z} + 1}.$$

i) Let $z(t) = -N - it$, where $0 \leq t \leq N$. Then for $n \geq N_0$,

$$|\tanh z(t)| \quad = \quad \left|\frac{e^{-2\pi N}e^{-2i\pi t} - 1}{e^{-2\pi N}e^{-2i\pi t} - 1}\right| \leq \frac{1 + e^{-2\pi N}}{1 - e^{-2\pi N}} \leq 2.$$

Moreover,

$$|0.5 + iz(t)|^x \quad = \quad |0.5 - iN + t|^x \geq N^x.$$

ii) Let $z(t) = N - it$, where $0 \leq t \leq N$. Then for $n \geq N_0$,

$$|\tanh z(t)| \quad = \quad \left|\frac{e^{2\pi N}e^{-2i\pi t} - 1}{e^{2\pi N}e^{-2i\pi t} + 1}\right| \leq \frac{e^{2\pi N} + 1}{e^{2\pi N} - 1} \leq 2.$$

Moreover

$$|0.5 + iz(t)|^x = |0.5 + iN + t|^x \geq N^x.$$

iii) Let $z(t) = t - iN$ where $-N \leq t \leq N$. Then

$$|\tanh z(t)| = \left| \frac{e^{2\pi t} e^{-2i\pi N} - 1}{e^{2\pi t} e^{-2i\pi N} + 1} \right| = \frac{e^{2\pi t} - 1}{e^{2\pi t} + 1} \leq 1.$$

Moreover

$$|0.5 + iz(t)|^x = |0.5 + it + N|^x \geq (N + 0.5)^x \geq N^x.$$

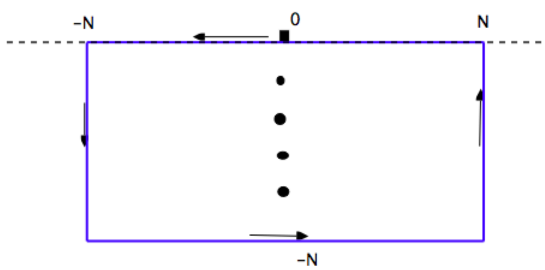
Since $x > 1$, we conclude that for $N \geq N_0$

$$\begin{aligned} \left| \sum_{j=1}^{3} \int_{\Gamma_N^j} f(z)dz \right| &\leq \sum_{j=1}^{2} \int_0^N |f(z_j(t))|dt + \int_{-N}^{N} |f(z_3(t))|dt \\ &\leq 2 \cdot \frac{2N}{N^x} + 2N \frac{1}{N^x} = \frac{6}{N^{x-1}} \to 0 \text{ as } N \to \infty. \end{aligned}$$

We conclude that

$$\frac{i}{2} \int_{-\infty}^{\infty} \frac{\tanh(\pi t)}{(0.5 + it)^x} \, dt = -\frac{i}{2} 2\pi i \sum_{k=0}^{\infty} \frac{1}{\pi} \, \frac{1}{(1+k)^x} = \zeta(x).$$

Note that the minus sign comes from the fact that the upper boundary of the rectangle R_N is run through from the right to the left.



12422. *Proposed by Mohammed Aassila, Strasbourg, France.* Let a, b, c be integers such that $a \neq 0$ and $an^2 + bn + c \neq 0$ for all positive integers n.
(a) Prove that if there is a positive integer k such that $b^2 - 4ac = k^2a^2$, then

$$\sum_{n=1}^{\infty} \frac{1}{an^2 + bn + c}$$

is rational.
(b)* Is the converse of **(a)** true?

Solution to problem 12422 in Amer. Math. Monthly 130 (2023), 862

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We solve (a). Put

$$R := \sum_{n=1}^{\infty} \frac{1}{an^2 + bn + c}.$$

Let r_1 and r_2 be the zeros of the polynomial $p(x) = ax^2 + bx + c$. Suppose that $a \neq 0$ and $b^2 - 4ac = k^2a^2$ for some $k \in \{1, 2, 3, \dots\}$. Then

$$r_1 = \frac{-b - ka}{2a} \text{ and } r_2 = \frac{-b + ka}{2a}$$

and $r_1 - r_2 = -k$. As an example we mention $a = 1$, $b = 5$ and $c = 4$, $k = 3$, $r_1 = -4$, $r_2 = -1$.

Now the partial fraction decomposition of $1/p(n)$ reads as

$$\frac{1}{an^2 + bn + c} = \frac{1}{a(n - r_1)(n - r_2)} = \frac{1}{a(r_1 - r_2)} \left(\frac{1}{n - r_1} - \frac{1}{n - r_2} \right).$$

Hence, for $n \geq n_0 > 1$ and n_0 chosen so that $n - r_j - 1 > 0$,

$$\begin{aligned}
S := \sum_{n=n_0}^{\infty} \frac{1}{an^2 + bn + c} &= \frac{1}{a(r_2 - r_1)} \sum_{n=n_0}^{\infty} \int_0^1 (x^{n-r_2-1} - x^{n-r_1-1})dx \\
&= \frac{1}{ak} \sum_{n=n_0}^{\infty} \int_0^1 x^{n-1}(x^{-r_2} - x^{-r_1})dx \overset{(1)}{=} \frac{1}{ak} \int_0^1 (x^{-r_2} - x^{-r_1}) \sum_{n=n_0}^{\infty} x^{n-1}dx \\
&= \frac{1}{ak} \int_0^1 x^{n_0-1} \frac{x^{-r_2} - x^{-r_1}}{1 - x} \, dx = \frac{1}{ak} \int_0^1 x^{n_0-1} x^{-r_2} \frac{1 - x^{r_2 - r_1}}{1 - x} \, dx \\
&= \frac{1}{ak} \int_0^1 x^{n_0 - r_2 - 1} \frac{1 - x^k}{1 - x} \, dx = \frac{1}{ak} \int_0^1 \sum_{j=0}^{k-1} x^{n_0 - 1 - r_2 + j} \, dx \\
&= \frac{1}{ak} \sum_{j=0}^{k-1} \frac{1}{n_0 - r_2 + j}.
\end{aligned}$$

Hence S is rational, and therefore R is rational, too.

Note that in (1) the interchanging $\int \sum = \sum \int$ is possible, since $x^{n-1}(x^{-r_2} - x^{-r_1})$ has constant sign.

12415. *Proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.* For a nonnegative integer n, evaluate

$$\sum_{j=0}^{2n}\sum_{k=\lfloor j/2\rfloor}^{j}\binom{2n+2}{2k+1}\binom{n+1}{2k-j}.$$

Solution to problem 12415 in Amer. Math. Monthly 130 (2023), 765

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Let $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and let

$$S_n := \sum_{j=0}^{2n}\sum_{k=\lfloor j/2\rfloor}^{j}\binom{2n+2}{2k+1}\binom{n+1}{2k-j}.$$

We show that

$$\boxed{S_n = 2^{3n+1}.}$$

First we interchange the two summations.

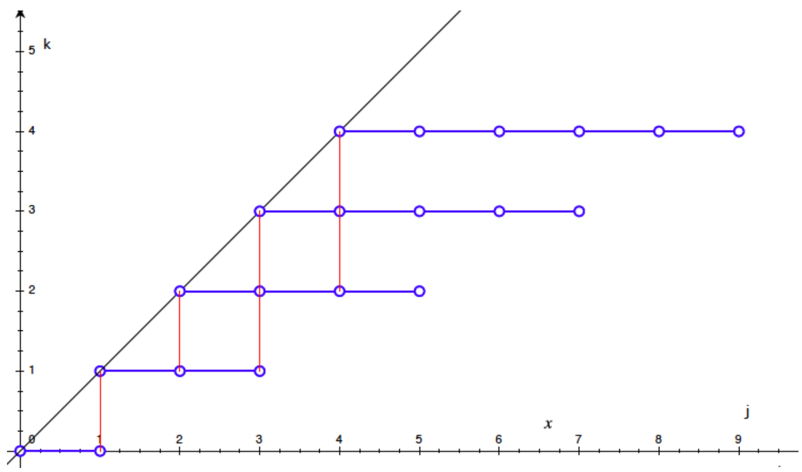


FIGURE 3. $k \le j \le 2k+1,\ k = 0, 1, 2, 3, 4$, or $\lfloor j/2\rfloor \le k \le j$ for $j = 0, 1, 2, 3, 4$

$$\begin{aligned} S_n &= \sum_{k=0}^{2n}\sum_{j=k}^{2k+1}\binom{2n+2}{2k+1}\binom{n+1}{2k-j} = \sum_{k=0}^{2n}\left[\binom{2n+2}{2k+1}\sum_{j=k}^{2k}\binom{n+1}{2k-j}\right] \\ &= \sum_{k=0}^{n}\left[\binom{2n+2}{2k+1}\sum_{m=0}^{k}\binom{n+1}{m}\right] \end{aligned}$$

It is well known that $\displaystyle\sum_{k=0}^{n}\binom{2n+2}{2k+1} = 2^{2n+1}$. In fact

$$2^{2n+2} = \sum_{k=0}^{2n+2}\binom{2n+2}{k} \quad \text{and} \quad 0 = (1+(-1))^{2n+2} = \sum_{k=0}^{2n+2}(-1)^k\binom{2n+2}{k}.$$

Substraction yields that

$$2^{2n+2} = 2\sum_{\substack{k=0\\k \text{ odd}}}^{2n+2}\binom{2n+2}{k} = 2\sum_{m=0}^{n}\binom{2n+2}{2m+1}.$$

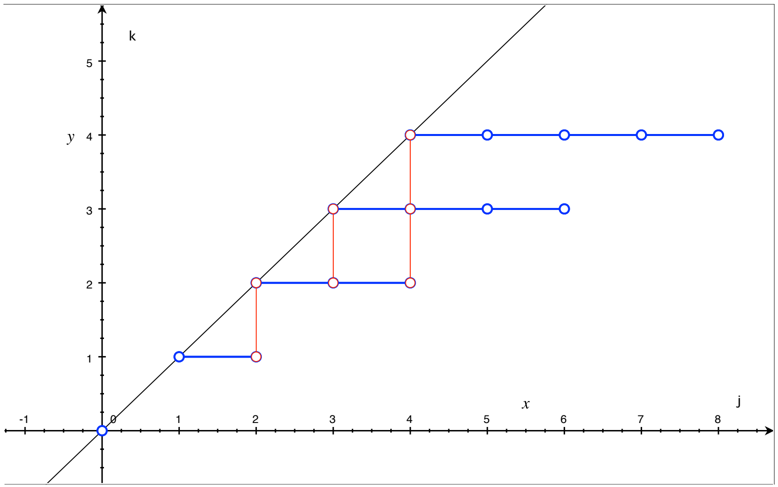
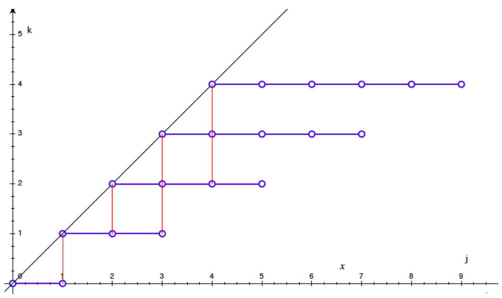
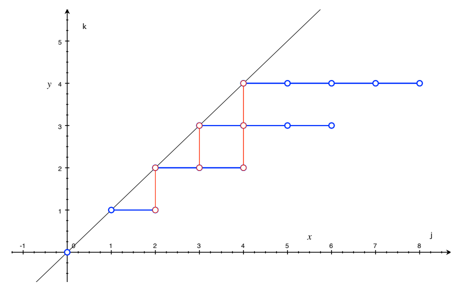


FIGURE 4. $k \leq j \leq 2k$, $k = 0, 1, 2, 3, 4$, or $\lceil j/2 \rceil \leq k \leq j$ for $j = 0, 1, 2, 3, 4$





Also,

$$\begin{aligned} 2^{n+1} &= (1+1)^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} = \sum_{j=0}^{k} \binom{n+1}{j} + \sum_{j=k+1}^{n+1} \binom{n+1}{j} \\ &= \sum_{j=0}^{k} \binom{n+1}{j} + \sum_{j=k+1}^{n+1} \binom{n+1}{n+1-j} \\ &\underset{i=n+1-j}{=} \sum_{j=0}^{k} \binom{n+1}{j} + \sum_{i=0}^{n-k} \binom{n+1}{i}. \end{aligned}$$

Hence

$$\begin{aligned} S_n &= \sum_{k=0}^{n} \binom{2n+2}{2k+1} \sum_{m=0}^{k} \binom{n+1}{m} \underset{k=n-j}{=} \sum_{j=0}^{n} \binom{2n+2}{2n-2j+1} \sum_{i=0}^{n-j} \binom{n+1}{i} \\ &= \sum_{j=0}^{n} \binom{2n+2}{(2n+2)-(2n-2j+1)} \sum_{i=0}^{n-j} \binom{n+1}{i} \\ &= \sum_{j=0}^{n} \binom{2n+2}{1+2j} \sum_{i=0}^{n-j} \binom{n+1}{i} \underset{j\to k}{=} \sum_{k=0}^{n} \binom{2n+2}{2k+1} \sum_{i=0}^{n-k} \binom{n+1}{i}. \end{aligned}$$

Addition yields

$$\begin{aligned}2S_n &= \sum_{k=0}^{n}\binom{2n+2}{2k+1}\sum_{j=0}^{k}\binom{n+1}{j}+\sum_{k=0}^{n}\binom{2n+2}{2k+1}\sum_{i=0}^{n-k}\binom{n+1}{i}\\ &= \sum_{k=0}^{n}\binom{2n+2}{2k+1}\left(\sum_{j=0}^{k}\binom{n+1}{j}+\sum_{i=0}^{n-k}\binom{n+1}{i}\right)\\ &= 2^{2n+1}\cdot 2^{n+1}=2^{3n+2}.\end{aligned}$$

Hence $S_n = 2^{3n+1}$.

Remarks

(1)Note that

$$S_n \underset{j=k-m}{=} \sum_{k=0}^{n}\left[\binom{2n+2}{2k+1}\sum_{j=0}^{k}\binom{n+1}{k-j}\right]$$

This has the form $\sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} b_{k-j}$, which is a little bit different from the Cauchy product

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j}.$$

(2) Replacing $\lfloor j/2\rfloor$ by $\lceil j/2\rceil$ yields the same result

$$R_n := \sum_{j=0}^{2n}\sum_{k=\lceil j/2\rceil}^{j}\binom{2n+2}{2k+1}\binom{n+1}{2k-j}=2^{3n+1}$$

(see below), alhough the associated index-grid is different (see figure 3 and 4).

Just note that

$$R_n = \sum_{k=0}^{2n}\sum_{j=k}^{2k}\binom{2n+2}{2k+1}\binom{n+1}{2k-j}=\sum_{k=0}^{2n}\sum_{j=k}^{2k+1}\binom{2n+2}{2k+1}\binom{n+1}{2k-j}=S_n$$

12407. *Proposed by an anonymous contributor, New Delhi, India.* Let r be a positive real number. Evaluate

$$\int_0^\infty \frac{x^{r-1}}{\left(1+x^2\right)\left(1+x^{2r}\right)}\,dx.$$

Solution to problem 12407 in Amer. Math. Monthly 130 (2023), **

Raymond Mortini and Rudolf Rupp

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Given $r>0$, let

$$I(r):=\int_0^\infty \frac{x^{r-1}}{(1+x^2)(1+x^{2r})}\,dx.$$

We show that

$$\boxed{I(r)=\frac{\pi}{4r}}.$$

First it is clear that the integral converges since at ∞ we have that the integrand $f_r(x)$ is similar to $1/x^{r+3}$ and at 0 $f_r(x)$ is similar to x^{r-1}, where $r-1>-1$. We make the change of the variable $x\to 1/y$. Then

$$\begin{aligned} I(r) &= \int_0^\infty \frac{y^{1-r}}{(1+y^{-2})(1+y^{-2r})}\frac{dy}{y^2}=\int_0^\infty \frac{y^{1+r}}{(1+y^2)(1+y^{2r})}\,dy\\ &= \int_0^\infty \frac{(y^2+1-1)y^{r-1}}{(1+y^2)(1+y^{2r})}\,dy=\int_0^\infty \frac{y^{r-1}}{1+y^{2r}}\,dy-I(r). \end{aligned}$$

Hence

$$2I(r)=\frac{1}{r}\arctan(y^r)\Big|_0^\infty=\frac{\pi}{2r},$$

from which we deduce that $I(r)=\pi/(4r)$.

12406. *Proposed by Raymond Mortini, University of Luxembourg, Esch-sur-Alzette, Luxembourg, and Rudolf Rupp, Nuremberg Institute of Technology, Nuremberg, Germany.* For fixed $p \in \mathbb{R}$, find all functions $f : [0, 1] \to \mathbb{R}$ that are continuous at 0 and 1 and satisfy $f\left(x^2\right) + 2pf(x) = (x + p)^2$ for all $x \in [0, 1]$.

Solution to problem 12406 in Amer. Math. Monthly 130 (2023), 679

Raymond Mortini and Rudolf Rupp

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Let $p \in \mathbb{R}$. Consider the functional equation

$$f(x^2) + 2pf(x) = (x + p)^2. \tag{16}$$

We claim that all solutions of (16) on $[0, 1]$ and continuous at $\{0, 1\}$ are actually continuous on $[0, 1]$ and are given by

$$f(x) = x + \frac{p^2}{1 + 2p}$$

whenever $p \neq -1/2$.

• If $p = -1/2$, then $f(x^2) - f(x) = (x - \frac{1}{2})^2$ has no solution on $[0, 1]$ (independently of being continuous or not) since for $x = 1$, we would get $0 = f(1) - f(1) = 1/4$.

• If $p = 0$, then $f(x^2) = x^2$ implies that on $[0, 1]$ one has $f(x) = x$.

• Let $p \neq -1/2$. We first determine the polynomial solutions. So let q be a polynomial solving (16). Then the degree of q is at most 1. Say $q(x) = ax + b$. Pulling into the functional equation yields

$$ax^2 + b + 2p(ax + b) = x^2 + 2px + p^2$$

or equivalently

$$(a - 1)x^2 + 2p(a - 1)x + b(1 + 2p) - p^2 = 0.$$

Hence $a = 1$ and $b = \frac{p^2}{1+2p}$.

It is straighforward to check that $q(x) = x + \frac{p^2}{1+2p}$ is indeed a solution to (16). We conclude that all polynomial solutions are given by the linear function q above.

Next we determine the general solution (16). So let f be a solution on $[0, 1]$ continuous at $0, 1$. Now put $h(x) := f(x) - q(x)$. Then h satisfies on $[0, 1]$ the functional equation (of Schroeder type)

$$h(x^2) = -2ph(x). \tag{17}$$

Of course this implies that $h(0) = 0$.

i) Let $p < -1/2$ or $p \geq 1/2$. Via induction

$$h(x^{2^n}) = (-2p)^n h(x).$$

Since h is continuous at 0, and $x^{2^n} \to 0$ for $0 < x < 1$, $h(0) = 0$, and $|2p|^n \to \infty$ respectively $(-2p)^n = (-1)^n$ if $p = 1/2$, we deduce that $h(x) = 0$ for $0 < x < 1$, too.

ii) If $0 < |p| < 1/2$, we rewrite (17) as

$$h(\sqrt{x}) = -\frac{1}{2p} h(x). \tag{18}$$

Via induction

$$h(x^{1/2^n}) = \left(-\frac{1}{2p}\right)^n h(x).$$

Since h is assumed to be continuous at 1 and $h(1) = 0$ by (17), $\left(\frac{1}{2p}\right)^n \to \infty$ implies that

$$0 = h(1) = \infty \cdot h(x),$$

and so $h(x) = 0$ for $x > 0$.

We conclude that for $p \neq -1/2$, the general solution to (16) on $[0, 1]$, and continuous at $0, 1$, is given by our poynomial

$$q(x) = x + \frac{p^2}{1 + 2p}.$$

Thus the solution is completely established.

The Schröder type functional equations $f(x^m) = rf(x)$ are analyzed in detail in [6].

12398. *Proposed by Lawrence Glasser, Clarkson University, Potsdam, NY.* Evaluate

$$\sum_{n=0}^{\infty} \operatorname{csch}(2^n).$$

Solution to problem 12398 in Amer. Math. Monthly 130 (2023), 587

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We suppose that this agglomeration *csch* of letters is nothing but $1/\sinh$. So let

$$S := \sum_{n=0}^{\infty} \frac{1}{\sinh 2^n}.$$

We prove that

$$\boxed{S = \tfrac{2}{e-1}}.$$

This is very simple though. Since $2\sinh x = e^x - e^{-x}$ and

$$(e^{2^n}+1)(e^{2^n}-1) = e^{2^{n+1}} - 1,$$

we obtain

$$\begin{aligned} S &= 2\sum_{n=0}^{\infty} \frac{1}{e^{2^n} - e^{-2^n}} = 2\sum_{n=0}^{\infty} \frac{e^{2^n} + 1 - 1}{e^{2^{n+1}} - 1} \\ &= 2\sum_{n=0}^{\infty} \left(\frac{1}{e^{2^n}-1} - \frac{1}{e^{2^{n+1}}-1} \right) = \frac{2}{e-1}. \end{aligned}$$

Another possibility would be to use the formula

$$\frac{1}{\sinh x} = \coth(x/2) - \coth x.$$

Then

$$S = \sum_{n=0}^{\infty} \big(\coth(2^{n-1}) - \coth 2^n\big) = \coth(1/2) - 1 = \frac{2e^{-1/2}}{e^{1/2} - e^{1/2}} = \frac{2}{e-1}.$$

12389. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f(x) = \sum_{n=1}^{\infty} |\sin(nx)|/n^2$. Prove $\lim_{x\to 0^+} f(x)/(x\ln x) = -1$.

Solution to problem 12389 in Amer. Math. Monthly 130 (2023), 386

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Our tool will be the fact that for $H_n := \sum_{j=1}^{n} \frac{1}{j}$ we have $H_n - \log n \searrow \gamma$, where γ is the Euler-Mascheroni constant. First note that

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{N}{(N+j)^2} &\leq \sum_{j=1}^{\infty} \frac{N}{(N+j)(N+j-1)}\\ &= N\sum_{j=1}^{\infty}\Big(\frac{1}{N+j-1} - \frac{1}{N+j}\Big)\\ &= 1.\end{aligned}$$

Fix $0 < x < 1$ and let $N := N(x) := \left\lfloor \frac{1}{x} \right\rfloor$. Let $\varepsilon \in]0, 1/2]$. Since $|\sin y| \leq y$ for $y \geq 0$, we obtain for

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|\sin(nx)|}{n^2} &\leq \sum_{n=1}^{N} \frac{nx}{n^2} + \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq x \sum_{n=1}^{N} \frac{1}{n} + \frac{1}{N}\\ &\leq x\Big(H_N - \log N - \gamma\Big) + x\gamma + x\log N + \frac{1}{N}.\end{aligned}$$

Hence, for x small enough, N is big, and so

$$H(x) := \frac{\sum_{n=1}^{\infty} \frac{|\sin(nx)|}{n^2}}{x\log(1/x)} \leq \frac{1}{\log(1/x)} + \frac{\gamma}{\log(1/x)} + \frac{\log\left\lfloor \frac{1}{x}\right\rfloor}{\log(1/x)} + \frac{1}{x\log(1/x)\left\lfloor \frac{1}{x}\right\rfloor}.$$

We conclude that

$$0 \leq \limsup_{x\to 0} H(x) \leq 0 + 0 + 1 + 0 = 1.$$

Now we estimate $\liminf_{x\to 0} H(x)$. Let $\varepsilon \in]0, 1/2]$. Since $x \mapsto (\sin x)/x$ is decreasing on $[0, \pi/2]$, we see that for $0 < u \leq \varepsilon$

$$\frac{\sin u}{u} \geq \frac{\sin\varepsilon}{\varepsilon}.$$

For $0 < x < \varepsilon$ put $N := N(x) := \left\lfloor \frac{\varepsilon}{x} \right\rfloor$. Then, $N > 0$ and for $n \leq N$ we have

$$nx \leq Nx = \left\lfloor \frac{\varepsilon}{x} \right\rfloor x \leq \frac{\varepsilon}{x} x = \varepsilon,$$

and so

$$\frac{\sin(nx)}{nx} \geq \frac{\sin\varepsilon}{\varepsilon}.$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|\sin(nx)|}{n^2} &\geq \frac{\sin\varepsilon}{\varepsilon} \sum_{n=1}^{N} \frac{nx}{n^2} = x\,\frac{\sin\varepsilon}{\varepsilon} \sum_{n=1}^{N} \frac{1}{n}\\ &\geq x\,\frac{\sin\varepsilon}{\varepsilon} \log N.\end{aligned}$$

We deduce that for $0 < x < \varepsilon$

$$\begin{aligned} H(x) &\geq \frac{\sin\varepsilon}{\varepsilon}\,\frac{\log\left\lfloor\frac{\varepsilon}{x}\right\rfloor}{\log(1/x)} \geq \frac{\sin\varepsilon}{\varepsilon}\,\frac{\log(\frac{\varepsilon}{x}-1)}{\log(1/x)} \\ &= \frac{\sin\varepsilon}{\varepsilon}\,\frac{\log(\varepsilon-x)-\log x}{-\log x}. \end{aligned}$$

Since

$$\lim_{x\to 0}\frac{\log(\varepsilon-x)}{-\log x} = \log\varepsilon\cdot 0$$

we conclude that

$$\liminf_{x\to 0} H(x) \geq \frac{\sin\varepsilon}{\varepsilon}.$$

Now $\varepsilon \to 0$ yields that $\liminf\limits_{x\to 0} H(x) \geq 1$. Consequently $\lim_{x\to 0} H(x) = 1$.

12388. *Proposed by Antonio Garcia, Strasbourg, France.* Let α be a real number. Evaluate

$$\int_0^\infty \frac{(\ln x)^2 \arctan(x)}{1-2(\cos\alpha)x+x^2}dx.$$

Solution to problem 12388 in Amer. Math. Monthly 130 (2023), 385

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For $a \in [0, 2\pi]$, let

$$I(a) := \int_0^\infty \frac{(\log x)^2 \arctan x}{1-2x\cos a + x^2}\, dx.$$

We prove that

$$I(a) = \begin{cases} \pi \dfrac{a}{\sin a} \dfrac{(2\pi - a)(\pi - a)}{12} & \text{if } 0 < a < 2\pi,\ a \neq \pi \\[2ex] \dfrac{\pi^3}{6} & \text{if } a = 0 \text{ or } a = 2\pi \\[2ex] \dfrac{\pi^3}{12} & \text{if } a = \pi. \end{cases}$$

If a is arbitrary, we replace a by $a - 2k\pi$, where $k \in \mathbb{Z}$ is chosen so that $2k\pi \le a < 2(k+1)\pi$.

First we let "disappear" the arctangent: the substitution $u = 1/x$, $dx = -1/u^2$ and the formula $\arctan(1/x) + \arctan x = \pi/2$ for $x > 0$ yield

$$I(a) = \int_0^\infty \frac{(\log u)^2 \left(\frac{\pi}{2} - \arctan u\right)}{1 - 2\frac{1}{u}\cos a + \frac{1}{u^2}} \frac{du}{u^2} = -I + \frac{\pi}{2} \int_0^\infty \frac{(\log x)^2}{1-2x\cos a + x^2}\, dx,$$

and so

$$I(a) = \frac{\pi}{4} \int_0^\infty \frac{(\log x)^2}{1-2x\cos a + x^2}\, dx.$$

Using again the transformation $u = 1/x$, we obtain that

$$\int_0^1 \frac{(\log x)^2}{1-2x\cos a + x^2}\, dx = \int_1^\infty \frac{(\log x)^2}{1-2x\cos a + x^2}\, dx,$$

and so

$$\boxed{I(a) = \frac{\pi}{2} \int_0^1 \frac{(\log x)^2}{1-2x\cos a + x^2}\, dx.}$$

Next we use that for $a \notin \{k\pi : k \in \mathbb{Z}\}$

$$\frac{1}{1-2x\cos a + x^2} = \frac{A}{x - e^{ia}} - \frac{\overline{A}}{x - e^{-ia}},$$

where $A = -\dfrac{i}{2\sin a}$. Hence, in that case,

$$\begin{aligned} I(a) &= 2\mathrm{Re}\left(\frac{\pi}{2} A \int_0^1 \frac{(\log x)^2}{x - e^{ia}}\, dx\right) \\ &= \frac{\pi}{2\sin a} \mathrm{Im} \int_0^1 \frac{(\log x)^2}{x - e^{ia}}\, dx \\ &= \frac{\pi}{2\sin a} \mathrm{Im}\,(-e^{ia}) \int_0^1 \frac{(\log x)^2}{1 - xe^{-ia}}\, dx \end{aligned}$$

Since $\sum_{n=0}^{\infty} x^n (\log x)^2$ is an $L^1(0,1)$-majorant, we have $\int \sum = \sum \int$. Thus

$$I(a) = -\frac{\pi}{2\sin a}\,\mathrm{Im}\Big(\sum_{n=0}^{\infty} e^{-ia(n+1)} \int_0^1 x^n (\log x)^2\,dx\Big).$$

By twice partial integration,

$$\int_0^1 x^n (\log x)^2\,dx = \frac{2}{(n+1)^3}.$$

We conclude that

$$I(a) = -\frac{\pi}{2\sin a}\,\mathrm{Im}\Big(\sum_{n=0}^{\infty} e^{-ia(n+1)} \frac{2}{(n+1)^3}\Big) = \frac{\pi}{\sin a}\sum_{n=0}^{\infty}\frac{\sin(n+1)a}{(n+1)^3}.$$

Let

$$h(a) := \sum_{n=0}^{\infty}\frac{\sin(n+1)a}{(n+1)^3}.$$

Then

$$h'(a) = \sum_{n=0}^{\infty}\frac{\cos(n+1)a}{(n+1)^2}.$$

Since $\frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ is the Fourier series of the function $(x-\pi)^2$, $0 \le x < 2\pi$, extended 2π-periodically, we see that for $0 < a < 2\pi$,

$$h'(a) = \frac{(a-\pi)^2}{4} - \frac{\pi^2}{12}.$$

As $h(0) = 0$, we deduce that for $0 < a < 2\pi$,

$$h(a) = \frac{(a-\pi)^3}{12} - \frac{\pi^2}{12}\,a + \frac{\pi^3}{12} = \frac{a^3 - 3\pi a^2 + 2\pi^2 a}{12}.$$

Consequently, for $0 < a < 2\pi$, $a \ne \pi$,

$$I(a) = \pi\,\frac{a}{\sin a}\,\frac{(2\pi - a)(\pi - a)}{12}.$$

Now let $a_n \searrow 0$ and $f_n(a) := \dfrac{(\log x)^2}{1 - 2x\cos a_n + x^2}$. As f_n is positive and increases to $\frac{(\log x)^2}{(1-x)^2}$, we deduce from Beppo-Levi's monotone convergence theorem that $I(a_n) \to I(0)$. Hence $I(0) = \pi^3/6$.

Moreover, $I(b_n) \to \frac{\pi^3}{12}$ as $b_n \nearrow \pi$. This is also the value of

$$I(\pi) = \frac{\pi}{2}\int_0^1 \frac{(\log x)^2}{(1+x)^2}\,dx.$$

Just write

$$\frac{(\log x)^2}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n-1} n x^{n-1} (\log x)^2,$$

and use again that $\int \sum = \sum \int$. Finally, $I(2\pi) = I(0) = \pi^3/6$.

12380. *Proposed by Dorin Mărghidanu, Alexandru Ioan Cuza National College, Corabia, Romania.* Let m, n, and p be positive integers, and let a, b, and c be nonnegative real numbers with $a+b+c=3$. Prove

$$\sqrt[m]{a+\sqrt[n]{b+\sqrt[p]{c}}}+\sqrt[m]{b+\sqrt[n]{c+\sqrt[p]{a}}}+\sqrt[m]{c+\sqrt[n]{a+\sqrt[p]{b}}}\leq 3\sqrt[m]{1+\sqrt[n]{2}},$$

and determine when equality occurs.

Solution to problem 12380 in Amer. Math. Monthly 130 (2023), 285

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Lemma Let f, g, h be positive increasing functions on $[0,\infty[$ satisfying for every $x_j\geq 0$ and $0\leq t_j\leq 1$ with $\sum_{j=1}^n t_j=1$ the concavity inequality

$$f\Big(\sum_{j=1}^n t_jx_j\Big)\geq\sum_{j=1}^n t_jf(x_j) \tag{19}$$

and similarly for g,h. Let $M_2, P_j\in[0,\infty[\times[0,\infty[$. Then the function G given by

$$G(M_2):=G(x,y):=f(x+g(y))$$

also satisfies

$$G\Big(\sum_{j=1}^n t_jP_j\Big)\geq\sum_{j=1}^n t_jG(P_j),$$

Similarly, if $M_3, Q_j\in[0,\infty[\times[0,\infty[\times[0,\infty[$, then the function H given by

$$H(M_3):=H(x,y,z):=f(x+g(y+h(z)))$$

satisfies

$$H\Big(\sum_{j=1}^n t_jQ_j\Big)\geq\sum_{j=1}^n t_jH(Q_j).$$

Proof

$$\begin{aligned}
G\Big(\sum_{j=1}^n t_jx_j,\sum_{j=1}^n t_jy_j\Big) &= f\Big(\sum_{j=1}^n t_jx_j+g\Big(\sum_{j=1}^n t_jy_j\Big)\Big)\\
&\geq f\Big(\sum_{j=1}^n t_jx_j+\sum_{j=1}^n t_jg(y_j)\Big)\\
&= f\Big(\sum_{j=1}^n t_j(x_j+g(y_j))\Big)\\
&\geq \sum_{j=1}^n t_jf(x_j+g(y_j))\\
&= \sum_{j=1}^n t_jG(P_j).
\end{aligned}$$

Now applying this, we get

$$\begin{aligned} H(\sum_{j=1}^n t_jQ_j) &= f\Big(\sum_{j=1}^n t_jx_j + g\Big(\sum_{j=1}^n t_jy_j + h(\sum_{j=1}^n t_jz_j)\Big)\Big) \\ &\geq f\Big(\sum_{j=1}^n t_jx_j + \sum_{j=1}^n t_jg(y_j + h(z_j))\Big) \\ &= f\Big(\sum_{j=1}^n t_j(x_j + g(y_j + h(z_j))\Big) \\ &\geq \sum_{j=1}^n t_jf(x_j + g(y_j + h(z_j))) \\ &= \sum_{j=1}^n t_jH(Q_j). \end{aligned}$$

Now we are ready to give the solution to the problem. Let

$$S(a,b,c) := \sqrt[m]{a + \sqrt[n]{b + \sqrt[p]{c}}} + \sqrt[m]{b + \sqrt[n]{c + \sqrt[p]{a}}} + \sqrt[m]{c + \sqrt[n]{a + \sqrt[p]{b}}}.$$

For $M := (x,y,z) \in \mathbb{R}^3, x,y,z \geq 0$, let

$$f(M) := f(x,y,z) := \sqrt[m]{x + \sqrt[n]{y + \sqrt[p]{z}}}.$$

By Lemma, for $P := (a,b,c)$, $Q = (b,c,a)$ and $R = (c,a,b)$ we have

$$\frac{1}{3}\big(f(P) + f(Q) + f(R)\big) \leq f\left(\frac{P+Q+R}{3}\right). \tag{20}$$

Since $a + b + c = 3$, we deduce that

$$f(P) + f(Q) + f(R) \leq 3 \cdot f(1,1,1) = 3\sqrt[m]{1 + \sqrt[n]{2}}.$$

In case $mnp > 1$, at least one function $\sqrt[r]{x}$ for $r \in \{m,n,p\}$ is strictly concave and we have strict inequality in (19) whenever not all the x_j are the same and $0 < t_j < 1$. The proof of the Lemma in particular then yields that equality holds in (20) only if $P = Q = R$, and so $a = b = c$. Thus, due to $a + b + c = 3$, we deduce that $a = b = c = 1$. Hence

$$S(a,b,c) = 3\sqrt[m]{1 + \sqrt[n]{2}}$$

if and only if $(a,b,c) = (1,1,1)$.

If $m = n = p = 1$, then $f(M)$ is linear in $\mathbb{R}^3$, and so for all a,b,c with $a+b+c = 3$ we have equality:

$$S(a,b,c) = 3(a+b+c) = 9 = 3\sqrt[m]{1 + \sqrt[n]{2}}.$$

12372. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* For $\alpha > 0$, evaluate

$$\int_0^1 \frac{\ln|x^\alpha - (1-x)^\alpha|}{x}\, dx.$$

Solution to problem 12372 in Amer. Math. Monthly 130 (2023), 187

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For $a > 0$, let $I := \displaystyle\int_0^1 \frac{\log|x^a - (1-x)^a|}{x}\, dx$, which is a double improper integral (with singularities at $0, 1/2$). We show that

$$\boxed{I = -\frac{a^2+2}{12a}\,\pi^2}.$$

To this end, we first note that $x^a \le (1-x)^a$ if and only if $0 \le x \le 1/2$. Hence, by substituting $x \to 1-x$ in the second integral

$$\begin{aligned}
I &= \int_0^{1/2} \frac{\log\big((1-x)^a - x^a\big)}{x}\, dx + \int_0^{1/2} \frac{\log\big(x^a - (1-x)^a\big)}{x}\, dx\\
&= \int_0^{1/2} \frac{\log\big((1-x)^a - x^a\big)}{x}\, dx + \int_0^{1/2} \frac{\log\big((1-x)^a - x^a\big)}{1-x}\, dx\\
&= \int_0^{1/2} \frac{\log\big((1-x)^a - x^a\big)}{x(1-x)}\, dx\\
&= \int_0^{1/2} \frac{\log\Big(1 - \big(\frac{x}{1-x}\big)^a\Big) + a\log(1-x)}{x(1-x)}\, dx.
\end{aligned}$$

Next we substitute $x/(1-x) = y$. Equivalently, $x = y/(1+y)$. Note that $0 \to 0$ and $1/2 \to 1$, $dx = \frac{1}{(1+y)^2} dy$ and $1 - x = \frac{1}{1+y}$. Hence

$$I = \int_0^1 \frac{\log(1-y^a) - a\ \log(1+y)}{\frac{y}{(1+y)^2}}\ \frac{1}{(1+y)^2}\, dy.$$

Consequently,

$$I = \int_0^1 \frac{1}{x} \log\left(\frac{1-x^a}{(1+x)^a}\right)\, dx.$$

Using partial integration for $\int_\varepsilon^{1-\eta}$ with $u' := 1/x$ and $v = \log\big(\frac{1-x^a}{(1+x)^a}\big)$, and passing to the limits $\varepsilon, \eta \to 0$, we obtain

$$\begin{aligned}
I &= 0 + a\int_0^1 \left(\frac{x^{a-1}}{1-x^a} + \frac{1}{1+x}\right) \log x\, dx\\
&= a\int_0^1 \left(\sum_{n=0}^\infty x^{a-1}x^{na} + \sum_{n=0}^\infty (-1)^n x^n\right) \log x\, dx.
\end{aligned}$$

Note that I has the form $I = \int \sum$. Now let us calculate $J := \sum \int$.

To do this, we apply for $\beta > -1$ the formula

$$\int_0^1 x^\beta \log x \, dx = -\frac{1}{(\beta+1)^2},$$

(which can easily be obtained by partial integration $u = \log x, v' = x^\beta$). Hence

$$J/a = -\frac{1}{a^2} \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(n+1)^2} \tag{21}$$

$$= -\frac{1}{a^2} \frac{\pi^2}{6} - \frac{\pi^2}{12} \tag{22}$$

$$= -\frac{a^2+2}{12a^2} \pi^2. \tag{23}$$

To finish the proof, we need to show that $\int \sum = \sum \int$. As the summands in the first sum $\sum_{n=0}^{\infty} x^{a-1} x^{na} \log x$ do not change sign, we may use Beppo-Levi's theorem. In the second sum, $\sum_{n=0}^{\infty} (-1)^n x^n \log x$, we have absolute convergence, in particular any rearrangement converges (to the same function), and so we apply Beppo-Levi to the sum over the odd integers and the sum over the even integers. Thus $\int \sum_{even} = \sum_{even} \int$ and $\int \sum_{odd} = \sum_{odd} \int$. Similarily to (21), it can be shown that the values of $\sum_{odd} \int$ and $\sum_{even} \int$ are finite. Hence

$$\int \sum = \int \sum_{even} - \int \sum_{odd} = \sum_{even} \int - \sum_{odd} \int = \sum \int.$$

12375. *Proposed by Hongwei Chen, Christopher Newport University, Newport News, VA.* Let

$$I_n = \int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^n dx.$$

Problem 12288 [2021, 946] in this MONTHLY asked for a proof that $I_2 = \pi/5$. Prove that I_n is a rational multiple of π whenever n is a positive integer.

Solution to problem 12375 in Amer. Math. Monthly 130 (2023), ??

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A change of the variable $x \to 1/x$ yields that

$$J := \int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^n dx = \int_0^\infty \frac{(x^2 - \sin^2 x)^n}{x^{2n+2}}\, dx.$$

Now we "linearize" the trigonometric powers: using $\sin^2 x = (1/2)(1 - \cos 2x)$, we obtain

$$\begin{aligned} J &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\left(x^2 - \frac{1}{2} + \frac{1}{2}\cos(2x)\right)^n}{x^{2n+2}}\, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^{2n+2}} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2^j}\left(x^2 - \frac{1}{2}\right)^{n-j} \cos^j(2x)\, dx. \end{aligned}$$

Noticing that

$$\cos^j(2x) = \frac{1}{2^j}\sum_{k=0}^{j} \binom{j}{k} \cos(2(j-2k)x),$$

we finally obtain that with $I := 2J$

$$I = \int_{-\infty}^{\infty} \sum_{\ell=0}^{n} \frac{1}{x^{2n+2}}\; p_\ell(x)\cos(2\ell x)\, dx$$

where p_ℓ is a polynomial of degree at most $2n$ and with rational coefficients.

Next we consider the functions

$$f(z) := \sum_{\ell=0}^{n} \frac{1}{z^{2n+2}}\; p_\ell(z) e^{2i\ell z}$$

and

$$F(z) := f(z) - \frac{p(z)}{z^{2n+2}},$$

where $\frac{p(z)}{z^{2n+2}} = \frac{q(z)}{z^{2n+2}} + \frac{r}{z}$ is the principal part of the meromorphic function f. Note that $\deg p \le 2n+1$, $\deg q \le 2n$, and $r \in \mathbb{Q} + i\,\mathbb{Q}$.

In particular, F has a holomorphic extension to the origin, hence is an entire function. Therefore $\int_\Gamma F(z)dz = 0$, where Γ is the boundary of the half-disk $|z| \le R$, $\operatorname{Im} z \ge 0$, consisting of the half circle Γ_R and the interval $[-R, R]$. Hence, by letting $R \to \infty$ and taking real parts,

$$0 = \operatorname{Re} \lim_{R\to\infty} \int_{\Gamma_R} F(z)dz + I.$$

By Jordan's Lemma, $\limsup_{R\to\infty} \left|\int_{\Gamma_R} e^{inz} dz\right| < \infty$. Hence, by noticing that the differences of the degrees of the polynomials in the denominator and numerator if f is bigger than 2,

$$\lim_{R\to\infty} \int_{\Gamma_R} F(z)dz = 0 + 0 + \lim_{R\to\infty} \int_{\Gamma_R} \frac{r}{z} dz = i\, r\, \pi.$$

We conclude that the value of the original integral J is rational.

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12362. *Proposed by Antonio Garcia, Strasbourg, France.* Evaluate

$$\lim_{n\to\infty}\int_0^{\pi/2}\frac{n}{\left(\sqrt{2}\cos x\right)^n+\left(\sqrt{2}\sin x\right)^n}\,dx.$$

Solution to problem 12362 in Amer. Math. Monthly 129 (2022), 986

Raymond Mortini

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We reduce the present problem to Problem 12340, telling us that for each $f:[0,1]\to\mathbb{R}$ continuous,

$$\lim_{n\to\infty}\frac{n}{2^n}\int_0^1\frac{f(x)}{x^n+(1-x)^n}\,dx=\frac{\pi}{4}\,f\left(\frac{1}{2}\right). \tag{24}$$

First we note that one may replace of course n by t, $t\to\infty$. Later we shall take $t=n/2$. As a result we obtain

$$\boxed{\lim_{n\to\infty}\int_0^{\pi/2}\frac{n}{\left(\sqrt{2}\cos x\right)^n+\left(\sqrt{2}\sin x\right)^n}\,dx=\frac{\pi}{2}} \tag{25}$$

To see this, let $u:=\sin x$. Then, $dx=(1-u^2)^{-1/2}\,du$ and so with

$$I_n:=\int_0^{\pi/2}\frac{n}{\left(\sqrt{2}\cos x\right)^n+\left(\sqrt{2}\sin x\right)^n}\,dx,$$

we obtain

$$I_n=n\int_0^1\frac{(1-u^2)^{-1/2}}{2^{n/2}\,(1-u^2)^{n/2}+2^{n/2}\,(u^2)^{n/2}}\,dx$$

Now let $y:=u^2$. Then $du=\frac{1}{2\sqrt{y}}\,dy$, and so

$$I_n=\frac{n}{2\cdot 2^{n/2}}\int_0^1\frac{y^{-1/2}(1-y)^{-1/2}}{(1-y)^{n/2}+y^{n/2}}\,dy$$

Let $g_\varepsilon(y)=(y+\varepsilon)^{-1/2}(1-y+\varepsilon)^{-1/2}$ and $g:=g_0$. Then

$$\frac{n}{2\cdot 2^{n/2}}\int_0^1\frac{g_\varepsilon(y)}{(1-y)^{n/2}+y^{n/2}}\,dy\le I_n. \tag{26}$$

Next we estimate from above. Let $x\in[0,1]$ satisfy $|x-1/2|\ge\delta$, where $\delta>0$ is small. Then, for $t\ge 1$,

$$x^t+(1-x)^t\ge(1/2+\delta)^t+(1/2-\delta)^t.$$

Hence

$$\frac{t}{2^t}\,\frac{1}{x^t+(1-x)^t}\le\frac{t}{(1+2\delta)^t+(1-2\delta)^t}=:m_t\to 0\text{ as }t\to\infty$$

Now for $\varepsilon>0$, choose δ so small that $|g(x)-g(1/2)|<\varepsilon$ for $|x-1/2|\le\delta$. Then

$$\begin{aligned}\frac{t}{2^t}\int_0^1\frac{g(x)}{x^t+(1-x)^t}\,dx &\le \frac{t}{2^t}\int_{|x-1/2|\le\delta}\frac{g(1/2)+\varepsilon}{x^t+(1-x)^t}\,dx+m_t\int_{|x-1/2|>\delta}g(x)\,dx\\ &\le \frac{t}{2^t}\int_0^1\frac{g(1/2)+\varepsilon}{x^t+(1-x)^t}\,dx+m_t||g||_1\underset{t\to\infty}{\longrightarrow}\frac{\pi}{4}(g(1/2)+\varepsilon).\end{aligned}$$

Together with (26), we obtain that

$$\frac{\pi}{4}\,g_\varepsilon(1/2)\le\liminf_n I_n\le\limsup_n I_n\le\frac{\pi}{4}(g(1/2)+\varepsilon).$$

Hence, by letting $\varepsilon\to 0$,

$$\lim_n I_n=\frac{\pi}{4}\,g(1/2)=\pi/2.$$

12347. *Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bîrlad, Romania*. Let a and b be real numbers with $0 < a < 1 < b$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$.

Solution to problem 12347 in Amer. Math. Monthly 129 (2022), 786

Raymond Mortini

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We show that on $\mathbb{R}$ there are exactly 4 continuous solutions to the functional equation

(27) $$f(f(x)) - (a+b)f(x) + abx = 0,$$

whenever $f(0) = 0$ and $0 < a < 1 < b$. Namely

$$F_1(x) = ax,\ F_2(x) = bx,\ F_3(x) = \begin{cases} ax & \text{if } x \leq 0 \\ bx & \text{if } x > 0 \end{cases} \text{ and } F_3(x) = \begin{cases} bx & \text{if } x \leq 0 \\ ax & \text{if } x > 0. \end{cases}$$

It is easy to check that F_j are solutions. Now suppose that f is a solution.

i) f is injective: let $f(x) = f(y)$. Then

$$abx = -f(f(x)) + (a+b)f(x) = -f(f(y)) + (a+b)f(y) = aby$$

and so $x = y$.

ii) f is strictly increasing: monotonicity implies that $M^{\pm} := \lim_{x \to \pm\infty} f(x)$ exists in $[-\infty, \infty]$. Now $M^{\pm}$ cannot be finite, since (1) and continuity would imply that $f(M^{\pm}) - (a+b)M^{\pm} + \pm\infty = 0$, which is impossible. But $M^+ \neq -\infty$, either, since otherwise

$$f(f(x)) + abx = (a+b)f(x) \to -\infty \text{ as } x \to \infty,$$

and so $\lim\limits_{x \to \infty} f(f(x)) = -\infty$. Hence, with $y := f(x) \to -\infty$, we deduce that

$$\lim_{y \to -\infty} f(y) = -\infty = \lim_{x \to \infty} f(x),$$

contradicting the monotonicity of f. We conclude that f is strictly increasing, $f(x) \geq 0$ for $x \geq 0$, $f(x) \leq 0$ for $x \leq 0$, and $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$.

iii) The inverse $h := f^{-1} : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

(28) $$h(h(y)) - \left(\frac{1}{a} + \frac{1}{b}\right) h(y) + \frac{1}{ab}\, y = 0.$$

Just take $x := h(h(y))$ in (27) and note that $h \circ f = f \circ h = \text{id}$. Then

$$y - (a+b)h(y) + ab\, h(h(y)) = 0.$$

Now divide by ab. We also deduce the following identity:

(29) $$\big[f(y) - ay\big] + ab\big[f^{-1}(y) - (1/a)\, y\big] = 0.$$

In particular f^{-1} is increasing, too.

iv) The only fixed point of f is 0: let $f(s) = s$. If $s \neq 0$, then, by (27) $s - (a+b)s + abs = 0$. Thus $1 + ab = a + b$, or equivalently, $b(a-1) = a - 1$. That is, $b = 1$ (since $a < 1$). A contradiction. We conclude that for $x > 0$ either $f(x) < x$ or $f(x) > x$ for every $x > 0$.

v) Let $f_n := \underbrace{f \circ \cdots \circ f}_{n\text{-times}}$ be the n-th iterate of f [5].

[5] We never use the exponent n to designate the n-th iterate when working with functions, as the risk to mix it up with the n-th power is too big.

• Suppose that there is $x_0 > 0$ such that $f(x_0) < x_0$. Then $f_n(x) \to 0$ for every $x \geq 0$. Indeed, by iv), $0 < f(x) < x$ for $x > 0$. Hence

$$f_{n+1}(x) = f(f_n(x)) < f_n(x)$$

and so $M(x) := \lim_{n\to\infty} f_n(x)$ exists for every $x > 0$. Plugging $f_n(x)$ into the functional equation (27), yields

$$M(x) - (a+b)M(x) + abM(x) = 0.$$

Consequently, $M(x)(1+ab-(a+b)) = 0$. But $1+ab-(a+b) = (1-a)+b(a-1) = (1-a)(1-b) \neq 0$. Hence $M(x) = 0$.
• Suppose that there is $x_0 > 0$ such that $f(x_0) > x_0$. Then, by iv) $f(x) > x$ for every $x > 0$ and the sequence $(f_n(x))$ of iterates is increasing for each $x > 0$. As its limit $M(x)$ can't be finite, in particular not 0, we see that $\lim_{n\to\infty} f_n(x) = \infty$ for every $x > 0$.

vi) For each $x \in \mathbb{R}$ we obtain the following three terms difference equations:

$$f_{n+2}(x) - (a+b)f_{n+1}(x) + abf_n(x) = 0,$$

with initial condition $f_0(x) := x$ and $f_1(x) := f(x)$.

The associated characteristic polynomial is $p(z) = z^2 - (a+b)z + ab$, which has as roots a and b. Hence, there exist real coefficients A_x and B_x depending on the initial value x such that

$$f_n(x) = A_x a^n + B_x b^n. \tag{30}$$

If $f(x) < x$ for every $x > 0$, then $\lim_n f_n(x) = 0$ implies that $B_x = 0$, because $b > 1$ and $0 < a < 1$. Hence $f(x) = A_x a$. As the initial value $f_0(x)$ equals x, we deduce from (30) that $A_x = x$. Thus, for $x > 0$, $f(x) = ax$ whenever there exists $x_0 > 0$ with $f(x_0) < x_0$.

If $f(x) > x$ for every $x > 0$, then $x > f^{-1}(x)$ (note that by iii) $h := f^{-1}$ is increasing). Hence, the difference equations,

$$h_{n+2}(x) - \left(\frac{1}{a} + \frac{1}{b}\right) h_{n+1}(x) + \frac{1}{ab}\, h_n(x) = 0 \tag{31}$$

with initial values $h_0(x) = x$ have for $x > 0$ the solutions

$$h_n(x) = C_x \frac{1}{a^n} + D_x \frac{1}{b^n} \tag{32}$$

for real coefficients C_x and D_x. Using (31), we see as above that $\lim_{n\to\infty} h_n(x) = 0$. Hence $C_x = 0$. Thus $h(x) = h_1(x) = D_x \frac{1}{b}$. As $h_0(x) = x$, we deduce from (32) that $D_x = x$.

To some up, $f^{-1}(x) = h(x) = x/b$ and so $f(x) = bx$ for every $x > 0$ whenever there exists $x_0 > 0$ with $f(x_0) > x_0$.

vii) The case for negative arguments follows from the observation that if f is a solution to (27), then the function g given by $g(x) = -f(-x)$ is a solution, too:

$$\begin{aligned} g(g(x)) - (a+b)g(x) + abx &= -f(-f(-x)) + (a+b)f(-x) + abx \\ &= -\Big(f(f(-x)) - (a+b)f(-x) + ab(-x)\Big) = 0. \end{aligned}$$

12340. *Proposed by Antonio Garcia, Strasbourg, France.* Let $g : [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = Cg(1/2)$$

for some constant C (independent of g) and determine the value of C.

Solution to problem 12340 in Amer. Math. Monthly 129 (2022), 686

Raymond Mortini and Rudolf Rupp

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As $g : [0,1] \to \mathbb{R}$ is continuous, $||g||_\infty = \max\{|g(x)| : 0 \le x \le 1\} < \infty$. Let

$$I_n := \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n}\, dx.$$

We claim that $\lim_{n\to\infty} I_n = (\pi/4)g(1/2)$.

To see this, we split the integral into two parts and use two different change of variables:

$$\begin{aligned}
I_n &= \frac{n}{2^n} \int_0^{1/2} \underbrace{\frac{g(x)}{x^n + (1-x)^n}\, dx}_{x=:\frac{1}{2}-\frac{s}{2n}} + \frac{n}{2^n} \int_{1/2}^1 \underbrace{\frac{g(x)}{x^n + (1-x)^n}\, dx}_{x=:\frac{1}{2}+\frac{s}{2n}} \\
&= \frac{n}{2^n} \int_0^n \frac{g(\frac{1}{2} - \frac{s}{2n})}{(\frac{1}{2} - \frac{s}{2n})^n + (\frac{1}{2} + \frac{s}{2n})^n} \frac{1}{2n} ds + \frac{n}{2^n} \int_0^n \frac{g(\frac{1}{2} + \frac{s}{2n})}{(\frac{1}{2} + \frac{s}{2n})^n + (\frac{1}{2} - \frac{s}{2n})^n} \frac{1}{2n} ds \\
&= \frac{1}{2} \int_0^n \frac{g(\frac{1}{2} - \frac{s}{2n}) + g(\frac{1}{2} + \frac{s}{2n})}{(1-\frac{s}{n})^n + (1+\frac{s}{n})^n}\, ds.
\end{aligned}$$

Note that $n \mapsto (1+\frac{s}{n})^n$ is increasing; so the integrand is dominated for $s \ge 1$ by

$$\frac{||g||_\infty}{(1+\frac{s}{2})^2} \le ||g||_\infty 4s^{-2}.$$

Hence, as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} I_n &= \frac{1}{2} 2g(1/2) \int_0^\infty \frac{ds}{e^{-s} + e^s}\, ds \\
&= g(1/2) \int_0^\infty \frac{e^s}{1 + (e^s)^2}\, ds \\
&= g(1/2) \big[\arctan e^s\big]_0^\infty = g(1/2)\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= \frac{\pi}{4} g(1/2).
\end{aligned}$$

Generalizations appear in [8].

12338. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^\infty \frac{\cos(x)-1}{x\ (e^x-1)}\,dx = \frac{1}{2}\ln\bigl(\pi\ \mathrm{csch}(\pi)\bigr).$$

Solution to problem 12338 in Amer. Math. Monthly 129 (2022), 686

Raymond Mortini and Rudolf Rupp

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Let $f(x) := \dfrac{\cos x - 1}{x(e^x-1)}$ and $I := \int_0^\infty f(x)dx$. Note that $\lim_{x\to 0} f(x) = 1/2$, that f is bounded, and that the integral converges (absolutely). Using the Laplace transform of f, we are going to show that

$$I = \frac{1}{2}\log\left(\frac{\pi}{\sinh\pi}\right).$$

So let

$$F(s) := \int_0^\infty e^{-sx} f(x)dx.$$

Also this integral converges absolutely and uniformly in $s \geq 0$, as the integrand is dominated on $[1,\infty[$ by $4e^{-x}$. Moreover, F is continuous on $[0,\infty[$ with $F(0) = I$. Now, by a similar reason,

$$G(s) := -\int_0^\infty x e^{-sx} f(x)dx$$

is absolutely convergent, as the integrand is dominated on $[1,\infty[$ by $4xe^{-x}$. Hence $F'(s) = G(s)$. Moreover, by considering for $x > 0$ the geometric series for $(1-e^{-x})^{-1}$,

$$G(s) \quad = \quad -\int_0^\infty \frac{e^{-(s+1)x}}{1-e^{-x}}(\cos x - 1)\ dx = \int_0^\infty \sum_{k=0}^\infty e^{-(s+1+k)x}(1-\cos x)\ dx.$$

As all the summands are positive, Beppo Levi's monotone convergence theorem for Lebesgue integrals implies that $\int\sum = \sum\int$. Hence, by using that for $a > 0$

$$\int_0^\infty e^{-ax}\cos x dx = \frac{a}{a^2+1},$$

we obtain

$$\begin{aligned} G(s) \quad &= \quad \sum_{k=0}^\infty \int_0^\infty e^{-(s+1+k)x}(1-\cos x)\ dx = \sum_{k=0}^\infty \left(\frac{1}{s+1+k} - \frac{s+1+k}{(s+k+1)^2+1}\right) \\ &\overset{n=k+1}{=} \sum_{n=1}^\infty \frac{1}{(s+n)^3+(s+n)}. \end{aligned}$$

The convergence being absolut and uniform on $[0,\infty[$ (a majorant is given by $\sum_{n=1}^\infty n^{-3}$), we can integrate termwise to re-obtain F. Note that a primitive P of G on $[0,\infty[$ is given by

$$\begin{aligned} P(s) \quad &= \quad \sum_{n=1}^\infty \int \frac{1}{(s+n)^3+(s+n)}ds = \sum_{n=1}^\infty \log\frac{s+n}{\sqrt{(s+n)^2+1}} \\ &= \quad -\frac{1}{2}\sum_{n=1}^\infty \log\frac{(s+n)^2+1}{(s+n)^2} = -\frac{1}{2}\sum_{n=1}^\infty \log\left(1+\frac{1}{(s+n)^2}\right). \end{aligned}$$

Now $F = P + c$ for some constant c. Since P is uniformly convergent, it easily follows that $\lim_{s\to\infty} P(s) = 0$ (just take a tail uniformly small, and use that the limit of the remaining finitely many summands is 0). But also $\lim_{s\to\infty} F(s) = 0$, because $|F(s)| \leq ||f||_\infty \int_0^\infty e^{-st} = ||f||_\infty/s$.

Hence $c = 0$ and so $F(0) = -\dfrac{1}{2}\displaystyle\sum_{n=1}^\infty \log\left(1+\frac{1}{n^2}\right)$. Next we use that

$$\sinh(\pi z) = \pi z \prod_{n=1}^\infty \left(1+\frac{z^2}{n^2}\right).$$

So,

$$\log\sinh(\pi) = \log\pi + \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right).$$

Hence

$$F(0) = I = -\frac{1}{2}\log\sinh(\pi) + \frac{1}{2}\log\pi = \frac{1}{2}\log\frac{\pi}{\sinh\pi}.$$

12338. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^\infty \frac{\cos(x)-1}{x\ (e^x-1)}\,dx = \frac{1}{2}\ln\big(\pi\ \mathrm{csch}(\pi)\big).$$

A different solution to problem 12338 in Amer. Math. Monthly 129 (2022), 686

Raymond Mortini and Rudolf Rupp

Let $I := \displaystyle\int_0^\infty \frac{\cos x - 1}{x(e^x-1)}\,dx$. We show that

$$I = \frac{1}{2}\log\left(\frac{\pi}{\sinh \pi}\right).$$

Recall that $\mathbb{N} = \{0,1,2,\dots\}$. First we develop for $x>0$ the integrand into a double, absolutely convergent series (so this is independent of the arrangement):

$$\begin{aligned} g(x) := \frac{\cos x-1}{x(e^x-1)} = \frac{\cos x - 1}{x}\,\frac{e^{-x}}{1-e^{-x}} &= \sum_{n=1}^\infty \frac{(-1)^n}{(2n)!}x^{2n-1}\sum_{k=1}^\infty e^{-kx} = \sum_{k=1}^\infty\sum_{n=1}^\infty \underbrace{\frac{(-1)^n}{(2n)!}x^{2n-1}\,e^{-kx}}_{:=a_{kn}} \\ &= \sum_{k=1}^\infty \sum_{n \text{ even}} \frac{1}{(2n)!}x^{2n-1}\,e^{-kx} - \sum_{k=1}^\infty\sum_{n \text{ odd}} \frac{1}{(2n)!}x^{2n-1}\,e^{-kx}. \end{aligned}$$

Note that $\lim_{x\to 0} g(x) = 1/2$, but that both absolutely convergent double series at the right vanish at 0. Beppo Levi's monotone convergence theorem for Lebesgue integrals applied twice, gives

$$\int\sum_{k\geq 2}\sum_{n \text{ even}} = \sum_{k\geq 2}\sum_{n \text{ even}}\int \quad \text{and} \quad \int\sum_{k\geq 2}\sum_{n \text{ odd}} = \sum_{k\geq 2}\sum_{n \text{ odd}}\int. \tag{33}$$

As the calculations below show, the sums $\sum_{n \text{ odd}}\int|a_{kn}|$ and $\sum_{n \text{ even}}\int|a_{kn}|$ converge for $k\geq 2$, but diverge for $k=1$, though $\sum_n\int a_{1n}$ converges. Moreover, $\sum_{k\geq 2}\sum_{n \text{ odd}}\int$ and $\sum_{k\geq 2}\sum_{n \text{ even}}\int$ are finite; hence (by (33)) ,

$$\sum_{k\geq 2}\sum_n\int = \int\sum_{k\geq 2}\sum_n. \tag{34}$$

So, at the end, by adding in (34) the term $\sum_n\int a_{1,n}$, respectively $\int\sum_n a_{1,n}$ (which coincide, too; see addendum) we see that

$$\int\sum_k\sum_n = \sum_n\sum_k\int.$$

To complete the calculations, we use that for $m\in\mathbb{N}$ and $k\in\mathbb{N}\setminus\{0\}$,

$$\int_0^\infty x^m e^{-kx}dx = m!/k^{m+1}.$$

Hence

$$\sum_{k=1}^\infty\sum_{n=1}^\infty \frac{(-1)^n}{(2n)!}\int_0^\infty x^{2n-1}\,e^{-kx}\,dx = \sum_{k=1}^\infty\sum_{n=1}^\infty \frac{(-1)^n}{(2n)!}\frac{(2n-1)!}{k^{2n}} =$$
$$\sum_{k=1}^\infty\left(\sum_{n=1}^\infty \frac{(-1)^n}{2n}\frac{1}{k^{2n}}\right) = -\frac{1}{2}\sum_{k=1}^\infty \log\left(1+\frac{1}{k^2}\right),$$

where the last identity comes from the fact that for $0\leq y\leq 1$

$$h(y) := \sum_{n=1}^\infty \frac{(-1)^n}{2n}y^{2n} = -\frac{1}{2}\log(1+y^2)$$

(note that for $y = 1$ there is no absolute convergence). Next we use that

$$\sinh(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right).$$

So,

$$\log \sinh(\pi) = \log \pi + \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right).$$

Hence

$$I = -\frac{1}{2} \log \sinh(\pi) + \frac{1}{2} \log \pi = \frac{1}{2} \log \frac{\pi}{\sinh \pi}$$

Addendum

(35) $$J := \int_0^{\infty} e^{-x} \frac{\cos x - 1}{x} dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} = -\frac{1}{2} \log 2.$$

First we note that, as above, $\sum_n \int a_{1,n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} = -\frac{1}{2} \log 2$. To show that $J = -\frac{1}{2} \log 2$, we interprete this as the Laplace transform $L(q)(s)$ of the function $q(x) = (\cos x - 1)/x$ evaluated at $s = 1$. By a well-known formula, if $L(F(t))(s) = f(s)$, then

$$L(q)(s) = L(\frac{F(t)}{t})(s) = \int_s^{\infty} f(u) du,$$

where

$$f(s) = \int_0^{\infty} e^{-st} (\cos t - 1)\, dt = \frac{1}{s^3 + s}.$$

Hence $L(q)(s) = -\frac{1}{2} \log(1 + s^{-2})$ and so $J = L(q)(1)) = -\frac{1}{2} \log 2$.

Another way to calculate the Laplace transform $J(s) := L(q)(s)$ of q is to take derivatives:

$$J'(s) = -\int_0^{\infty} e^{-st} (\cos t - 1) dt = f(s).$$

Note that $\frac{d}{ds} \int = \int \frac{d}{ds}$, since both integrands are locally (in s) dominated by $L^1[0, \infty[$ functions.

Remark This integral J appears also on the web, see [38].

12312. *Proposed by Martin Tchernookov, University of Wisconsin, Whitewater, WI.* Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ such that, for all positive x,

$$f(x)\left(f(x)-\frac{1}{x}\int_0^x f(t)\,dt\right) \geq \left(f(x)-1\right)^2.$$

Solution to problem 12312, AMM 129 (3) (2022), p. 286

Gerd Herzog, Raymond Mortini

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We show that the constant function 1 is the only solution

Let $y = y(x) := \int_0^x f(t)dt$ and suppose that the continuous function $f : [0, \infty[\to \mathbb{R}$ satisfies on $]0, \infty[$

$$f(x)\left(f(x)-\frac{1}{x}\int_0^x f(t)dt\right) \geq (f(x)-1)^2.$$

Then

(36) $$y'\left(2-\frac{y}{x}\right) \geq 1 \text{ for } x > 0 \text{ and } y(0) = 0.$$

Note that this implies that $y'(0) = 1$, because, by letting $x \to 0$,

$$y'(0)(2-y'(0)) \geq 1 \iff (y'(0)-1)^2 \leq 0$$

Let the function $w : [0, \infty[\to \mathbb{R}$ be given by

$$w(x) := \begin{cases} \frac{y(x)}{x} & \text{if } x > 0 \\ y'(0) & \text{if } x = 0. \end{cases}$$

Then $w \in C([0, \infty[) \cap C^1(]0, \infty[)$. We claim that

(37) $$w(x) = 1 \text{ for every } x \geq 0,$$

from which we conclude that $y(x) = x$ and so $f(x) = y'(x) = 1$ for $x \geq 0$.

To see this, note that by (36), $w(x) \neq 2$. Since w is continuous on $[0, \infty[$, $w(0) = 1$, and w does not take the value 2, we have that $w(x) < 2$ for each $x > 0$. Hence, for $x > 0$,

$$\begin{aligned} w'(x) &= \frac{xy'(x)-y(x)}{x^2} \geq \frac{1}{x}\left(\frac{1}{2-w(x)} - w(x)\right) \\ &= \frac{1}{x}\cdot\frac{(1-w(x))^2}{2-w(x)} \end{aligned} \tag{38}$$

Thus we may deduce from (38) that $w' \geq 0$; that is w is increasing [6].

Now suppose that (37) is not true.

Case 1 There is $x_0 > 0$ with $w(x_0) < 1$. This is not possible, though, as w is increasing, but $w(0) = 1$.

Case 2 There is $x_0 > 0$ with $w(x_0) > 1$. As w is increasing, $w > 1$ for $x \geq x_0$. Note that we already know that $w < 2$. Since the map $t \mapsto \frac{(1-t)^2}{2-t}$ is increasing on $[1, 2[$, we deduce from (38) that for $x \geq x_0$

$$w'(x) \geq \frac{1}{x}\cdot\frac{(1-w(x_0))^2}{2-w(x_0)} =: c\frac{1}{x}.$$

Hence, by integration, for $x \geq x_0$,

$$w(x) \geq w(x_0) + c\log(x/x_0) \to \infty \ (x \to \infty).$$

An obvious contradiction. □

[6] in the weak sense; or funnily called nondecreasing, a very ambiguous word.

12308. *Proposed by Cezar Lupu, Yanqi Lake BIMSA and Tsinghua University, Beijing, China.* What is the minimum value of $\int_0^1 \left(f'(x)\right)^2\, dx$ over all continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f(x)\, dx = \int_0^1 x^2 f(x)\, dx = 1$?

Solution to problem 12308, AMM 129 (3) (2022), p. 285 , by

Raymond Mortini

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We show that the minimal value is given by 105/2 and is obtained by the polynomial $f(x) = -105/16x^4 + 105/8x^2 - 33/16$

Let p be any polynomial. Then, by Cauchy-Schwarz,

$$\left(\int_0^1 f'p\; dx\right)^2 \leq \left(\int_0^1 f'^2 dx\right)\left(\int_0^1 p^2 dx\right).$$

A primitives of $f'p$ is given by $fp - \int fp'dx$. Now choose p so that $p(0) = p(1) = 0$ and $p'(x) = \alpha x^2 + \beta$. To this end, put

$$p(x) = ax(x^2 - 1).$$

Then

$$I := \int_0^1 f'p\; dx \quad = \quad fp\Big|_0^1 - \int_0^1 f(3ax^2 - a)dx = -3a + a = -2a$$

Moreover,

$$\int_0^1 p^2 dx = a^2 \int_0^1 (x^6 + x^2 - 2x^4)dx = a^2\left(\frac{1}{7} + \frac{1}{3} - \frac{2}{5}\right).$$

Hence

$$\int f'^2 dx \geq \frac{4a^2}{a^2\left(\frac{1}{7} + \frac{1}{3} - \frac{2}{5}\right)} = \frac{105}{2}.$$

Equality in the Cauchy-Schwarz inequality is given whenever $f' = p$. Thus

$$f(x) = \frac{a}{4}x^4 - \frac{a}{2}x^2 + c.$$

Now a and c have to be chosen so that $\int f = \int x^2 f = 1$. This yields the linear system

$$-7a + 60c = 60$$
$$-27a + 140c = 420$$

whose solution is $a = -105/4$ and $c = -33/16$. Consequently

$$f(x) = -105/16x^4 + 105/8x^2 - 33/16.$$

Note that

$$f'(x)^2 = \big(-\frac{105}{4}x(x^2 - 1)\big)^2.$$

12326. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every fixed $y \in \mathbb{R}$, $f(x+y) - f(x)$ is a polynomial in x. Prove that f is a polynomial function.

Solution to problem 12326, AMM 129 (5) (2022), p. 487

Raymond Mortini, Peter Pflug, Amol Sasane

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By considering the symmetric function $p(x,y) := f(x+y) - f(x) - f(y)$ we get from the assumption that as well $p(\cdot, y)$ and $p(x, \cdot)$ are polynomials in their variables separately. Hence, by [39], $p(x,y)$ is a polynomial.

Case 1 $f \in C^1(\mathbb{R})$. Write $p(x,y) = \sum a_{i,j} x^i y^j$ with symmetrical coefficients and $a_{0,0} = -f(0)$ (the sum being finite of course) If we take $y = 0$, then for all x

$$-f(0) = f(x+0) - f(x) - f(0) = a_{0,0} + \sum a_{i,0} x^i.$$

Hence $a_{i,0} = 0$ for all $i \geq 1$. Due to symmetry, we also have $a_{0,j} = 0$ for all $j \geq 1$. Thus we have only coefficients $a_{i,j}$ for $i, j \geq 1$. Consequenlty

$$\frac{f(x+y) - f(x) - (f(y) - f(0)))}{y} = \sum_{i,j \geq 1} a_{i,j} x^i y^{j-1}.$$

As f is assumed to be differentiable, we may take $y \to 0$ and get

$$f'(x) - f'(0) = \sum_{i \geq 1} a_{i,1} x^i.$$

Integration yields

$$f(x) - f(0) - xf'(0) = \sum_{i \geq 1} a_{i,1} \frac{x^{i+1}}{i+1}.$$

Thus f is a polynomial.

Case 2 $f \in C(\mathbb{R})$. Let $F(x) := \int_0^x f(t)dt$ be a primitive of f. Then with

$$G(x,y) := F(x+y) - F(x) - F(y)$$

$$\begin{aligned}
G(x,y) &= \int_0^{x+y} f(t)dt - \int_0^x f(t)dt - \int_0^y f(t)dt \\
&\underset{t=y+s}{=} \int_{-y}^{x} f(y+s)ds - \int_0^x f(t)dt - \int_0^y f(t)dt \\
&= \int_{-y}^{0} f(y+s)ds + \int_0^x \big(f(y+s) - f(s)\big)ds - \int_0^y f(t)dt \\
&\underset{t=y+s}{=} \int_0^y f(t)dt + \int_0^x \big(f(y+s) - f(s)\big)ds - \int_0^y f(t)dt \\
&= \int_0^x p(y,s)ds + f(y)x
\end{aligned}$$

which is a polynomial in x. Again, by symmetry, and the Carroll argument, G is a polynomial. Hence, by Case 1, F is a polynomial and so does $f = F'$.

A detailed analysis of the functional equation $f(x+y) - f(x) - f(y) = p(x,y)$ (and based on these methods) appears in [5].

12290. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Find all analytic functions $f:\mathbb{C}\to\mathbb{C}$ that satisfy

$$|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$$

for all real numbers x and y.

Solution to problem 12290 in Amer. Math. Monthly 128 (2021), 946

Raymond Mortini and Rudolf Rupp

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We show that all solutions are given by az, $b\sin(kz)$ and $c\sinh(kz)$ where $a,b,c\in\mathbb{C}$ and $k\in\mathbb{R}$.

First we note that any solution f necessarily satisfies $f(0)=0$. Now let $h(z):=|f(z)|^2=(f\overline{f})(z)$. Since $f_x=f'$ and $f_y=if_x=if'$, we see that $f_{xy}=(f')_y=i(f')_x=if''$. Moreover $(\overline{f})_x=\overline{f_x}$. Hence

$$\begin{aligned} h_{xy} &= (f_x\overline{f}+f\overline{f}_x)_y = f_{xy}\overline{f}+f_x\overline{f}_y+f_y\overline{f}_x+f\overline{f}_{xy}\\ &= 2\mathrm{Re}(f_{xy}\overline{f})+0 = 2\mathrm{Re}(if''\overline{f}) = -2\mathrm{Im}(f''\overline{f}). \end{aligned}$$

Now $|f(z)|^2=|f(x)|^2+|f(iy)|^2$ implies that the mixed derivative of the right hand side is 0. We conclude that $\mathrm{Im}(f''\overline{f})=0$ in $\mathbb{C}$. Let $U=\mathbb{C}\setminus Z(f)$, where $Z(f)=\{z\in\mathbb{C}: f(z)=0\}$. Then on U, this is equivalent to

$$0=\mathrm{Im}\left(\frac{f''}{f}|f|^2\right)=\mathrm{Im}\left(\frac{f''}{f}\right).$$

Thus, a necessary condition for $f\not\equiv 0$ being a solution is that f''/f is a real constant λ. The differential equation $f''=\lambda f$ in $\mathbb{C}$ has the solutions $az+d$ if $\lambda=0$, or $\alpha e^{\sqrt{\lambda}\,z}+\beta e^{-\sqrt{\lambda}\,z}$ if $\lambda>0$, and $\alpha e^{i\sqrt{|\lambda|}\,z}+\beta e^{-i\sqrt{|\lambda|}\,z}$ if $\lambda<0$. Since $f(0)=0$, we have $d=0$ and $\beta=-\alpha$. So, with $k:=\sqrt{|\lambda|}$,

$$f(z)=az,\ c\sinh kz \text{ if } \lambda>0 \text{ and } c\sin kz \text{ if } \lambda<0.$$

It is now easy to check that these are solutions indeed (wlog for $k=1$):

$$\begin{aligned} \sin(x+iy) &= \cos(iy)\sin x+\cos x\sin(iy)\\ &= \frac{e^{-y}+e^{y}}{2}\sin x - i\cos x\frac{e^{-y}-e^{y}}{2}\\ &= \cosh y\sin x + i\cos x\sinh y \end{aligned}$$

$$\begin{aligned} |\sin z|^2 &= \sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\\ &= \sin^2 x\cosh^2 y+(1-\sin^2 x)\sinh^2 y\\ &= \sin^2 x(\cosh^2 y-\sinh^2 y)+\sinh^2 y\\ &= \sin^2 x+\sinh^2 y\\ &= \sin^2 x+|\sin^2(iy)|. \end{aligned}$$

as $\sin(iy)=i\sinh y$

12288. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx = \frac{\pi}{5}.$$

Solution to problem 12288 in Amer. Math. Monthly 128 (2021), 946

Raymond Mortini and Rudolf Rupp

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A change of the variable $x \to 1/x$ yields that

$$J := \int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 \, dx = \int_0^\infty \frac{(x^2 - \sin^2 x)^2}{x^6} \, dx.$$

Note that

$$(x^2 - \sin^2 x)^2 = x^4 - 2x^2 \sin^2 x + \sin^4 x.$$

Now we "linearize" the trigonometric powers: $\sin^2 x = (1/2)(1 - \cos 2x)$ and $\sin^4 x = (3/8) - (1/2)\cos 2x + (1/8)\cos 4x$. Thus $J = I/2$, where

$$I := \int_{\mathbb{R}} \frac{\frac{3}{8} + x^4 - x^2 + (x^2 - \frac{1}{2})\cos(2x) + \frac{1}{8}\cos(4x)}{x^6} \, dx.$$

Next we consider the meromorphic function

$$f(z) := \frac{\frac{3}{8} + z^4 - z^2 + (z^2 - \frac{1}{2})e^{2iz} + \frac{1}{8}e^{4iz}}{z^6}.$$

Then we add in the numerator the polynomial

$$p(z) := i\left(\frac{1}{2}z - \frac{4}{3}z^3 + \frac{2}{5}z^5\right),$$

that is we consider the function

$$F(z) := f(z) + \frac{p(z)}{z^6}.$$

Note that this polynomial is chosen so that F has a removable singularity at $z = 0$ (in other words, $-\frac{p(z)}{z^6}$ is the principal part in the Laurent expansion of f around the origin). Hence $\int_\Gamma F(z)dz = 0$, where Γ is the boundary of the half-disk $|z| \leq R$, $\operatorname{Im} z \geq 0$, consisting of the half circle Γ_R and the interval $[-R, R]$. Hence, by letting $R \to \infty$ and taking real parts,

$$0 = \operatorname{Re} \lim_{R\to\infty} \int_{\Gamma_R} F(z)dz + I.$$

By Jordan's Lemma, $\limsup_{R\to\infty} \int_{\Gamma_R} |e^{inz}||dz| < \infty$. Hence,

$$\lim_{R\to\infty} \int_{\Gamma_R} F(z)dz = 0 + 0 + i \lim_{R\to\infty} \int_{\Gamma_R} \frac{\frac{2}{5}z^5}{z^6} dz = -\frac{2\pi}{5}.$$

We conclude that the value of the original integral J is $\pi/5$.

12256. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Prove

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx = -\frac{5}{8}\zeta(3),$$

where $\zeta(3)$ is Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$.

Solution to problem 12256 in Amer. Math. Monthly 128 (2021), 478

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Using that $4ab = (a+b)^2 - (a-b)^2$, we obtain

$$4\int_0^1 \frac{\log(1+x)\log(1-x)}{x}dx = \int_0^1 \frac{\log^2(1-x^2)}{x}dx - \int_0^1 \frac{\log^2 \frac{1+x}{1-x}}{x}dx =: I_1 - I_2.$$

For I_1, we make the substitution $1-x^2 = t^2$. Hence, due to $-xdx = tdt$,

$$I_1 = \int_0^1 \frac{\log^2 t^2}{1-t^2} t\, dt$$

Using that $\int \sum = \sum \int$ (Lebesgue), and twice integration by parts,

$$I_1 = 4\sum_{n=0}^{\infty}\int_0^1 t^{2n+1}\log^2 t\, dt = 8\sum_{n=0}^{\infty}\frac{1}{(2n+2)^3} = \xi(3).$$

For the second one, I_2, we make the substitution $t = \frac{1+x}{1-x}$. Then $x = \frac{t-1}{t+1}$ and $dx = \frac{2}{(t+1)^2}dt$. Hence

$$I_2 = 2\int_1^{\infty}\frac{\log^2 t}{1-t^2}\, dt \overset{t=1/s}{=} 2\int_0^1 \frac{\log^2 s}{1-s^2}\, ds = 2\sum_{n=0}^{\infty}\int_0^1 s^{2n}\log^2 s\, ds = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = 4\frac{7}{8}\xi(3).$$

Consequently, $4I = (1-\frac{7}{2})\xi(3) = -\frac{5}{2}\xi(3)$ and so

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}dx = -\frac{5}{8}\xi(3).$$

11684. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France, and Rudolf Rupp, Georg-Simon-Ohm Hochschule Nürnberg, Nuremberg, Germany.* For complex a and z, let $\phi_a(z) = (a-z)/(1-\overline{a}z)$ and $\rho(a,z) = |a-z|/|1-\overline{a}z|$.

(**a**) Show that whenever $-1 < a, b < 1$,

$$\max_{|z|\leq 1} |\phi_a(z) - \phi_b(z)| = 2\rho(a,b), \text{ and}$$

$$\max_{|z|\leq 1} |\phi_a(z) + \phi_b(z)| = 2.$$

(**b**) For complex α, β with $|\alpha| = |\beta| = 1$, let

$$m(z) = m_{a,b,\alpha,\beta}(z) = |\alpha\phi_a(z) - \beta\phi_b(z)|.$$

Determine the maximum and minimum, taken over z with $|z| = 1$, of $m(z)$.

original statement

Given $a, b, \alpha, \beta \in \mathbb{C}$ with $|a| < 1$, $|b| < 1$ and $|\alpha| = |\beta| = 1$, let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ and $\rho(a,b) = |a-b|/|1-\overline{a}b|$ the pseudohyperbolic distance between a and b.

i) Show that whenever $a, b \in]-1, 1[$,

$$M^- := \max_{|z|\leq 1} |\varphi_a(z) - \varphi_b(z)| = 2\rho(a,b)$$

and

$$M^+ := \max_{|z|\leq 1} |\varphi_a(z) + \varphi_b(z)| = 2.$$

ii) Determine

$$M := \max_{|z|=1} |\alpha\varphi_a(z) - \beta\varphi_b(z)|$$

and

$$m := \min_{|z|=1} |\alpha\varphi_a(z) - \beta\varphi_b(z)|.$$

Solution to problem 11684 AMM 120 (2013), 76

Raymond Mortini, Rudolf Rupp

i) That $M^+ = 2$ is easy: just take $z = 1$ and evaluate:

$$|\varphi_a(1) + \varphi_b(1)| = |-1-1| = 2.$$

Since $M^+ \leq 2$, we are done.

ii) We first observe that ϕ_b is its own inverse. Let $c = (b-a)/(1-a\overline{b})$ and $\lambda = -(1-a\overline{b})/(1-\overline{a}b)$. Since ϕ_b is a bijection of the unit circle onto itself,

$$\max_{|z|=1} |\alpha\varphi_a(z) - \beta\varphi_b(z)| = \max_{|z|=1} |\alpha\overline{\beta}\varphi_a(\varphi_b(z)) - z| = \max_{|z|=1} |\alpha\overline{\beta}\lambda\varphi_c(z) - z|.$$

The same identities hold when replacing the maximum with the minimum.

Put $\gamma := \alpha\overline{\beta}\lambda$ and let $-\pi < \arg\gamma \leq \pi$. For $|z| = 1$ we obtain

$$H(z) := |\gamma\phi_c(z) - z| = \left|\gamma\frac{z(c\overline{z}-1)}{1-\overline{c}z} - z\right|$$

$$= \left|\gamma\frac{1-c\overline{z}}{1-\overline{c}z} + 1\right| = \left|\gamma\ \frac{w}{\overline{w}} + 1\right|,$$

where

$$w = 1 - c\overline{z} = 1 - c\frac{1}{z}.$$

If z moves on the unit circle, then w moves on the circle $|w-1| = |c|$. Let $w = |w|e^{i\theta}$. Then (see figure 5) the domain of variation of θ is the interval $[-\theta_m, \theta_m]$ with $|\theta_m| < \pi/2$ and $\sin\theta_m = |c| = \rho(a,b)$. Now

$$H(z) = |\gamma e^{2i\theta} + 1| = 2|\cos(\tfrac{\arg\gamma}{2} + \theta)|.$$

Hence,

$$M = \max_{|z|=1} H(z) = 2\max\{|\cos(\tfrac{\arg\gamma}{2} + \theta)| : |\theta| \le \arcsin(\rho(a,b))\}$$

and

$$m = \min_{|z|=1} H(z) = 2\min\{|\cos(\tfrac{\arg\gamma}{2} + \theta)| : |\theta| \le \arcsin(\rho(a,b))\}.$$

In particular, if $a, b \in\,]-1, 1[$ and $\alpha = \beta = 1$, then $\gamma = -1$, and so (using the maximum principle at $*$)

$$M^- \overset{*}{=} \max_{|z|=1} H(z) = 2\max\{|\sin\theta| : |\theta| \le \arcsin(\rho(a,b))\} = 2\rho(a,b).$$

If $a, b \in\,]-1, 1[$ and $\alpha = 1, \beta = -1$, then $\gamma = 1$, and so

$$M^+ \overset{*}{=} \max_{|z|=1} H(z) = 2\max\{|\cos\theta| : |\theta| \le \arcsin(\rho(a,b))\} = 2.$$

We note that $m = 0$, that is $H(z_0) = 0$ for some z_0 with $|z_0| = 1$, if and only if $\gamma\phi_c$ has a fixed point on the unit circle (namely z_0). This is equivalent to the condition $|\cos(\frac{\arg\gamma}{2})| \le |c|$. Moreover, $M = 2$ if and only if $|\sin(\frac{\arg\gamma}{2})| \le |c|$.

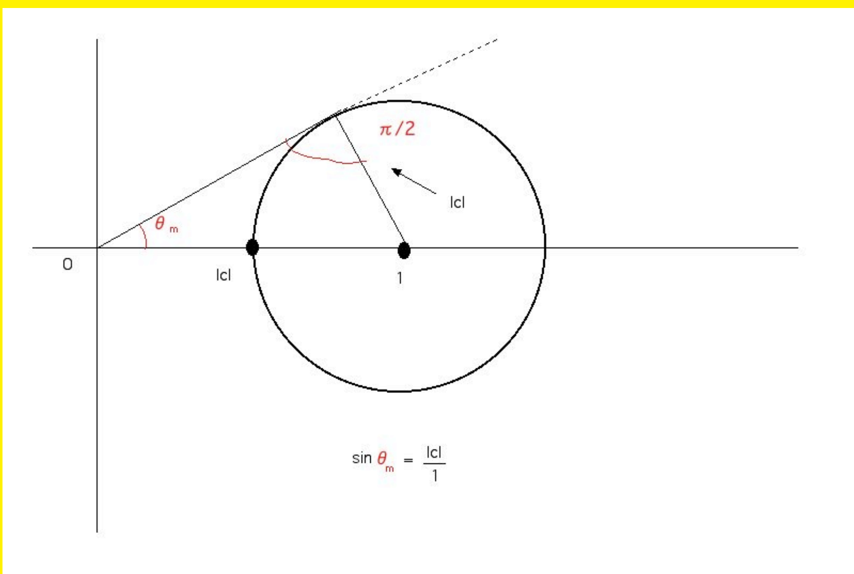
!


FIGURE 5. The domain of variation of $\arg w$

Solution by the proposers.
(**b**) Observe that ϕ_a is its own inverse. Let $c = (b-a)/(1-a\overline{b})$ and let

$$\lambda = -\frac{1 - a\overline{b}}{1 - \overline{a}b}.$$

Since ϕ_b is a bijection of the unit circle onto itself,

$$\max_{|z|=1} |\alpha\phi_a(z) - \beta\phi_b(z)| = \max_{|z|=1} |\alpha\overline{\beta}\phi_a(\phi_b(z)) - z| = \max_{|z|=1} |\alpha\overline{\beta}\lambda\phi_c(z) - z|.$$

The same identities hold when the maximum is replaced by the minimum. Put $\gamma = \alpha\overline{\beta}\lambda$, and let $-\pi < \arg\gamma \le \pi$. For $|z| = 1$, let $H(z) = |\gamma\phi_c(z) - z|$. We have

$$H(z) = \left|\gamma\,\frac{z(c\bar{z}-1)}{1-\bar{c}z} - z\right| = \left|\gamma\,\frac{1-c\bar{z}}{1-\bar{c}z} - 1\right| = \left|\gamma\,\frac{w}{\overline{w}} + 1\right|,$$

where $w = 1 - c\bar{z} = 1 - c/z$. As z moves around the unit circle, w moves around the circle $|w-1| = |c|$. Write $w = |w|e^{i\theta}$. Note that θ varies on the interval $[-\theta_m, \theta_m]$, where $|\theta_m| < \pi/2$ and $\sin\theta_m = |c| = \rho(a,b)$. Now

$$H(z) = |\gamma e^{2i\theta} + 1| = 2\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right|.$$

Hence

$$\max_{|z|=1} H(z) = 2\max\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a,b)\right\} \qquad (*)$$

and

$$\min_{|z|=1} H(z) = 2\min\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a,b)\right\}.$$

(**a**) Specialize $(*)$ by taking $a, b \in (-1,1)$ and $\alpha = \beta = 1$, so that $\gamma = -1$. By the maximum principle, the maximum on the disk is achieved on the boundary, so

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\max\left\{|\sin\theta| : |\theta| \le \arcsin\rho(a,b)\right\} = 2\rho(a,b).$$

For the other part of (**a**), instead specialize $(*)$ by taking $a, b \in (-1,1)$ and $\alpha = 1$, $\beta = -1$, so that $\gamma = 1$. This gives

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2\max\left\{|\cos\theta| : |\theta| \le \arcsin\rho(a,b)\right\} = 2.$$

Also solved by P. P. Dályay (Hungary) and R. Stong. Part (a) only by A. Alt, D. Beckwith, D. Fleischman, O. P. Lossers (Netherlands), and T. Smotzer.

11584. *Proposed by Raymond Mortini and Jérôme Noël, Université Paul Verlaine, Metz, France.* Let $\langle a_j \rangle$ be a sequence of nonzero complex numbers inside the unit circle such that $\prod_{k=1}^{\infty} |a_k|$ converges. Prove that

$$\left| \sum_{j=1}^{\infty} \frac{1-|a_j|^2}{a_j} \right| \leq \frac{1-\prod_{j=1}^{\infty}|a_j|^2}{\prod_{j=1}^{\infty}|a_j|}.$$

Solution to problem 11584 AMM 118 (2011), 558

Raymond Mortini, Jérôme Noël

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By the Schwarz-Pick inequality, $\frac{(1-|z|^2)|B'(z)|}{1-|B(z)|^2} \leq 1$ for any holomorphic self-map of the unit disk. Then, if we let B be the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\overline{a}_n z}$$

associated with the zeros (a_n), we get:

$$\frac{|B'(0)|}{1-|B(0)|^2} \leq 1.$$

But

$$\frac{B'(z)}{B(z)} = -\sum_{n=1}^{\infty} \frac{1-|a_j|^2}{(1-\overline{a}_j z)(a_j - z)}.$$

Hence

$$\left| \sum_{j=1}^{\infty} \frac{1-|a_j|^2}{a_j} \right| = \frac{|B'(0)|}{|B(0)|} \leq \frac{1-|B(0)|^2}{|B(0)|} = \frac{1-\prod_{j=1}^{\infty}|a_j|^2}{\prod_{j=1}^{\infty}|a_j|}.$$

Motivation for posing this as a problem to AMM: We are interested in a direct elementary proof.

11578. *Proposed by Roger Cuculière, Clichy la Garenne, France.* Let E be a real normed vector space of dimension at least 2. Let f be a mapping from E to E, bounded on the unit sphere $\{x \in E\colon \|x\| = 1\}$, such that whenever x and y are in E, $f(x + f(y)) = f(x) + y$. Prove that f is a continuous, linear involution on E.

Solution to problem 11578 in Amer. Math. Monthly 118 (2011), 464

Raymond Mortini

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Lemma 2. *Let $0 < ||x|| < 1$ and $s \in S$. Let s' be the (second) uniquely determined intersection point of the half-line starting at s and passing through x with S. Then the map $Q : S \to [0, \infty[$, $s \mapsto ||x - s||/||x - s'||$ is a nonconstant continuous map.*

Proof. Q obviously is continuous. If we suppose that Q is constant κ, then this constant is necessarily 1 (just interchange s with s'). Now $x = (1 - t)s + ts'$. Thus $x - s = t(s' - s)$ and $x - s' = (1 - t)(s - s')$ and so $Q(s) = t/(1 - t)$. Hence $1 = \kappa = \frac{t}{1-t}$. So $t = 1/2$. Now $x/||x||$ and $-x/||x||$ belong to S and with $t = (1 - ||x||)/2$ we have $x = (1 - t)\frac{x}{||x||} + t\frac{-x}{||x||}$. So $t = 1/2$ implies that $x = 0$. □

Lemma 3. *The unit sphere S is connected whenever* $\dim E \geq 2$.

Proof. Let $x, y \in S$, $x \neq y$. If x is linear independent of y, then the segment $\{tx + (1 - t)y : 0 \leq t \leq 1\}$ does not pass through the origin; hence

$$t \mapsto \frac{tx + (1 - t)y}{||tx + (1 - t)y||}$$

is a path joining y with x on S.

If $y = \lambda x$ for some $\lambda \in \mathbb{R}$, then we use the hypothesis that $\dim E \geq 2$ to guarantee the existence of a vector u linear independent of x. Thus $v := u/||u|| \in S$. By the first case, we may join x with v and then v with y by a path in S. □

The first step is to show that $f(0) = 0$.

(1) Let $x = 0$, $y = -f(0)$. Then $f(f(-f(0))) = f(0) - f(0) = 0$;

(2) Let $x = y = 0$. Then $f(f(0)) = f(0)$;

(3) Let $x = -f(y)$. Then $f(0) = f(-f(y)) + y$. With $y = 0$ this gives $f(0) = f(-f(0))$.

(4) Applying f yields $f(f(0)) = f(f(-f(0))) \underset{(1)}{=} 0$. Thus, by (2), $f(0) = 0$.

(5) Let $x = 0$. Then $f(f(y)) = f(0) + y = y$. Hence f is an involution.

(6) f is additive since

$$f(x + y) \underset{(5)}{=} f(x + f(f(y))) = f(x) + f(y).$$

(7) Next we show that f is $\mathbb{Q}$-homogeneous by induction. Indeed, by (5),

$$f((n + 1)x) = f(nx + x) = f(\underbrace{nx}_{X} + f(\underbrace{f(x)}_{Y})) = f(nx) + f(x).$$

Thus $f(mx) = mf(x)$ for every $m \in \mathbb{N}$.

Now

$$0 = f(0) = f(-x + x) \underset{(5)}{=} f(-x + f(f(x))) = f(-x) + f(x).$$

Thus $f(-x) = -f(x)$. Hence, for $p \in \mathbb{Z}$, we have $f(px) = pf(x)$.

Next, if $n \in \mathbb{N}$, then

$$nf(\frac{x}{n}) = f(\frac{x}{n}) + (n - 1)f(\frac{x}{n}) = f(\frac{x}{n}) + f(\frac{n-1}{n}x)$$

$$= f\Bigg(\underbrace{\frac{x}{n}}_{X} + f\Big(\underbrace{f(\frac{n-1}{n}x)}_{Y}\Big)\Bigg) \underset{(5)}{=} f\left(\frac{x}{n} + \frac{n-1}{n}x\right) = f(x)$$

Hence $f\left(\frac{x}{n}\right) = \frac{1}{n} f(x)$. Therefore $f\left(\frac{p}{n}\right) = \frac{p}{n} f(x)$ for $p \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(8) By hypothesis, $||f(s)|| \leq C$ for every $s \in S$. Let $0 < ||x|| < 1$. Consider, as in Lemma 2, the map $H : S \to [0, \infty[, s \mapsto ||x - s||/||x - s'||$. H is continuous and non-constant. Since dim $E \geq 2$, S is connected by Lemma 3. Hence $H(S)$ is an interval. In particular, there is $s \in S$ such that $r := ||x - s||/||x - s'||$ is rational. Thus, with $t = r/(1 + r)$,

$$x = (1 - t)s + ts'$$

is a rational convex-combination of two elements in the sphere.

Since f is $\mathbb{Q}$-linear, we conclude that

$$||f(x)|| \leq (1 - t)||f(s)|| + t||f(s')|| \leq (1 - t)C + tC = C.$$

Now let $x \in E$ be arbitrary. Choose a null-sequence ϵ_n of positive numbers so that $q_n := ||x|| + \epsilon_n$ is rational. Then, $||x/q_n|| \leq 1$. Since f is $\mathbb{Q}$-linear, we obtain

$$||f(x)|| = q_n ||f(x/q_n)|| \leq q_n C.$$

Letting n tend to infinity, we get

$$||f(x)|| \leq C||x||.$$

Thus f is continuous at the origin. Since f is additive, we deduce that f is continuous everywhere; just use $f(x_0 + x) = f(x_0) + f(x) \to f(x_0)$ if $x \to 0$.

(9) It easily follows now that f is homogeneous: if $\alpha \in \mathbb{R}$, choose a sequence (r_n) of rational numbers converging to α. Then, due to continuity,

$$f(\alpha x) = \lim_n r_n f(x) = \alpha f(x).$$

To sum up, we have shown that f is a continuous linear involution.

Remarks

If $n = 1$, then the unit sphere S is just a two point set, and so every function is automatically bounded on S. There exist, though, non-continuous linear involutions in $\mathbb{R}$. To this end, let $\mathcal{B}$ be a Hamel basis of the $\mathbb{Q}$-vector space $\mathbb{R}$, endowed with the usual Euclidean norm. We may assume that $\mathcal{B}$ is dense in $\mathbb{R}$. Fix two elements b_0 and $b_1 \in \mathcal{B}$. Let f be defined by $f(b_0) = b_1$, $f(b_1) = b_0$ and $f(b) = b$ if $b \in \mathcal{B} \setminus \{b_0, b_1\}$. Linearly extend f (in a unique way). Then, obviously, f is a linear involution. But f is not continuous at b_0. In fact, let $(b_k)_{n \geq 2} \in \mathcal{B}^{\mathbb{N}}$ converge to b_0. Then $f(b_k) = b_k \to b_0 = f(b_1) \neq f(b_0)$.

11548. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f be a twice-differentiable real-valued function with continuous second derivative, and suppose that $f(0) = 0$. Show that

$$\int_{-1}^{1} (f''(x))^2\,dx \geq 10\left(\int_{-1}^{1} f(x)\,dx\right)^2.$$

Solution to problem 11548 in Amer. Math. Monthly 118 (2011), 85

Raymond Mortini and Jérôme Noël

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Let $f \in C^2([-1,1])$, $f(0) = 0$. Then

$$\left(\int_{-1}^{1} f(x)dx\right)^2 \leq \frac{1}{10}\int_{-1}^{1} (f''(x))^2 dx.$$

Moreover, the constant $1/10$ is best possible.

Solution We consider the auxiliary integral

$$I = \frac{1}{2}\left[\int_0^1 (t-1)^2 f''(t)dt + \int_{-1}^{0} (1+t)^2 f''(t)\right].$$

We first show that $I = \int_{-1}^{1} f(t)dt$. In fact, twice integration by parts yields:

$$\int_0^1 (t-1)^2 f''(t)dt = -f'(0) - 2\int_0^1 (t-1)f'(t)dt = -f'(0) + 2\int_0^1 f(t)dt,$$

as well as

$$\int_{-1}^{0} (t+1)^2 f''(t)dt = f'(0) - 2\int_{-1}^{0} (t+1)f'(t)dt = f'(0) + 2\int_{-1}^{0} f(t)dt.$$

This proves the first claim. Now we use the Cauchy-Schwarz inequality to estimate I:

$$\left(\int_0^1 (t-1)^2 f''(t)dt\right)^2 \leq \int_0^1 (t-1)^4 dt \int_0^1 (f''(t))^2 dt = \frac{1}{5}\int_0^1 (f''(t))^2 dt,$$

and similarly for the second integral. Hence, by using that $(A+B)^2 \leq 2(A^2+B^2)$, we obtain

$$I^2 \leq 2\frac{1}{4}\left(\frac{1}{5}\int_0^1 (f''(t))^2 dt + \frac{1}{5}\int_{-1}^{0} (f''(t))^2 dt\right) = \frac{1}{10}\int_{-1}^{1} (f''(t))^2 dt.$$

The constant $1/10$ is obtained for the function

$$f(t) = \begin{cases} \frac{1}{12}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 & \text{if } -1 \leq t \leq 0 \\ \frac{1}{12}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 & \text{if } 0 \leq t \leq 1. \end{cases}$$

Indeed, this follows from the fact that in the Cauchy-Schwarz inequality we actually have equality if the functions are colinear: $p''(t) = (1+t)^2$ if $-1 \leq t \leq 0$ and $p''(t) = (1-t)^2$ if $0 \leq t \leq 1$. A computation then shows that

$$\left(\int_{-1}^{1} p(x)dx\right)^2 = \frac{1}{10}\int_{-1}^{1} (p''(x))^2 dx = \frac{1}{25}.$$

Remark If $f \in C^2([-1,1])$ satisfies $f(1) = f(-1) = f'(1) = f'(-1) = 0$, then the inequality above holds, too. In fact.

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} 1\cdot f(x)dx = xf(x)|_{-1}^{1} - \int_{-1}^{1} xf'(x)dx =$$

$$= -\frac{1}{2}x^2 f'(x)|_{-1}^{1} + \frac{1}{2}\int_{-1}^{1} x^2 f''(x)dx = \frac{1}{2}\int_{-1}^{1} x^2 f''(x)dx$$

Thus, by Cauchy-Schwarz,

$$\left(\int_{-1}^{1} f(x)dx\right)^2 \leq \frac{1}{4}\int_{-1}^{1} x^4 dx \int_{-1}^{1} (f''(x))^2 dx = \frac{1}{4}\cdot\frac{2}{5}\int_{-1}^{1} (f''(x))^2 dx.$$

11456. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France.* Find

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right).$$

Solution to problem 11456 AMM 116 (2009), 747

Raymond Mortini

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$$a_m := 1 - \frac{1}{m} + \frac{5}{4}\frac{1}{m^2} = \frac{1 + \left(\frac{2m-1}{2}\right)^2}{m^2}$$

$$\prod_{m=1}^{n} a_m = \frac{\prod_{m=1}^{n}\left(1 + \left(\frac{2m-1}{2}\right)^2\right)}{\prod_{m=1}^{n} m^2} \, \frac{\prod_{m=1}^{2n} m^2}{\prod_{m=1}^{n}(2m-1)^2 \prod_{m=1}^{n}(2m)^2}$$

$$= \frac{\prod_{m=1}^{n}\left(\frac{1}{(2m-1)^2} + \frac{1}{4}\right)\ (2n)!^2}{4^n (n!)^4} = \frac{\prod_{m=1}^{n}\left(\frac{4}{(2m-1)^2} + 1\right)\ (2n)!^2}{16^n (n!)^4}.$$

Now, by Stirlings formula,

$$\frac{(2n)!}{4^n n!^2} \sim \frac{(2n)^{2n} e^{-2n}\sqrt{4\pi n}}{(n^n e^{-n}\sqrt{2\pi n})^2 2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

Since $\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$, we have

$$\lim_n n \prod_{m=1}^{n} a_m = \frac{\cos(\pi i)}{\pi} = \frac{\cosh \pi}{\pi}.$$

We note that

$$\sqrt{\prod_{m=1}^{n} a_m} = \frac{1}{n!}\prod_{m=1}^{n}\left| i - \frac{2m-1}{2}\right| = (n+1)\frac{2}{\sqrt{5}}\frac{|f^{(n+1)}(0)|}{(n+1)!},$$

where $f(z) = (1-z)^{i+\frac{1}{2}}$, an interesting function in the Wiener algebra (its Taylor coefficients behave like $n^{-3/2}$ by the above calculations).

11402. *Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.* Let $f: [0,1] \to [0,\infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval (0,1) on the x-axis and the other two vertices on the graph of f.

Solution to problem 11402, AMM 115 (10), (2008), p. 949

Raymond Mortini

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The problem obviously is equivalent to show the existence of two points $0 < a < b < 1$ with $f(a) = f(b) = b - a$, or in other words, find $0 < a < b < 1$ with $b - f(b) = a$ and $f(b) = f(a)$.

To this end, consider the function $h(x) := f(x - f(x)) - f(x)$, where we have continuously extended f by the value 0 for $x < 1$ and $x > 1$. Then h is continuous. We have to show that h admits a zero b in $]0,1[$ with $f(b) < b$. Then $a := b - f(b) \in]0,1[$ and $b - a = f(b) = f(a)$.

To do this, we prove that h takes positive and negative values on $[0,1]$. Since $h(0) = h(1) = 0$, the continuity of h implies that h has a zero b in $]0,1[$. Our construction will guarantee that $f(b) < b$.

Let ξ_0 be the largest fixed point of f (note hat $0 \leq \xi_0 < 1$). For later purposes, we note that $f(x) \leq x$ whenever $\xi_0 \leq x \leq 1$. If $\xi_0 = 0$, we let x_0 be the be the smallest point for which $f(x_0) = M := \max_{x\in[0,1]} f(x)$. Note that $x_0 \in]0,1[$. Finally, let $x_1 \in [\xi_0, 1[$ be the largest point with $f(x_1) = M_1 := \max_{x\in[\xi_0,1]} f(x)$. Then $0 < x_0 \leq x_1 < 1$. Since the function $x - f(x)$ is 0 at ξ_0 and 1 at 1, the intermediate value theorem for continuous functions implies that there exists $y_1 \in]\xi_0, 1[$ such that $y_1 - f(y_1) = x_1$. Since $f > 0$, $y_1 > x_1$. Thus

$$h(y_1) = f(y_1 - f(y_1)) - f(y_1) = f(x_1) - f(y_1) = M_1 - f(y_1) > 0.$$

On the other hand, $h(\xi_0) = f(\xi_0 - f(\xi_0)) - f(\xi_0) = 0 - f(\xi_0) < 0$ if $\xi_0 > 0$, and if $\xi_0 = 0$, then, $h(x_0) = f(x_0 - f(x_0)) - f(x_0) < 0$ (since $x_0 - f(x_0)$ is left from the smallest maximal point x_0 of f.)

In both cases, there exists b such that $h(b) = 0$. Since $\xi_0 < b < y_1$ if $\xi_0 > 0$ and $0 < x_0 < b < y_1$ if $\xi_0 = 0$, we see that $f(b) < b$.

11333. *Proposed by Pablo Fernández Refolio, Universidad Autónoma de Madrid, Madrid, Spain.* Show that

$$\prod_{n=2}^{\infty}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right)=\pi.$$

Solution to problem 11333, AMM 114 (10), (2007), p. 926

Raymond Mortini

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Let

$$P_N=\prod_{n=2}^{N}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right).$$

a) We have the following equalities:

$$\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^2-1}=\frac{(N+1)^{N^2-1}}{N^{N(N+2)}}\,(N!)^2,$$

b)

$$\prod_{n=2}^{N}\left(\frac{n+1}{n-1}\right)^{n}=\frac{(N+1)^N N^{N+1}}{2(N!)^2}.$$

Hence

$$\sqrt{P_N}=\frac{(N+1)^{N^2-1}}{N^{N(N+2)}}\,(N!)^2\,\frac{(N+1)^{N/2}\,N^{(N+1)/2}}{\sqrt{2}N!}=$$

$$\left(\frac{N+1}{N}\right)^{N^2-1}\frac{N^{N^2-1}}{N^{N(N+2)}}\,N!\,\frac{(N+1)^{N/2}\,N^{(N+1)/2}}{\sqrt{2}}=$$

$$\left(\frac{N+1}{N}\right)^{N^2-1}N!\,\frac{(N+1)^{N/2}}{\mathbf{N^{N/2}}\sqrt{2}}\,\frac{N^{(N+1)/2}\mathbf{N^{N/2}}}{N^{2N+1}}=$$

$$\left(\frac{N+1}{N}\right)^{N^2-1}N!\,\frac{\left(1+\frac{1}{N}\right)^{N/2}}{\sqrt{2}}\,\frac{\sqrt{N}}{N^{N+1}}.$$

We are now using Stirling's formula telling us that $n!\sim e^{-n}n^n\sqrt{2\pi n}$. Hence

$$\sqrt{P_N}\sim\frac{\sqrt{e}}{\sqrt{2}}N^Ne^{-N}\sqrt{2\pi N}\left(\frac{N+1}{N}\right)^{N^2-1}\frac{\sqrt{N}}{N^{N+1}}=$$

$$\sqrt{e}\sqrt{\pi}e^{-N}\left(\frac{N+1}{N}\right)^{N^2-1}.$$

But $a_N:=e^{-N}\left(\frac{N+1}{N}\right)^{N^2-1}\to\frac{1}{\sqrt{e}}$ as $N\to\infty$; in fact, by taking logarithms we obtain $\log a_n=(N^2-1)\log(1+\frac{1}{N})-N\sim N^2\log(1+\frac{1}{N})-N=N^2(\frac{1}{N}-\frac{1}{2N}\pm\cdots)-N\sim-\frac{1}{2}$. Hence $\sqrt{P_N}\to\sqrt{\pi}$ and so $P_N\to\pi$.

11226. *Proposed by Franck Beaucoup, Ottawa, Canada, and Tamás Erdélyi, Texas A& M University, College Station, TX.* Let $a_1, \dots, a_n$ be real numbers, each greater than 1. If $n \geq 2$, show that there is exactly one solution in the interval (0, 1) to

$$\prod_{j=1}^{n}(1 - x^{a_j}) = 1 - x.$$

Solution to problem 11226, AMM 113 (5), (2006), p. 460

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Let $h(x) = \prod_{j=1}^n (1 - x^{a_j})$. Then $h'(x)/h(x) = -\sum_{j=1}^n \frac{a_j x^{a_j - 1}}{1 - x^{a_j}}$ and hence

$$h'(x) = -\sum_{j=1}^{n} a_j x^{a_j - 1} \prod_{k \neq j}(1 - x^{a_k}).$$

Clearly $h'(0) = h'(1) = 0$. Let

$$f(x) = (1 - x)^{-1} \prod_{j=1}^{n}(1 - x^{a_j})$$

if $0 \leq x < 1$. Note that $f(0) = 1$ and $\lim_{x \to 1} f(x) = -h'(1) = 0$. Thus, if we show that $f'(0) > 0$ and that the derivative of f has a unique zero in the open interval $]0, 1[$, we are done (that is we can then conclude by the intermediate value theorem that there is a unique x_0 with $0 < x_0 < 1$ so that $f(x_0) = 1$, and hence $h(x_0) = 1 - x_0$.)

Now, $f'(x)/f(x) = \frac{1}{1-x} + h'(x)/h(x)$. In particular, $f'(0) = 1$. Thus we have to look for $x \in]0, 1[$ so that $g(x) := \sum_{j=1}^n a_j x^{a_j - 1} \frac{1-x}{1-x^{a_j}} = 1$. But $g(0) = 0$, and, by de l'Hopital's rule, $\lim_{x \to 1} g(x) = n$. The intermediate value theorem yields the existence of x. The uniqueness of such an x follows from the fact that g is strictly increasing. This is due to the fact that the function $\frac{x^{a-1} - x^a}{1 - x^a}$ is strictly increasing on $]0, 1[$ whenever $a > 1$.

The latter follows from the fact that

$$\frac{d}{dx}\frac{x^{a-1} - x^a}{1 - x^a} = \frac{x^{a-2}\big((a-1) + x^a - ax\big)}{(1 - x^a)^2}$$

and that $k(x) := a - 1 + x^a - ax \geq 0$ for $0 \leq x \leq 1$, because $k(0) = a - 1 > 0$, $k(1) = 0$ and $k'(x) = a(x^{a-1} - 1) \leq 0$.

11210. *Proposed by Michael S. Becker, University of South Carolina at Sumter, Sumter, SC.* Show that

$$\prod_{n=0}^{\infty} \frac{(2n+1)^4}{(2n+1)^4 - (2/\pi)^4} = \frac{2e\sec(1)}{e^2+1}.$$

Solution to problem 11210, AMM 113 (3), (2006), p. 267

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We note that

$$p_n := \frac{(2n+1)^4 - (2/\pi)^4}{(2n+1)^4} = \left(1 - \left[\frac{2}{\pi(2n+1)}\right]^4\right) =$$

$$\left(1 - \frac{4}{\pi^2(2n+1)^2}\right)\left(1 + \frac{4}{\pi^2(2n+1)^2}\right).$$

Multiplying in the numerator and denominator (which is 1) with the "missing" factors

$$\left(1 - \frac{4}{\pi^2(2n)^2}\right)\left(1 + \frac{4}{\pi^2(2n)^2}\right)$$

we obtain

$$P := \prod_{n=0}^{\infty} p_n = \prod_{k=1}^{\infty} \frac{\left(1 - \frac{4}{\pi^2 k^2}\right)\left(1 + \frac{4}{\pi^2 k^2}\right)}{\left(1 - \frac{1}{\pi^2 k^2}\right)\left(1 + \frac{1}{\pi^2 k^2}\right)}.$$

Using the standard infinite product representation of the sinus

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 k^2}\right),$$

we obtain

$$P = \frac{\frac{\sin 2}{2}\,\frac{\sin(2i)}{2i}}{\frac{\sin 1}{1}\,\frac{\sin i}{i}} = \cos 1 \cosh 1 = (\cos 1)\,\frac{e^2+1}{2}.$$

11202. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Prove that if $\langle a_n \rangle$ is a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n < \infty$, then for all p in $(0, 1)$

$$\lim_{n\to\infty} n^{1-1/p} \left(a_1^p + \cdots + a_n^p\right)^{1/p} = 0.$$

Solution to problem 11202, AMM 113 (2), (2006), p. 179

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The assertion is an immediate consequence of Hölder's inequality: Wlog let $0 \le a_j \le 1$ and let $q \in]0,1[$ be such that $p + q = 1$ (note that $p \in]0,1[$.)

$$n^{p-1}\sum_{j=1}^{n} a_j^p = n^{p-1}\left(\sum_{j=1}^{N} a_j^p + \sum_{j=N+1}^{n} a_j^p \cdot 1\right) \le$$

$$\frac{N}{n^{1-p}} + \left(\sum_{j=N+1}^{n} (a_j^p)^{1/p}\right)^p \left(\sum_{j=N+1}^{n} 1^{1/q}\right)^q \frac{1}{n^{1-p}} \le$$

$$\frac{N}{n^{1-p}} + \left(\sum_{j=N+1}^{\infty} a_j\right)^p \frac{n^q}{n^{1-p}} = \frac{N}{n^{1-p}} + \left(\sum_{j=N+1}^{\infty} a_j\right)^p \le \epsilon$$

if N and $n > N$ is sufficiently big.

11185. *Proposed by Rainer Brück, University of Dortmund, Dortmund, Germany, and Raymond Mortini, University of Metz, Metz, France.* Find all natural numbers n and positive real numbers α such that the integral

$$I(\alpha, n) = \int_0^\infty \log\left(1 + \frac{\sin^n x}{x^\alpha}\right) dx$$

converges.

Solution to problem 11185 AMM 112 (2005), 840

Rainer Brück, Raymond Mortini

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We claim that

$I(\alpha,p)$ converges if and only if $(\alpha,p) \in \,]1,\infty[\, \times \mathbb{N}$ or $(\alpha,p) \in \left]\frac{1}{2},1\right] \times (2\mathbb{N}+1)$.

First we discuss the behaviour of the integrand at the origin. For $\alpha > 0$ we have $\left|\log\left(1+\frac{\sin^p x}{x^\alpha}\right)\right| \le \log\left(1+x^{-\alpha}\right)$. Substituting $\frac{1}{x}$ by t, we obtain

$$\int_0^1 \log\left(1+x^{-\alpha}\right) dx = \int_1^\infty \frac{\log\left(1+t^\alpha\right)}{t^2}\, dt\,,$$

and this integral is convergent. Hence, our integral $I(\alpha,p)$ converges at 0 for every $\alpha > 0$ and $p \in \mathbb{N}$.

Now we discuss the behaviour at infinity. Since $\lim\limits_{t\to 0} \frac{\log(1+t)}{t} = 1$, we see that at infinity

$$A(x) := \log\left(1+\frac{\sin^p x}{x^\alpha}\right) \sim \frac{\sin^p x}{x^\alpha} =: B(x)\,.$$

Hence $\int_1^\infty A(x)\,dx$ converges absolutely if and only if $\int_1^\infty B(x)\,dx$ does. Note that by Riemann's convergence test $\int_1^\infty |B(x)|\,dx \le \int_1^\infty \frac{dx}{x^\alpha} < \infty$ whenever $\alpha > 1$. Hence, $\int_1^\infty A(x)\,dx$ is absolutely convergent for $\alpha > 1$.

Now suppose that $0 < \alpha \le 1$. On the intervals $J_k := \left[\frac{\pi}{6}+2k\pi, \frac{\pi}{2}+2k\pi\right]$, $k \ge 1$, we have $|\sin x| \ge \frac{1}{2}$ and $x \ge 1$. Hence $\frac{\sin^p x}{x^\alpha} \ge \frac{2^{-p}}{x} \ge \frac{2^{-p}}{2\pi(k+1)}$. Therefore,

$$\int_{J_k} |B(x)|\,dx \ge \frac{1}{3} \cdot \frac{2^{-p-1}}{k+1}\,.$$

Since $\int_1^\infty |B(x)|\,dx \ge \sum_{k=1}^\infty \int_{J_k} |B(x)|\,dx$, we see that $\int_1^\infty |B(x)|\,dx$ and hence $\int_1^\infty |A(x)|\,dx$ diverges (absolutely) for $0 < \alpha \le 1$. In particular, $\int_1^\infty A(x)\,dx$ diverges whenever p is even, since in that case $|A(x)| = A(x)$.

To continue, we may thus assume that $p = 2n+1$ is odd. We use that for every $\alpha > 0$ and $n \in \mathbb{N}$ the integral $\int_1^\infty \frac{\sin^{2n+1} x}{x^\alpha}\,dx$ converges. Indeed, let $I_m(x) := \int_1^x \frac{\sin^m t}{t^\alpha}\,dt$ and let F_m be a primitive of $\sin^m t$ with $F_m(1) = 0$. For m odd, F_m is periodic, hence bounded. By partial integration we obtain

$$I_{2n+1}(x) = \frac{F_{2n+1}(x)}{x^\alpha} + \alpha \int_1^x \frac{F_{2n+1}(t)}{t^{\alpha+1}}\,dt\,,$$

and we conclude that $I_{2n+1}(x)$ converges as $x \to \infty$.

Now we use the Taylor development

$$\log(1+u) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k} u^k + \frac{(-1)^{m-1}}{m} u^m \left(1+\varepsilon(u)\right)\,,$$

where ε is a continuous function of u and $\varepsilon(0) = 0$. In particular, $|\varepsilon(u)| < 1$ whenever $|u| \le \delta$ with $\delta > 0$ sufficiently small. Now, we set $u = u(x) = \frac{\sin^{2n+1} x}{x^\alpha}$, where $x > 0$ is so large that

$|u| \le \delta$. Then for sufficiently large real numbers $M > N$, we have

$$I := \int_N^M \log\left(1 + \frac{\sin^{2n+1} x}{x^\alpha}\right) dx = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k} \int_N^M \left(\frac{\sin^{2n+1} x}{x^\alpha}\right)^k dx$$

$$+ \frac{(-1)^{m-1}}{m} \int_N^M \left(\frac{\sin^{2n+1} x}{x^\alpha}\right)^m (1 + \varepsilon(u(x)))\, dx =: \sum_{k=1}^{m-1} I_k + \widetilde{I}_m\,.$$

Choosing $m \in \mathbb{N}$ such that $m\alpha > 1$ and $(m-1)\alpha \le 1$, the boundedness of $\varepsilon(u)$ yields the absolute convergence of the last integral $\widetilde{I}_m$. If $\frac{1}{2} < \alpha \le 1$, then $m = 2$ and hence $I = I_1 + \widetilde{I}_2$. But I_1 and $\widetilde{I}_2$ converge, and hence I converges. If $0 < \alpha \le \frac{1}{2}$, then $m \ge 3$ and at least a third integral I_2 above appears. That integral is divergent, since the exponent of the sin is an even one (note that by the choice of m, the exponent of x is still at most 1). Since all those divergent integrals I_{2q} come up with the same sign, we finally get the divergence of $I_1 + I_2 + \cdots + I_{m-1}$, and thus I diverges.

Finally, we note that the example $p = 1$ and $\alpha = \frac{1}{2}$ yields examples of functions f and g such that at infinity, $f \sim g$, but for which $\int_0^\infty f(x)\, dx$ diverges and $\int_0^\infty g(x)\, dx$ converges, namely $f(x) = \log\left(1 + \frac{\sin x}{\sqrt{x}}\right)$ and $g(x) = \frac{\sin x}{\sqrt{x}}$.

11147. *Proposed by Pamela Gorkin, Bucknell University, Lewisburg, PA, and Raymond Mortini, Université Paul Verlaine, Metz, France.* For each nonzero integer n let $a_n = i\pi n/(i\pi n - 1)$, and $a_n^* = 1/\overline{a_n}$. Note that a_n^* is the reflection of a_n in the unit circle. Show that the expression

$$\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-a_n} - \frac{1}{z-a_n^*}\right)$$

converges uniformly on compact subsets of $\mathbb{C}\setminus\{1\}$ to a zero-free meromorphic function.

Solution to problem 11147 AMM 112 (2005), 366

Pamela Gorkin, Raymond Mortini

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Let $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$ be the atomic inner function. Put

$$f = \frac{1/e - S}{1-(1/e)S}.$$

Then f is an inner function (that is it has radial limts of modulus one almost everywhere). Since f does not have radial limit zero, it must be a pure Blaschke product (see Garnett, p.76), that is

$$f(z) = e^{i\theta} z \prod_{n\in\mathbb{Z}\setminus\{0\}} \frac{|a_n|}{a_n}\,\frac{a_n - z}{1-\overline{a}_n z}.$$

Its zeros are exactly the numbers a_n for $n \in \mathbb{Z}\setminus\{0\}$, including the the origin. Since the derivative of S is $S'(z) = -S(z)\frac{2}{(1-z)^2}$, it follows that the derivative of f does not vanish either. But

$$\frac{S'(z)}{S(z)} = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{z-a_n} - \frac{1}{z-a_n^*}\right).$$

11136. *Proposed by Raymond Mortini, Université de Metz, Metz, France.* Prove that there exists a sequence $\langle \lambda_n \rangle$ of distinct complex numbers in the closed unit disk D and a summable sequence $\langle a_n \rangle$ in ℓ^1 such that, for every continuous function u on D that is harmonic on the interior of D and satisfies $u(0) = 0$,

$$\sum_n a_n u(\lambda_n) = 0.$$

Solution to problem 11136 AMM 112 (2005), 181

Raymond Mortini

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Let $D_n = D(\lambda_n, r_n)$ be a sequence of pairwisw disjoint, closed disks contained in the open unit disk $\mathbb{D}$ such that the area measure of $\mathbb{D} \setminus \bigcup D_n$ is zero. Noticing that by the mean-value area theorem for harmonic functions

$$\int\int_{D(\lambda,r)} u(z)dA(z) = \pi r^2 u(\lambda),$$

we obtain the assertion

$$0 = u(0) = \int\int_{\mathbb{D}} u(z)dA(z) = \sum_n \int\int_{D_n} u(z)dA(z) = \pi \sum_n r_n^2 u(\lambda).$$

Remark The problem was motivated by the question, circulating in England, and communicated to me by Joel F. Feinstein, whether the set of exponentials $\{e^{i\lambda z} : \lambda \in \mathbb{C}\}$ is countably linear independent! The method for the proof above presumably appeared for the first time in a paper of J. Wolff [Comptes Rendus Acad. Sci. Paris 173 (1921), 1056-1058].

11070. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata", Rome, Italy.* Let f and g be two commuting analytic maps from a nonempty open connected set $D \subseteq \mathbb{C}$ into D. Suppose that $z_0 \in D$ is a fixed point of both f and g, and that neither $f'(z_0)$ nor $g'(z_0)$ is a root of unity. Suppose also that there exists an integer $N \geq 1$ such that $f^{(k)}(z_0) = g^{(k)}(z_0) = 0$ for $1 \leq k \leq N-1$, while $f^{(N)}(z_0) = g^{(N)}(z_0) \neq 0$. Prove that the restrictions of f and g to D are equal.

Solution to problem 11070, AMM 111 (2004), p. 258

Raymond Mortini

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Let $\mathbb{N} = \{1, 2, \cdots\}$ and $f, g \in C^n(\Omega)$. Then the result follows from the following formula:

$$(f \circ g)^{(n)}(z) = \sum_{j=1}^{n} f^{(j)}(g(z)) \Bigg(\sum_{\substack{k \in \mathbb{N}^j \\ |k|=n}} C_k^n \, g^{(k)}(z) \Bigg), \qquad (\mathrm{Mo}_n)$$

where $k = (k_1, k_2, \ldots, k_j) \in \mathbb{N}^j$ is an ordered multi-index with $k_1 \leq k_2 \leq \cdots \leq k_j$, $|k| = \sum_{i=1}^{j} k_i$, $g^{(k)} = g^{(k_1)} g^{(k_2)} \ldots g^{(k_j)}$ and $C_k^n = \dfrac{1}{\prod_i [A_k(i)!]} \dbinom{n}{k}$. Here $A_k(i)$ denotes the cardinal of how often i appears within the ordered index k and $\binom{n}{k} = \frac{n!}{k_1! k_2! \ldots k_j!}$.

This formula has many advantages vis-å-vis the Faa di Bruno formula

$$(f \circ g)^{(n)} = \sum \binom{n}{p} (f^{(p)} \circ g) \prod_{j=1}^{n} \left(\frac{g^{(j)}}{j!} \right)^{p_j},$$

where $p_j \in \{0, 1, 2, \cdots\}$, $p = p_1 + p_2 + \cdots + p_n$ and $p_1 + 2p_2 + \cdots + np_n = n$, since one immediately can write down all the factors that occur without solving the above equations for p_j.

Case 1: Let $f(z_0) = g(z_0) = z_0$, $A := f'(z_0) = g'(z_0) \neq 0$, $A^p \neq 1 \ \forall p \in \mathbb{N}$ and $f \circ g = g \circ f$.

In order to show that $f \equiv g$ it is enough to prove that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all n. The proof is done inductively:

$n = 2$: Since $(f \circ g)'' = (f'' \circ g) g'^2 + (f' \circ g) g''$ and $f \circ g = g \circ f$ we get: $f''(z_0) A^2 + A g''(z_0) = g''(z_0) A^2 + A f''(z_0)$. Hence $f''(z_0)(A-1) = g''(z_0)(A-1)$. Since $A \neq 1$ we obtain that $f''(z_0) = g''(z_0)$.

$n \to n+1$:

$$(f \circ g)^{(n+1)} = (f' \circ g) g^{(n+1)} + \sum_{j=2}^{n} (f^{(j)} \circ g) \sum_{\substack{k \in \mathbb{N}^j \\ |k|=n+1}} C_k^{n+1} g^{(k)} + (f^{(n+1)} \circ g)(g')^{n+1}$$

Evaluating at z_0 and noticing that, by induction hypotheses, all derivatives appearing in the middle term coincide at z_0 with those when f is replaced by g, we get that

$$A g^{(n+1)}(z_0) + f^{(n+1)}(z_0) A^{n+1} = A f^{(n+1)}(z_0) + g^{(n+1)}(z_0) A^{n+1}.$$

Hence $f^{(n+1)}(z_0)(A^n - 1) = g^{(n+1)}(z_0)(A^n - 1)$, from which we conclude that $f^{(n+1)}(z_0) = f^{(n+1)}(z_0)$, because $A^n \neq 1$.

Case 2: $f(z_0) = g(z_0) = z_0$, $f^{(j)}(z_0) = g^{(j)}(z_0) = 0$ for $1 \leq j < n_0$, but $f^{(n_0)}(z_0) = g^{(n_0)}(z_0) \neq 0$ and $f \circ g = g \circ f$.

Suppose that $f^{(j)}(z_0) = g^{(j)}(z_0)$ has been shown to be true for $j < n$, where $n = pn_0 + q$, with $0 \leq q < n_0$ and $p \geq 1$. We show that this holds then for $j = n$.

Let $N = n_0^2 + (p-1)n_0 + q$ and consider $(f \circ g)^{(N)}(x_0)$. All the terms in $(\mathrm{Mo})_N$ with $j < n_0$ disappear, since $f^{(j)}(g(z_0)) = f^{(j)}(z_0) = 0$. Moreover, as we are going to show, all other terms, excepted the term for $j = n_0$ and the index $k = (n_0, \cdots, n_0, pn_0 + q) \in \mathbb{N}^{n_0}$, coincide for f and g; hence can be thrown off when regarding the equality $(f \circ g)^{(N)} = (g \circ f)^{(N)}$. Thus that equality is equivalent to

$$f^{(n_0)}(g(z_0)) (g^{(n_0)})^{n_0 - 1}(z_0) g^{(pn_0+q)}(z_0) = g^{(n_0)}(f(z_0)) (f^{(n_0)})^{n_0 - 1}(z_0) f^{(pn_0+q)}(z_0)$$

But this implies of course that $f^{(pn_0+q)}(z_0) = g^{(pn_0+q)}(z_0)$, which is what we were after.

That one can restrict to this single index $k = (n_0, \cdots, n_0, pn_0 + q) \in \mathbb{N}^{n_0}$ is seen as follows: Let $k' \in \mathbb{N}^{n_0}$, be an ordered index with $|k'| = |k| = (n_0 - 1)n_0 + pn_0 + q = N$. Suppose that the last coordinate of k' (which is the maximum) is strictly bigger than the last coordinate of k. Then at least one of the previous coordinates of k' must be strictly smaller than n_0. But the associated derivatives of g (resp f) vanish at z_0. Thus this term does not appear in the formula for $(f \circ g)^{(N)}(z_0)$. On the other hand, if the last coordinate of k' is strictly less than $pn_0 + q$ (hence all of the coordinates of k'), then by induction all the associated derivatives of g (in $(f \circ g)^{(N)}$) coincide with those for f (in $(g \circ f)^{(N)}$) at z_0. Thus these terms can be thrown away.

Now let $k' \in \mathbb{N}^j$ with $n_0 < j \leq N$ and $|k'| = N$. Then the maximum of the coordinates of k' is strictly less than $pn_0 + q$, since otherwise $|k'| \geq (j-1)n_0 + pn_0 + q \geq n_0^2 + pn_0 + q > N$, a contradiction. Thus, as above, also these terms can be thrown away.

10991. *Proposed by Raymond Mortini, Départment de Mathematiques, Université de Metz, Ile du Saucy, France.* For complex $a, z \in \mathbb{D} = \{s : \ |s| < 1\}$, let $F(a, z) = (a + z)/(1 + \bar{a}z)$ be a map of $\mathbb{D}$ onto $\mathbb{D}$. Let $\rho(a, b) = |(a - b)/(1 - \bar{a}b)|$ be the pseudohyperbolic distance.

(**a**) Prove that there exists a function $C\colon \ \mathbb{D} \to \mathbb{R}^+$ so that $\rho(F(a, z), F(b, z)) \leq C(z)\rho(a, b)$ for every $a, b, z \in \mathbb{D}$.

(**b**) Find the minimal value of $C(z)$ for which this bound holds.

No own Solution to problem 10991, AMM 110 (2003), p. 155

10890. *Proposed by Raymond Mortini, Université de Metz, France.* Let d_1 and d_2 be two metrics on a nonempty set X with the property that every ball in (X, d_1) contains a ball in (X, d_2) and vice versa. Must d_1 and d_2 generate the same topology?

Solution to problem 10890, AMM 108 (2001), p. 668

Raymond Mortini

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Let d denote the Euclidean metric on $\mathbb{R}$ and let f be an injective real-valued function on $\mathbb{R}$. It is easy to see that the function $\rho(x, y) = |f(x) - f(y)|$ defines a second metric on $\mathbb{R}$, i.e. satisfies the axioms

(D1) $\rho(x, y) \geq 0, \rho(x, y) = 0 \iff x = y$,
(D2) $\rho(x, y) = \rho(y, x)$
(D3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in \mathbb{R}$.

Let $B_d(x_0, \epsilon)$ resp. $B_\rho(x_0, \epsilon)$ denote the open balls of radius ϵ and center x_0 with respect to the distances d and ρ.

Let us now additionally assume that f is increasing, one-sided continuous but not continuous, and has only a finite number of discontinuities. This guarantees that $I := f(\mathbb{R})$ is a union of non-degenerated intervals, with pairwise disjoint closures. The inverse function $f^{-1} : I \to \mathbb{R}$ then is continuous on I. Fix x_0. Hence for every $\epsilon > 0$ there exists $\delta > 0$ such that $B_\rho(x_0, \delta) \subseteq B_d(x_0, \epsilon)$.

Let x_0 be a point at which f is, say, left-continuous. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x < x_0$, $d(x, x_0) < \delta$ implies $\rho(x, x_0) = |f(x) - f(x_0)| < \epsilon$. Let $x_1 = x_0 - \frac{1}{2}\delta$. Then $B_d(x_1, \delta/2) \subseteq B_\rho(x_0, \epsilon)$.

Thus each ball in the d-metric contains a ball in the ρ-metric, and vice-versa.

It is clear that the identity map id: $(\mathbb{R}, \rho) \to (\mathbb{R}, d)$, although being continuous, has no continuous inverse. Note that id : $(\mathbb{R}, d) \to (\mathbb{R}, \rho)$ is continuous at x_0 if and only if f is continuous at x_0. Thus the two topologies are distinct.

Remark If we additionally assume that (X, d_j) are topological vector spaces, then the answer is yes. This is due to the fact that these topologies can be generated by translation invariant metrics d'_1 and d'_2. In fcat, $\forall \varepsilon > 0\ \exists \delta > 0 : B_{d'_1}(x_0, \delta) \subseteq B_{d'_2}(0, \varepsilon/2)$. In particular, x_0 and $-x_0$ are in $B_{d'_2}(0, \varepsilon/2)$. Hence

$$B_{d'_1}(0, \delta) = -x_0 + B_{d'_1}(x_0, \delta) \subseteq B_{d'_2}(0, \varepsilon/2) + B_{d'_2}(0, \varepsilon/2) \subseteq B_{d'_2}(0, \varepsilon).$$

The problem was also solved by Matthias Bueger and Dietmar Voigt (Germany).

10857 [2001,172]. *Proposed by Harold Diamond, University of Illinois, Urbana IL.*
(a) Show that

$$\frac{x + \frac{x^3}{3!} + \cdots + \frac{x^{2n-1}}{(2n-1)!}}{1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}} < \tanh x < \frac{x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}}{1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}}.$$

(b) Show that

$$\frac{x + \frac{x^3}{3!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n+1}}{2(2n+1)!}}{1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}} < \tanh x < \frac{x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}}{1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+2}}{2(2n+2)!}},$$

whenever n is a natural number and $0 < x < 2n$.

Solution to problem 10857 (a), AMM 108 (2001), p. 172

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Let $C_{2n} = \sum_{j=0}^{n} \frac{x^{2j}}{(2j)!}$ and $S_{2n+1} = \sum_{j=0}^{n} \frac{x^{2j+1}}{(2j+1)!}$. We show that, for every $x > 0$, the sequence $(\frac{S_{2n+1}}{C_{2n}})$ is strictly decreasing, whereas $(\frac{S_{2n-1}}{C_{2n}})$ is stricly increasing. Since both sequences converge to $\tanh x$ we get that $\frac{S_{2n-1}}{C_{2n}} < \tanh < \frac{S_{2n+1}}{C_{2n}}$.

i) We have the following equivalences:

$$(\frac{S_{2n+1}}{C_{2n}}) \searrow \Longleftrightarrow \frac{S_{2n+1}}{S_{2n-1}} < \frac{C_{2n}}{C_{2n-2}} \Longleftrightarrow \frac{S_{2n-1} + \frac{x^{2n+1}}{(2n+1)!}}{S_{2n-1}} < \frac{C_{2n-2} + \frac{x^{2n}}{(2n)!}}{C_{2n-2}} \Longleftrightarrow$$

$$\Longleftrightarrow 1 + \frac{\frac{x^{2n+1}}{(2n+1)!}}{S_{2n-1}} < 1 + \frac{\frac{x^{2n}}{(2n)!}}{C_{2n-2}} \Longleftrightarrow xC_{2n-2} < (2n+1)S_{2n-1} \Longleftrightarrow$$

$$\sum_{j=0}^{n-1} \frac{x^{2j+1}}{(2j)!} < (2n+1) \sum_{j=0}^{n-1} \frac{x^{2j+1}}{(2j+1)!} \tag{1}$$

But $\frac{1}{(2j)!} < (2n+1)\frac{1}{(2j+1)!} \Longleftrightarrow 2j+1 < 2n+1$, which is true. Since $x > 0$ we get (1).

ii) That $(\frac{S_{2n-1}}{C_{2n}})$ is stricly increasing, is shown in exactly the same way.

To sum up, we get

$$\frac{C_{2n+2}}{C_{2n}} < \frac{S_{2n+1}}{S_{2n-1}} < \frac{C_{2n}}{C_{2n-2}}.$$

Continuous Additive Functions.

10854 [2001,171]. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, China.* Find every function $f:\ \mathbb{R} \to \mathbb{R}$ that is continuous at zero and satisfies

$$f(x+2f(y)) = f(x) + y + f(y)$$

for all real numbers x and y.

Solution to problem 10854 AMM 108 (2001), p. 171

Raymond Mortini

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Suppose that $f:\mathbf{R}\to\mathbf{R}$ is a function, continuous at the origin, and satisfying

$$f(x+2f(y)) = f(x) + f(y) + y \tag{1}$$

for all $x, y \in \mathbf{R}$. First, we shall show that f is continuous everywhere. In fact,

$$\begin{aligned} f\big(x+2f(x+2f(y))\big) &= f(x+2[f(x)+f(y)+y]) = f\big([x+2y+2f(y)]+2f(x)\big) = \\ &= f\big((x+2y)+2f(y)\big)+f(x)+x = f(x+2y)+f(y)+y+f(x)+x. \end{aligned} \tag{2}$$

On the other hand:

$$\begin{aligned} f\big(x+2f(x+2f(y))\big) &= f(x)+f(x+2f(y))+x+2f(y) = \\ &= f(x)+[f(x)+f(y)+y]+x+2f(y) = 2f(x)+3f(y)+y+x. \end{aligned} \tag{3}$$

By (2) and (3) we get that $f(x+2y) = f(x)+2f(y)\ \forall (x,y)\in\mathbf{R}^2$.

In particular, by setting $x=y=0$, we see that $f(0)=0$.
It easily follows that f is continuous at every point $x\in\mathbf{R}$.

So, in order to continue, we may assume that f is a continuous solution of (1).
Let $x=y$. Then

$$f(y+2f(y)) = y+2f(y). \tag{4}$$

First we shall determine all continuous solutions of (4). Let $g(y) = y+2f(y)$. Since g is continuous, $g(\mathbf{R})$ is either a singleton or a nondegenerate interval I. If g is constant, say $g\equiv c$, then $f(y) = \frac{c-y}{2}$ and so $c = f(y+2f(y)) = f(c)$, from which we conclude that $c=0$. Hence $f(y) = -\frac{y}{2}$. If g is not constant, take $z\in I$; that is $y+2f(y) = g(y) = z$ for some y. Then $f(z) = z$. Hence f is the identity on I. It follows that $3z = z+2f(z) = f(z+2f(z)) = g(z)$. Therefore $3z\in I$ and so $I = \langle m,\infty[$ for some $m\in\mathbf{R}\cup\{-\infty\}$. Thus $f(x)=x$ for every $x>m$. Since $g\geq m$, we have that $f\geq \frac{m-y}{2}$ on $]-\infty,m]$.

To prove the converse, choose $m\in\mathbf{R}$. Let f^* be any continuous function on $]-\infty,m]$ such that $f^*(y)\geq\frac{m-y}{2}$ for $y\leq m$ and so that $f^*(m)=m$. Then

$$\widetilde{f}(y) = \begin{cases} f^*(y) & \text{if } y\leq m \\ y & \text{if } y\geq m \end{cases} \tag{5}$$

is a continuous solution of (4).

We deduce that any continuous solution of (1) necessarily has the form (5) or equals $-\frac{1}{2}y$. We shall now show that only for $f^* = id$, we really get a solution of (1).

So let f be a continuous solution of (1). Then $f=\widetilde{f}$ for some f^*. Fix $x<m$. Take $y>m$ so that $x+2y>m$. Then
$f(x+2f(y)) = f(x+2y) = x+2y$ and $f(x)+y+f(y) = f^*(x)+2y$. Hence (1) implies that $f^*(x)=x$.

We conclude that f is a continuous solution of (1) if and only if $f(x) = x$ or $f(x) = -\frac{x}{2}$ on $\mathbf{R}$.

10768. ***Proposed by Sung Soo Kim, Hanyang University, Ansan, Kyunggi, Korea.***
(a) Show that there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f+g$ is not increasing for any differentiable function g.
(b) Show that there is a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f+g$ is not increasing for any continuously differentiable function g.
(c) Show that, for any continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, there is a real analytic function g such that $f+g$ is increasing.

Solution to problem 10768 AMM 106 (1999), 963

Raymond Mortini

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a) Let $f(x) = \sqrt{|x|}\sin\dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is continuous on $\mathbb{R}$. Let g be a differentiable function on $\mathbb{R}$. Then, in every neigborhood of 0, $h := f + g - g(0)$ takes negative and positive values. In fact, suppose that $h \geq 0$ on $[0, \varepsilon]$. Then $\frac{h(x)}{x} \geq 0$ on $[0, \varepsilon]$. But $\liminf_{x\to 0^+} \frac{h(x)}{x} = g'(0) + \liminf_{x\to 0^+} \frac{1}{\sqrt{x}} \sin\frac{1}{x} = -\infty$, a contradiction. Thus $f + g$ is not monotone on any interval centered at 0.

b) Let $f(x) = x^2 \sin\dfrac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $\mathbb{R}$, $f'(0) = 0$, but $f'(x) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$ takes arbitrarily large negative and positive values in any neighborhood U of 0. Let g be any $C^1(\mathbb{R})$ function. In particular, g' is bounded on every compact interval centered at 0. Hence $f' + g'$ takes arbitrary large negative and positive values in U. Thus $f + g$ is not monotone on any interval centered at 0.

c) We show that for every function $f \in C^1(\mathbb{R})$ there exists an entire function g (that is a function holomorphic on the whole plane), real-valued on $\mathbb{R}$, such that $f + g$ is increasing on $\mathbb{R}$. In fact, $f' + 2|f'| + 2\varepsilon \geq 2\varepsilon > 0$ on $\mathbb{R}$. Let $q = 2|f'| + 2\varepsilon$. Then q is continuous on $\mathbb{R}$. By Carleman's theorem (see [41] and [40, p.125].) there exists an entire function Q such that $\|q - Q\|_\infty \leq \varepsilon$, where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{R}$. Let $G(x) = \operatorname{Re} Q(x)$. Then $\|q - G\| \leq \varepsilon$. Moreover, the function $H(z) = \frac{1}{2}\big(Q(z) + \overline{Q(\overline{z})}\big)$ is analytic in $\mathbb{C}$, and H coincides on $\mathbb{R}$ with G.

Now it is easy to check that $f' + G \geq \varepsilon > 0$. Let g be a primitive of G. Then g is the trace of an entire function and $f + g$ is (strictly) increasing, since its derivative is strictly positive.

10747. *Proposed by Athanasios Kalakos, Athens, Greece.* **Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that are twice differentiable on an open interval containing 0, have exactly one real root, satisfy $f(1) = 1$, and satisfy $f'(f(t)) = 2f(t)$ for every $t \in \mathbb{R}$.**

Solution to problem 10747 AMM 106 (1999), p. 685

Raymond Mortini

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We claim that all differentiable solutions f of $f'(f(t)) = 2f(t)$, $t \in \mathbb{R}, f(1) = 1$, and having only one real root, have the form $f(t) = t^2$ for $t \geq 0$ and $f(t) = g(t)$ for $t < 0$, where g is an arbitrary differentiable function, defined on $]-\infty, 0]$ satisfying $g(t) > 0$ for $t < 0$ and $g(0) = g'(0) = 0$. The assumption, that f should be twice differentiable in a neighboorhood of 0, is not important.

Proof Let f be a solution of the problem. Put $h = f \circ f - f^2$. Then $h' = (f' \circ f)f' - 2f'f = f'(f' \circ f - 2f) \equiv 0$. Hence h is a constant, say C. Because $h(1) = 0$, we see that $C = 0$ and so $f \circ f = f^2$. Let $y \in f(\mathbb{R})$. Then $f(x) = y$ for some $x \in \mathbb{R}$. Therefore $f(y) = f(f(x)) = f^2(x) = y^2$. By hypothesis, $\{0, 1\} \subseteq f(\mathbb{R})$. By continuity we conclude that $[0, 1] \subseteq f(\mathbb{R})$. Since the left derivative at $x = 1$ is 2, the differentiability of f now implies that there exists points x_0 greater than 1 for which $f(x_0) > f(1) = 1$. Since $f_{n+1} = f^{2^n}$, we obtain that $f_{n+1}(x_0) = [f(x_0)]^{2^n} \to \infty$. Hence f is unbounded. By the intermediate value theorem, we then get that $[0, \infty] \subseteq f(\mathbb{R})$. Hence $f(x) = x^2$ for $x \geq 0$.

To determine the behaviour of f for negative values, we use the hypothesis that f should have only one zero. Since $f(0) = 0$, by continuity, we conclude that either $f(x) < 0$ for all $x < 0$ or $f(x) > 0$ for all $x < 0$. But $f(x_0) < 0$ for some (all) $x_0 < 0$ implies that $f(f(x_0)) = f^2(x_0) > 0$, a contradiction. Thus $f(x) > 0$ for $x > 0$.

It is easy to check that every function of the form $f(x) = x^2$ for $x \geq 0$ and $f(x) = g(x)$ for $x < 0$, where $g > 0$ is differentiable and satisfies $g(0) = g'(0) = 0$, is a solution of $f \circ f = f^2$. Hence, by differentiating, $f'(f(x))f'(x) = 2f'(x)f(x)$. If $f'(x) \neq 0$, then we are done. If $f'(x_0) = g'(x_0) = 0$ for some $x_0 < 0$, then we use the fact that $y := f(x_0) > 0$ and that for these positive values $f(y) = y^2$. Hence, $f'(f(x_0)) = 2f(x_0)$. So we obtain a solution of our functional equation.

10739. *Proposed by Oscar Ciaurri, Logroño, Spain.* Suppose that $f\colon [0,1] \to \mathbb{R}$ has a continuous second derivative with $f''(x) > 0$ on $(0,1)$, and suppose that $f(0) = 0$. Choose $a \in (0,1)$ such that $f'(a) < f(1)$. Show that there is a unique $b \in (a,1)$ such that $f'(a) = f(b)/b$.

Solution to problem 10739 AMM 106 (1999), p. 586

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Let $H(x) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$. Since $f''(x) > 0$, the function f is strictly convex and both its derivative and the quotient H are strictly increasing (see e.g. W. Walter, Analysis 1, Springer-Verlag, p. 303). Moreover, H is continuous on $]0,1]$. Note that $H(1) = f(1)$ and that $H(0) := \lim_{x\to 0} f'(x)$ exists in $[-\infty, f(1)]$. Hence, by the intermediate value theorem, there exists for every value w with $H(0) < w < H(1)$ a point $b \in]0,1[$ with $H(b) = w$. Now choose $a \in]0,1[$ such that $w := f'(a)$ satisfies $H(0) < w < H(1)$ (such a choice obviously is possible). Thus there exists $b \in]0,1[$ so that $\frac{f(b)}{b} = H(b) = f'(a)$. Choose $x_a \in]0,a[$ so that $H(a) = f'(x_a)$. Due to the monotonicity of f' we obtain: $H(a) = f'(x_a) < f'(a) = H(b)$. Since H is monotone, b is unique and satisfies $a < b < 1$.

10697. ***Proposed by José L. Diaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*** **Given n distinct nonzero complex numbers $z_1, z_2, \ldots, z_n$, show that**

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n}.$$

Solution to problem 10697 AMM 105 (1998), p. 955

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This is nothing but a Lagrange interpolatory argument:

In fact let $w_1, \cdots, w_n \in \mathbb{C}$. Then

$$p(z) = \sum_{k=1}^{n} w_k \frac{\prod_{j=1, j\neq k}^{n} (z - z_j)}{\prod_{j=1, j\neq k}^{n} (z_k - z_j)}$$

is the unique polynomial of degree at most $n-1$ satisfying $p(z_k) = w_k, \ k = 1, \cdots, n$. Now choose $w_k = 1$ for every k. Since $q(z) \equiv 1$ satisfies the interpolation $q(z_k) = w_k$, we obtain from uniqueness that $q = p$. Let $z = 0$. Then

$$1 = q(0) = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{(-z_j)}{z_k - z_j} = (-1)^{n-1} \sum_{k=1}^{n} \frac{\prod_{j=1, j\neq k}^{n} z_j}{\prod_{j=1, j\neq k}^{n} (z_k - z_j)}.$$

Dividing by $\prod_{j=1}^{n} z_j$, yields the assertion

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n-1}}{\prod_{j=1}^{n} z_j}.$$

10651. ***Proposed by W. K. Hayman, Imperial College, London, U.K..*** **If u_1 and u_2 are nonconstant real functions of two variables, and if u_1, u_2, and u_1u_2 are all harmonic in a simply connected plane domain D, prove that $u_2 = av_1 + b$, where v_1 is a harmonic conjugate of u_1 in D, and a and b are real constants.**

Solution to problem 10651 AMM 105 (1998), p. 271

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We prove a stronger version than in the formulated problem (see [11], which was based on this).

Proposition 1 *Let u and v be two non constant harmonic functions on a domain $D \subseteq \mathbb{C}$. Suppose that uv is harmonic. Then u has an harmonic conjugate $\tilde{u}$ on D and there are constants $a, b \in \mathbb{R}$ such that*

$$v = a\tilde{u} + b. \tag{1}$$

Remarks. (1) If u is a constant, then (1) is not true (because v may be chosen to be any harmonic function).

(2) If v is a constant then (1) is true for $a = 0$, provided a harmonic conjugate exists. A well known sufficient condition for the existence of a harmonic conjugate being that D is simply connected.

(3) Of course, if v is any harmonic function satisfying (1), then uv is harmonic.

Solution Let Δ be the Laplace operator. Because $\Delta u = \Delta v = 0$ we obtain:

$$0 = \Delta(uv) = \big(u_{xx}v + 2u_xv_x + v_{xx}\big) + \big(u_{yy}v + 2u_yv_y + v_{yy}\big) = 2\big(u_xv_x + u_yv_y\big).$$

Let $f = u_x - iu_y$ and $g = v_x - iv_y$. The harmonicity of u and v imply that f and g satisfy the Cauchy-Riemann differential equations; hence f and g are holomorphic. It is easy to see that $\text{Re } f\overline{g} = u_xv_x + u_yv_y$. Thus $\text{Re } f\overline{g} \equiv 0$ on D.

Let $Z(g) = \{z \in D : g(z) = 0\}$ denote the zero set of g. It is a discrete subset of D provided that $g \not\equiv 0$. Since v is assumed not to be a constant, we see that $g \not\equiv 0$. Then on $D \setminus Z(g)$ we have $\text{Re } \dfrac{f}{g} = \text{Re } \dfrac{f\overline{g}}{|g|^2}$. Thus $\text{Re } \dfrac{f}{g} \equiv 0$ on $D \setminus Z(g)$. This implies, in view of the analyticity, that $\frac{f}{g}$ is a pure imaginary constant, say $\frac{f}{g} \equiv i\lambda$ on $D \setminus Z(g)$. Hence $f = i\lambda g$ on D. The definitions of f and g now yield that $u_x = \lambda v_y$ and $u_y = -\lambda v_x$. Consequently, by the Cauchy-Riemann equations, the function $u + i\lambda v$ is holomorphic on D. In particular, u has an harmonic conjugate on D. (Note that we do not have assumed that D is simply connected.) Thus, for any other harmonic conjugate $\tilde{u}$ of u, we have $\lambda v = \tilde{u} + c$ for some constant $c \in \mathbb{R}$. Note that u not constant implies that $\lambda \neq 0$. Thus v has the desired form (1).

A natural question now is the following. Let u and v be two harmonic functions on a domain $D \subseteq \mathbb{C}$. Then $(u + iv)^2 = u^2 - v^2 + 2iuv$. Assume that $u^2 - v^2$ is harmonic. What can be said for v? We have the following result:

Proposition 2 *Assume that u, v and $u^2 - v^2$ are harmonic in a simply connected domain $D \subseteq C$. Then there exists $a \in \mathbb{R}$ and $\theta \in [0, 2\pi[$ such that*

$$v = \cos\theta\, u - \sin\theta\, \tilde{u} + a. \tag{2}$$

Conversely, every function v satisfying (2) for a harmonic function u has the property that $u^2 - v^2$ is harmonic.

Proof Because $\Delta u = \Delta v = 0$ we obtain:

$$0 = \Delta(u^2 - v^2) = 2\Big(u_x^2 + u_y^2 - (v_x^2 + v_y^2)\Big).$$

Hence $u_x^2 + u_y^2 = v_x^2 + v_y^2$. Again, let $f = u_x - iu_y$ and $g = v_x - iv_y$. As above, f and g are holomorphic on D. Moreover $|f|^2 = |g|^2$. Thus g is a rotation of f, say $g = e^{i\theta} f$.

Let $z_0 \in D$. Since D is simply connected, u and v have harmonic conjugates $\tilde{u}$ and $\tilde{v}$ respectively, satisfying $\tilde{u}(z_0) = \tilde{v}(z_0) = 0$. Let $F = u + i\tilde{u}$ and $G = v + i\tilde{v}$. Then, by Cauchy-Rieman, $F' = u_x + i\tilde{u}_x = u_x - iu_y = f$. Similiarly $G' = g$. Thus $G = e^{i\theta}F + c$ for some constant $c \in \mathbb{C}$. Taking real parts yields

$$v = \cos\theta\, u - \sin\theta\, \tilde{u} + a$$

for some real constant a. The converse is easy to check.

The above results are related to the following more general result:

Proposition 3. *Let h be an entire function and let $u : D \to \mathbb{R}$ and $v : D \to \mathbb{R}$ be two nonconstant harmonic functions in a simply connected domain D. Let $\tilde{u}$ be a harmonic conjugate of u in D. Then $h(u + iv) : D \to \mathbb{C}$ is harmonic if and only if $v = \pm\tilde{u} + a$ for a constant $a \in \mathbb{R}$.*

Proof Since h is holomorphic, we have, by Cauchy-Riemann, $h_y = ih_x$ and $h_x = h'$. Hence $h_{xx} = h'', h_{xy} = h_{yx} = ih''$ and $h_{yy} = -h''$. As above, let $f = u_x - iu_y$ and $g = v_x - iv_y$.Then

$$\Delta[h \circ (u + iv)] = h'' \circ (u + iv) \cdot [(|f|^2 - |g|^2) + 2i\mathrm{Re}\, f\overline{g}].$$

Obviously $h'' \circ q \not\equiv 0$ for any nonconstant continuous function q. Hence $h(u+iv)$ is harmonic if and only if $|f| = |g|$ and Re $f\overline{g} = 0$. By the paragraphs above we conclude that $f = i\lambda g$ for some $\lambda \in \mathbb{R}$. Hence $|\lambda| = 1$. Thus $u_x = \pm v_y$ and $-u_y = \pm v_x$. So v or $-v$ is a harmonic conjugate of u in D. Therefore $v = \pm\tilde{u} + a$.

To prove the converse, we have simply to note that the composition of a holomorphic function with a holomorphic or anti-holomorphic function is harmonic.

10638. *Proposed by Brian Conolly, Cambridge, UK.* For $0 \le \lambda \le 1$ and $m \ge 0$, let $S_m(\lambda) = \sum_{n\ge 1} e^{-\lambda n}(\lambda n)^{n-m}/n!$. Show that $S_0(\lambda) = \lambda/(1-\lambda)$, $S_1(\lambda) = 1$, $S_2(\lambda) = 1/\lambda - 1/2$, and $S_3(\lambda) = 1/\lambda^2 - 3/(4\lambda) + 1/6$.

Solution to problem 10638 AMM 105 (1998), p. 69

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In the following we present a solution to problem number 10638. We shall not only compute the functions $S_0, \cdots, S_3$, but we will give an explicit value for all $m \in \mathbb{N}$. To this end we need the following Lemma.

Lemma *Let $f(z) = ze^z$. Then f is invertible in a neighborhood of the origin in $\mathbb{C}$ and the inverse function has the Taylor representation*

$$f^{-1}(w) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-1)^{n-1}w^n,$$

which converges for $|w| < \frac{1}{e}$.

Proof By the residue theorem it is easy to see that whenever f is holomorphic and injective in a disque $D \subseteq \mathbb{C}$ (or even a simply connected domain), then

$$f^{-1}(w)n(\Gamma, f^{-1}(w)) = \frac{1}{2\pi i}\int_{\Gamma} z\,\frac{f'(z)}{f(z)-w}\,dz,$$

where Γ is an arbitrary cycle (=finite union of closed, piecewise C^1-curves) in D.
Applying this formula for $f(z) = ze^z$ and the disk $|z| < 2\delta$, δ small enough, we obtain :

$$\frac{d^n}{(dw)^n} f^{-1}(w) = \frac{n!}{2\pi i}\int_{|z|=\delta} z\,\frac{(z+1)e^z}{(ze^z - w)^{n+1}}\,dz.$$

Thus, for the power series $f^{-1}(w) = \sum_{n=0}^{\infty} a_n w^n$ we have $a_0 = 0$ and for $n \ge 1$:

$$a_n = \frac{1}{2\pi i}\int_{|z|=\delta} \frac{z+1}{z^n} e^{-nz}\,dz = \frac{1}{2\pi i}\sum_{k=0}^{\infty}(-1)^k \int_{|z|=\delta} \frac{z(nz)^k + (nz)^k}{z^n k!}\,dz = (-1)^{n-1}\frac{n^{n-1}}{n!}.$$

By d'Alembert's rule it is easy to check that the radius of convergence is $1/e$. ○

Proposition *For $0 < \lambda < 1$ and $m \in \mathbb{Z}$, let $g_m(\lambda) = \lambda^m S_m(\lambda)$, where*

$$S_m(\lambda) = \sum_{n=1}^{\infty} e^{-\lambda n}(\lambda n)^{n-m}/n!. \tag{1}$$

Then, for $m \in \{1, 2, \cdots\}$, g_m is a polynomial of degree m vanishing at the origin, say $g_m(\lambda) = -\sum_{n=1}^{m} b_{n,m}(-\lambda)^n$, and the coefficients $b_{n,m}$ are given by the recurrence relation

$$b_{n,m} = \frac{1}{n}(b_{n,m-1} + b_{n-1,m-1}), \quad b_{1,1} = 1. \tag{3}$$

Solving these difference equations yields

$$b_{n,m} = \sum_{j=1}^{n} \frac{1}{n!}\binom{n}{j}(-1)^{j-1}\left(\frac{1}{j}\right)^{m-n}. \tag{4}$$

Proof We note that, by Stirling's formula, the series $g_m(\lambda)$ converges locally uniformly in $0 \le \lambda < 1$, but does not converge whenever $\lambda = 1$ and $m = 0$. Note that $g_m(0) = 0$. Due to local uniform convergence, it is easy to see that, in order to obtain $g'_m(\lambda)$, one can differentiate the series for g_m term by term. This yields that for $m \in \mathbb{Z}$

$$g'_m(\lambda) = \frac{1-\lambda}{\lambda} g_{m-1}. \tag{5}$$

Later we shall show that $g_1(\lambda) = \lambda$. Hence, by induction on (5), it is clear that for $m = 1, 2, \cdots$ the function g_m is a polynomial vanishing at the origin, say $g_m(\lambda) = -\sum_{n=1}^{m} b_{n,m}(-\lambda)^n$. If we let $x = -\lambda$, then we obtain $\sum_{n=1}^{m} n b_{n,m} x^n = (1+x)\sum_{n=1}^{m} b_{n,m-1} x^n$. Comparing coefficients, finally yields (3).

This difference equation can be solved by the usual methods. May be Maple or Mathematica gives the solution. In any case, by the uniqueness of the solution, it suffices to show that (4) verifies the difference equation. Note also, that for $n > m$, the $b_{n,m}$ in (4) are 0. This follows from the fact that the p-th difference operator $D^p(a_n) = \sum_{j=0}^{p} \binom{n}{j}(-1)^j a_{n-j}$ vanishes identically whenever a_n is a polynomial (in n) of degree strictly less than p.

For the readers convenience, here are the coefficients for $m = 1, \cdots, 5$:

1				
1	$\frac{1}{2}$			
1	$\frac{3}{4}$	$\frac{1}{6}$		
1	$\frac{7}{8}$	$\frac{11}{36}$	$\frac{1}{24}$	
1	$\frac{15}{16}$	$\frac{85}{216}$	$\frac{25}{288}$	$\frac{1}{120}$

The case m=1 In that case we have

$$g_1(\lambda) = \lambda \sum_{n=1}^{\infty} e^{-\lambda n}(\lambda n)^{n-1}/n! = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(\lambda e^{-\lambda})^n.$$

Let $w = -\lambda e^{-\lambda}$. Now, for $w \in \mathbb{C}, |w| < \frac{1}{e}$, the function $h(w) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-1)^{n-1} w^n$ is, by Lemma 1, nothing but the inverse function of the holomorphic function $f(z) = ze^z$, $|z| < \delta$ for sufficiently small $\delta > 0$. Thus $g_1(\lambda) = \lambda$.

The case m=0 By (5) we see that $1 = g'_1(\lambda) = \frac{1-\lambda}{\lambda} g_0(\lambda)$. Hence, $g_0(\lambda) = \frac{\lambda}{1-\lambda}$.

Using (5) it is also easy to derive, inductively, the values of g_m for negative integers m. For example we get:

$$g_{-1}(\lambda) = \frac{\lambda}{(1-\lambda)^3}, \qquad g_{-2}(\lambda) = \frac{\lambda}{(1-\lambda)^5}(1+2\lambda), \qquad g_{-3}(\lambda) = \frac{\lambda}{(1-\lambda)^7}(1+8\lambda+6\lambda^2).$$

In general, one can convince oneself that for $m \in \mathbb{Z}, m < 0$, $g_m(\lambda)$ has the form $g_m(\lambda) = \frac{\lambda}{(1-\lambda)^{-2m+1}} Q_m(\lambda)$, where Q is a polynomial of degree $-m-1$ with value 1 at the origin and satisfying the differential equations

$$Q_{m-1}(\lambda) = \lambda(1-\lambda)Q'_m(\lambda) + (1-2m\lambda)Q_m(\lambda).$$

Due to lack of time we were not able to solve this explicitely. May be Maple and Mathematica will be helpfull.

10624. ***Proposed by William F. Trench, Trinity University, San Antonio TX.*** **Suppose that $a_0 > a_1 > a_2 > \cdots$ and $\lim_{n\to\infty} a_n = 0$. Define**

$$S_n = \sum_{j=n}^{\infty}(-1)^{j-n}a_j = a_n - a_{n+1} + a_{n+2} - \cdots.$$

Show that $\sum a_n S_n < \infty$ if and only if $\sum a_n^2 < \infty$.

Solution to problem 10624 AMM 104 (1997), p. 871

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By Leibniz's criteria, we know that S_n actually converges and that $S_n \geq 0$ for every $n \in \mathbb{N}$. Since $S_n = a_n - S_{n+1}$, we see that $S_n \leq a_n$ and so $\sum a_n S_n \leq \sum a_n^2$. Thus the convergence of $\sum a_n^2$ implies the convergence of $\sum a_n S_n$.

Now assume that $\sum a_n S_n = \sum (S_n + S_{n+1}) S_n$ (1) converges. Since all the terms are positive, we deduce the convergence of the sums $\sum S_n^2$ and $\sum S_{n+1} S_n$. A shift of the variable yields that $\sum S_{n+1}^2$ converges. Hence $\sum S_{n+1}(S_{n+1} + S_n)$ (2) converges. Summing (1) and (2) yields that $\sum a_n^2 = \sum (S_{n+1} + S_n)^2 = \sum (S_n + S_{n+1}) S_n + \sum S_{n+1}(S_{n+1} + S_n)$ is convergent.

10605. *Proposed by Jonathan M. Borwein and C. G. Pinner, Simon Fraser University, Burnaby, BC, Canada.* Let r and m be positive integers and define

$$P_r(m) = \prod_{n \neq m} \frac{n^r - m^r}{n^r + m^r}.$$

(a) Show that $P_1(m) = 0$ and that

$$P_3(m) = (-1)^{m+1}\frac{2}{3}(m!)^2 \prod_{n=1}^{m} \frac{n+m}{n^3+m^3}.$$

(b) Show that $P_2(m) = (-1)^{m+1}\pi m/\sinh(\pi m)$ and that, more generally, $P_{2s}(m)$ is given by

$$(-1)^{m+1}\frac{2^\epsilon m\pi}{s}(\sinh m\pi)^{(-1)^s}\prod_{j=1}^{s-1}\left(\cosh\left(2\pi m \sin\left(\frac{j\pi}{2s}\right)\right) - \cos\left(2\pi m \cos\left(\frac{j\pi}{2s}\right)\right)\right)^{(-1)^j}$$

where $\epsilon = (1 + (-1)^s)/2$.

Solution to problem 10605 (b) AMM 104 (1997), p. 567

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Write $\dfrac{n^{2s}-m^{2s}}{n^{2s}+m^{2s}} = \dfrac{1-(m/n)^{2s}}{1+(m/n)^{2s}}$. Let $y = \left(\dfrac{m}{n}\right)^2$. Then

$$\frac{1-y^s}{1+y^s} = \frac{\prod_{j=0}^{s-1}\left(1 - y\exp(-i\frac{2\pi j}{s})\right)}{\prod_{j=0}^{s-1}\left(1 - y\exp(-i\frac{\pi+2\pi j}{s})\right)}.$$

Since $\varepsilon \in \mathbb{C}$ is an s-root of 1 [resp. (-1)] if and only if $\overline{\varepsilon}$ is an s-root, we obtain:

$$\frac{1-y^s}{1+y^s} = \frac{(1-y)(1+y)\prod_{j=1}^{p-1}\left|1 - y\exp(-i\frac{2\pi j}{s})\right|^2}{\prod_{j=1}^{p-1}\left|1 - y\exp(-i\frac{\pi(2j+1)}{s})\right|^2}$$

if $s = 2p$ and

$$\frac{1-y^s}{1+y^s} = \frac{(1-y)\prod_{j=1}^{p}\left|1 - y\exp(-i\frac{2\pi j}{s})\right|^2}{(1+y)\prod_{j=1}^{p-1}\left|1 - y\exp(-i\frac{\pi(2j+1)}{s})\right|^2}$$

if $s = 2p+1$.

This can be written by a single formula:

$$\frac{1-y^s}{1+y^s} = (1-y)(1+y)^{(-1)^s}\prod_{k=1}^{s-1}\left|1 - y\exp(-i\frac{\pi k}{s})\right|^{2(-1)^k}. \tag{1}$$

In particular

$$\prod_{k=1}^{s-1}\left|1 - exp(-i\frac{\pi k}{s})\right|^{2(-1)^k} = \lim_{y\to 1}\frac{\frac{1-y^s}{1+y^s}}{(1-y)(1+y)^{(-1)^s}} = \frac{s}{2\cdot 2^{(-1)^s}}. \tag{2}$$

It is easy to check that

$$P := \prod_{k=1}^{s-1}(2\pi^2 m^2)^{(-1)^k} = \begin{cases} 1, & \text{if } s \text{ is odd} \\ \frac{1}{2\pi^2 m^2}, & \text{if } s \text{ is even.} \end{cases} \tag{3}$$

Now use the infinite product representation of the function $\sin \pi z$. This gives:

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

and

$$\prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right) = \frac{\sin i\pi z}{i\pi z} = \frac{\sinh \pi z}{\pi z}.$$

Moreover we have by de l'Hôpital's rule that

$$\prod_{n\neq m}^{\infty} \left(1 - \frac{m^2}{n^2}\right) = \lim_{z\to m} \frac{\sin \pi z}{\pi z} \Big/ 1 - \left(\frac{z}{m}\right)^2 = \frac{(-1)^{m+1}}{2}.$$

Finally we need that $|\sin z|^2 = \frac{1}{2}(\cosh 2y - \cos 2x)$ for $z = x + iy$.
Put all this together to get from (1)

$$2^{(-1)^s} \prod_{n\neq m} \frac{n^{2s} - m^{2s}}{n^{2s} + m^{2s}} = \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \cdot \prod_{k=1}^{s-1} \left| \prod_{n\neq m} \left(1 - \left(\frac{m}{n} exp(-i\frac{\pi k}{2s})\right)^2\right)\right|^{2(-1)^k} =$$

$$= \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \prod_{k=1}^{s-1} \left| \frac{\sin\bigl(\pi m \exp(-i\frac{\pi k}{2s})\bigr)}{\pi m\,\bigl(1 - \exp(-i\frac{\pi k}{s})\bigr)} \right|^{2(-1)^k} =$$

$$= \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \frac{\prod_{k=1}^{s-1} \left[\frac{1}{2}\left(\cosh\left(2\pi m \sin\frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos\frac{\pi k}{2s}\right)\right)\right]^{(-1)^k}}{\prod_{k=1}^{s-1} \left|1 - \exp(-i\frac{\pi k}{s})\right|^{2(-1)^k} \cdot \prod_{k=1}^{s-1} (\pi m)^{2(-1)^k}} =$$

$$= \frac{(-1)^{m+1}}{2} \left(\frac{\sinh \pi m}{\pi m}\right)^{(-1)^s} \frac{\prod_{k=1}^{s-1} \left[\cosh\left(2\pi m \sin\frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos\frac{\pi k}{2s}\right)\right]^{(-1)^k}}{\prod_{k=1}^{s-1} \left|1 - \exp(-i\frac{\pi k}{s})\right|^{2(-1)^k} \cdot \prod_{k=1}^{s-1} (2\pi^2 m^2)^{(-1)^k}} =$$

$$= \frac{(-1)^{m+1} (\sinh \pi m)^{(-1)^s} \prod_{k=1}^{s-1} \left[\cosh\left(2\pi m \sin\frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos\frac{\pi k}{2s}\right)\right]^{(-1)^k}}{(\pi m)^{(-1)^s} \frac{s}{2^{(-1)^s}} \cdot P}$$

Clearly

$$\frac{1}{(\pi m)^{(-1)^s}\, s \cdot P} = 2^{(1+(-1)^s)/2} \left(\frac{\pi m}{s}\right).$$

Putting $\varepsilon = (1 + (-1)^s)/2$, we get the final equality:

$$P_{2s} = \prod_{n\neq m} \frac{n^{2s} - m^{2s}}{n^{2s} + m^{2s}} =$$

$$= (-1)^{m+1} \frac{2^{\varepsilon} m\pi}{s} (\sinh \pi m)^{(-1)^s} \prod_{k=1}^{s-1} \left[\cosh\left(2\pi m \sin\frac{\pi k}{2s}\right) - \cos\left(2\pi m \cos\frac{\pi k}{2s}\right)\right]^{(-1)^k}.$$

If $s = 1$ we interpret the empty product as 1. This gives

$$P_2(m) = (-1)^{m+1} \pi m / \sinh(\pi m).$$

10588. ***Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.*** **Show that**

$$\prod_{j\geq 1} e^{-1/j}\left(1+\frac{1}{j}+\frac{1}{2j^2}\right)=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}},$$

where γ is Euler's constant.

Solution to problem 10588/10595 AMM 104 (1997), p. 456

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We show that

$$P=\prod_{n=1}^{\infty} e^{-\frac{1}{n}}\left(1+\frac{1}{n}+\frac{1}{2n^2}\right)=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}}.$$

Let

$$\Gamma(z)=\left[e^{\gamma z}z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{z/n}\right]^{-1}$$

be the Gamma function and let $\varepsilon=\frac{1}{2}(1+i)$. Then $\overline{\varepsilon}=1-\varepsilon$. Hence, as is well known,

$$\Gamma(\varepsilon)\Gamma(\overline{\varepsilon})=\Gamma(\varepsilon)\Gamma(1-\varepsilon)=\frac{\pi}{\sin\pi\varepsilon}.$$

Therefore

$$\frac{\sin\pi\varepsilon}{\pi}=e^{\gamma\varepsilon}\varepsilon\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon}{n}\right)e^{-\varepsilon/n}\times e^{\gamma\overline{\varepsilon}}\overline{\varepsilon}\prod_{n=1}^{\infty}\left(1+\frac{\overline{\varepsilon}}{n}\right)e^{-\overline{\varepsilon}/n}=$$

$$=e^{\gamma}\frac{1}{2}\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon}{n}\right)\left(1+\frac{\overline{\varepsilon}}{n}\right)e^{-1/n}=$$

$$=e^{\gamma}\frac{1}{2}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}+\frac{1}{2n^2}\right)e^{-1/n}.$$

Hence $P=\dfrac{2\sin\pi\varepsilon}{\pi e^{\gamma}}=\dfrac{2\cosh\pi/2}{\pi e^{\gamma}}$, which is the assertion.

6654. *Proposed by W. O. Egerland and C. E. Hansen, Aberdeen Proving Ground, Aberdeen, MD.*

Suppose ω is real, n is a positive integer greater than 1, and $a_1, a_2, \dots, a_n$ are complex numbers with $|a_k| < 1$ for $k = 1, 2, \dots, n$. Prove that the equation

$$e^{i\omega}(z - a_1)(z - a_2)\cdots(z - a_n) = z(1 - \bar{a}_1 z)(1 - \bar{a}_2 z)\cdots(1 - \bar{a}_n z)$$

has at least $n - 1$ roots on the unit circle.

Solution to problem 6654 AMM 98 (1991), p. 273

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Let $B(z) = e^{i\omega}\prod_{j=1}^{n}\frac{a_j - z}{1 - \bar{a}_j z}$ be a finite Blaschke product of degree $n \geq 2$ ($a_j \in \mathbb{D}$) and let $\varphi(z) = \frac{z_0 - z}{1 - \bar{z}_0 z}$, $z_0 \in \mathbb{D}$.

Then $\varphi^{-1} \circ B \circ \varphi$, called a conjugate of B, is again a finite Blaschke product of degree $n \geq 2$. (This follows from Rouché's theorem which shows that f is n to 1 and from the maximum principle.)

It is also easy to see that the relation

$$(\star) \qquad 1 = B(z)\overline{B\left(\frac{1}{\bar{z}}\right)}$$

implies that z_0, $z_0 \neq 0$, is a fixed point of B if and only if $1/\bar{z}_0$ is a fixed point of B.

Assume now that for some $z_0 \in \mathbb{D}$ we have $B(z_0) = z_0$. Then $f(0) = 0$. Hence, by Schwarz's lemma, $|f(z)| < |z|$ in $\mathbb{D}$. Therfore f, and hence B, cannot have further fixed points in $\mathbb{D}$. Thus, by $(\star)$, B can have at most two fixed points outside the unit circle T. Because B has n (resp. $n+1$) fixed points in $\mathbb{C}$ whenever $0 \in \{a_1, \dots, a_n\}$ (resp. $0 \notin \{a_1, \dots, a_n\}$) we can conclude that B has either $n-1$ or $n+1$ fixed points on T.

Remark. One can also conclude that B has a unique fixed point in $\mathbb{D}$ if and only if B is conjugate to a finite Blaschke product of degree n with $f(0) = 0$.

6648. *Proposed by Walter Rudin, University of Wisconsin, Madison.*

Let Ω be the region obtained by removing the points $0, 1, \infty$ from the Riemann sphere. Find all nonconstant holomorphic functions defined on Ω which map Ω into itself.

Solution to problem 6648 AMM 98 (1991), p. 63

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Answer: Let $\Omega = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Then there are, besides the constants, exactly the six functions

$$z, \quad \frac{1}{z}, \quad 1 - z, \quad \frac{1}{1-z}, \quad \frac{z}{z-1}, \quad \frac{z-1}{z}$$

which map Ω holomorphically into Ω.

Proof: Let f be a nonconstant holomorphic map of Ω into Ω. Because f omits the two points $w = 0, 1$, Picard's theorem tells us that none of the points $z = 0, 1, \infty$ is an essential singularity. Thus f is a rational function R of degree (order) $n \in \mathbb{N}$. Because $R^{-1}(\{0, 1\})$, which is a subset of $\{0, 1, \infty\}$, contains at least two different points, R can have at most one pole. This has then order n. By taking, if necessary, reflections, we may assume without loss of generality that ∞ is this pole. Thus R is a polynomial of degree n. The assumption on f now implies that both the points $z = 0$ and $z = 1$ must be n-fold w_0-points, where $w_0 \in \{0, 1\}$. This means that the derivative of the polynomial R has at least $2(n-1)$ zeros. This implies that $n = 1$.

Thus in the general case, f is a rational function of degree 1, hence a Möbius transform. Considering all permutations of $(0, 1, \infty)$ and constructing the Möbius transforms S with $S(\{0, 1, \infty\}) = \{0, 1, \infty\}$ yields the assertion.

E 3329. *Proposed by Michel Balazard, Faculté des Sciences, Limoges, France.*

Suppose f and g are differentiable real-valued functions defined on $(-\infty, +\infty)$. Must there exist a differentiable real-valued function h defined on $(-\infty, +\infty)$ such that $h' = f'g'$?

Solution to problem E3329 AMM 96 (1989), p. 445

Raymond Mortini

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This problem is well known, and its solution is **No.**

In fact, let $F(x) = \cos\frac{1}{x}$ for $x \neq 0$ and $F(0) = 0$. Then F admits a primitive f of the form

$$f(x) := \int_0^x 2t \sin\frac{1}{t}\, dt - x^2 \sin\frac{1}{x} \text{ for } x \neq 0$$

and $f(0) = 0$.

But F^2 does not have a primitive H, because otherwise

$$0 = F^2(0) = H'(0) = \lim_{x\to 0} \frac{1}{x}\int_0^x \cos^2\frac{1}{t}\, dt = \frac{1}{2},$$

which is a contradiction. Note that the last equality follows from the facts that

$$1 = \frac{1}{x}\int_0^x \left(\cos^2\frac{1}{t} + \sin^2\frac{1}{t}\right)\, dt$$

and that

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x}\int_0^x \left(\cos^2\frac{1}{t} - \sin^2\frac{1}{t}\right)\, dt &= \lim_{x\to 0} \frac{1}{x}\int_0^x \cos\frac{2}{t}\, dt \\
&= \lim_{x\to 0} \frac{1}{\frac{x}{2}}\int_0^{\frac{x}{2}} \cos\frac{1}{s}\, ds \\
&= f'(0) = 0.
\end{aligned}$$

E 3325. *Proposed by Walter Rudin, University of Wisconsin, Madison.*

Let us say that a function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

has property P_t if $\sum_2^\infty n|a_n| \leqslant t$. Prove:

(a) If P_t holds for some $t < \infty$, then f is continuous on the closed unit disc, i.e., on $\{z \in \mathbf{C} : |z| \leqslant 1\}$.

(b) If P_1 holds, then f is one-to-one on the closed unit disc.

(c) If $t > 1$, there exists a function f satisfying P_t which is not one-to-one in the open unit disc.

Solution to problem E3325 AMM 96 (1989), p. 445

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a) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfy

$$(\star) \qquad \sum_{n=2}^{\infty} n|a_n| \le t$$

for some $t > 0$. Then $(\star)$ implies that the power series for f' and that for f converge uniformly (and absolutely) on $\overline{\mathbf{D}} = \{z : |z| \le 1\}$. Hence f and f' have continuous extensions to $\overline{\mathbf{D}}$.

b) If $t = 1$, then $(\star)$ implies that for $z \in \overline{\mathbf{D}}$ we have

$$(1) \qquad |f'(z) - 1| \le 1.$$

In particular, we have $\operatorname{Re} f'(z) \ge 0$.

Let $z, w \in \overline{\mathbf{D}}$ and let $\xi(t) = z + t(w - z)$, $0 \le t \le 1$. By the identity theorem for power series, relation (1) implies that f' and hence $\operatorname{Re} f'$ does not vanish identically on the segment $[z, w]$ unless $f(z) \equiv z$. Thus we have for $z, w \in \overline{\mathbf{D}}$, $z \ne w$

$$\begin{aligned} |f(w) - f(z)| &= \left|\int_0^1 f'(\xi(t))(w - z)\, dt\right| \\ &\ge |w - z| \operatorname{Re} \int_0^1 f'(\xi(t))\, dt \\ &= |w - z| \int_0^1 \operatorname{Re} f'(\xi(t))\, dt \ne 0. \end{aligned}$$

Hence f is injective on $\overline{\mathbf{D}}$.

c) Let $t > 1$. Then the functions $f(z) = z - \frac{t}{2} z^2$ satisfy $(\star)$. Looking at the parabola $x\left(1 - \frac{t}{2}x\right)$, we see that its maximum is attained at $x = \frac{1}{t} \in (0, 1)$. Thus f cannot be injective.

Remark. This problem is well known [see P. Duren, *Univalent Funktions*, Exercise 24, § 2, page 73]. Related to this problem is Exercise 12 in Rudin's book *Real and Complex Analysis*, 3rd Edition, § 14, page 294.

2. Mathematics Magazine

2223. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.*

Evaluate

$$\int_0^1 \frac{(1-x)(\ln x)^2}{x^3+1}\,dx.$$

Solution to problem 2223 Math. Mag. 98 (3) 2025, p. 234

Raymond Mortini and Rudolf Rupp

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2218. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain.*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+4} - \cdots\right) = \frac{11\pi^2 + 12(\ln 2)^2}{96}.$$

Solution to problem 2218 Math. Mag. 98 (2) 2025, p. 146

Raymond Mortini

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2216. *Proposed by Aristides V. Doumas, National Technical University of Athens, Athens, Greece.*

Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{(1-x)^{2n}\,((1+x)^n - 1)}{x}\, dx.$$

Solution to problem 2216 Math. Mag. 98 (2) 2025, p. 146

Raymond Mortini, Rudolf Rupp and Roberto Tauraso

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2212. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \operatorname{arcsinh}\left(\frac{1}{\sqrt{n^2+k^2}}\right).$$

Solution to problem 2212 Math. Mag. 98 (1) 2025, p. 67

Raymond Mortini, Rudolf Rupp

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2210. *Proposed by Souvik Dey, Charles University, Prague, Czech Republic.*

Characterize those functions $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$, which are continuous for all metrics d_1 and d_2 on $\mathbb{R}$.

Solution to problem 2210 Math. Mag. 97 (2) 2024, p. 575

Raymond Mortini

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This does not seem to be a serious problem proposal. Let $X := (\mathbb{R}, d_1)$, where d_1 is the usual Euclidean metric $d_1(x, y) = |x - y|$. Now take as d_2 the discrete metric defined by $d_2(x, y) = 1$ if $x \neq y$ and $d_2(x, x) = 0$ (the triangle inequality is trivially satisfied). Since continuous functions map connected sets to connected sets, and since the only non-empty connected sets in $Y = (\mathbb{R}, d_2)$ are the singletons, any continuous map $f : X \to Y$ must be constant (since $f(\mathbb{R})$ is connected, hence a singleton). Conversely, every constant function $f : M \to \tilde{M}$, $x \mapsto c$, between any metric spaces $M, \tilde{M}$, is continuous since $f^{-1}[V] = M$ for every open set V in $\tilde{M}$ with $c \in V$, and $f^{-1}[V] = \emptyset$ for any open set V in $\tilde{M}$ with $c \notin V$.

2202. *Proposed by Joseph Santmyer, Las Cruces, NM.*

Evaluate

$$\int_0^{2\pi} \cos(\cos(t))\cosh(\sin(t))dt \quad \text{and} \quad \int_0^{2\pi} \sin(\cos(t))\cosh(\sin(t))dt.$$

Solution to problem 2202 Math. Mag. 97 (2) 2024, p. 434

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Using complex analysis, we prove that

$$\boxed{I_1 := \int_0^{2\pi} \cos(\cos t)\ \cosh(\sin t)\ dt = 2\pi}$$

and

$$\boxed{I_2 = \int_0^{2\pi} \sin(\cos t)\ \cosh(\sin t)\ dt = 0}.$$

First note that $\cosh z = \cos(iz)$. Hence, by using that

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\ \cos\left(\frac{x-y}{2}\right),$$

and

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\ \cos\left(\frac{x-y}{2}\right),$$

we obtain

$$\begin{aligned}\cos(\cos t)\ \cosh(\sin t) &= \cos(\cos t)\ \cos(i\sin t)\\ &= \cos\left(\frac{e^{it}+e^{-it}}{2}\right)\ \cos\left(\frac{e^{it}-e^{-it}}{2}\right)\\ &= \frac{1}{2}\left(\cos(e^{it})+\cos(e^{-it})\right),\end{aligned}$$

and

$$\begin{aligned}\sin(\cos t)\ \cosh(\sin t) &= \sin(\cos t)\ \cos(i\sin t)\\ &= \sin\left(\frac{e^{it}+e^{-it}}{2}\right)\ \cos\left(\frac{e^{it}-e^{-it}}{2}\right)\\ &= \frac{1}{2}\left(\sin(e^{it})+\sin(e^{-it})\right).\end{aligned}$$

Now, by Cauchy's residue theorem in complex analysis,

$$\begin{aligned}I_1 &= \frac{1}{2}\int_0^{2\pi}\left(\cos(e^{it})+\cos(e^{-it})\right)\ dt \overset{z=e^{it}}{=} \frac{1}{2i}\oint \frac{\cos z+\cos(1/z)}{z}\ dz\\ &= \frac{1}{2i}2\pi i\left(\operatorname{Res}\left[\frac{\cos z}{z},0\right]+\operatorname{Res}\left[\frac{\cos(1/z)}{z},0\right]\right) = \pi(1+1) = 2\pi,\end{aligned}$$

and

$$\begin{aligned}I_2 &= \frac{1}{2}\int_0^{2\pi}\left(\sin(e^{it})+\sin(e^{-it})\right)\ dt \overset{z=e^{it}}{=} \frac{1}{2i}\oint \frac{\sin z+\sin(1/z)}{z}\ dz\\ &= \frac{1}{2i}2\pi i\left(\operatorname{Res}\left[\frac{\sin z}{z},0\right]+\operatorname{Res}\left[\frac{\sin(1/z)}{z},0\right]\right) = \pi(0+0) = 0.\end{aligned}$$

2191. *Proposed by John Chapman (student), Yvonne Cheng (student), and Gregory Dresden, Washington & Lee University, Lexington, VA.*

For an integer $n > 1$, denote the area between the curves $y = \cos x$ and $y = \cos nx$ over $[0, \pi]$ by A_n. Find $\lim_{n\to\infty} A_n$.

Solution to problem 2191 Math. Mag. 97 (2) 2024, p. 223

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We prove that this area A_n tends to $\boxed{8/\pi \sim 2.5464790\ldots}$.

To this end, we use the Fourier series for $f(x) := |\sin x|$

(39) $$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1},\ x \in \mathbb{R},$$

which can easily be obtained by noticing that that f is even and π-periodic, and so

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_k \cos(2kx)$$

with

$$a_k = \frac{2}{\pi}\int_0^\pi f(x)\cos(2kx)dx = \frac{2}{\pi}\int_0^\pi \sin x\cos(2kx)dx \ = \frac{1}{\pi}\int_0^\pi \big(\sin(1+2k)x + \sin(1-2k)x\big)dx.$$

Note that the desired area A_n is given by $\int_0^\pi |\cos x - \cos(nx)|dx$. By the addition theorem

$$|\cos x - \cos(nx)| = 2|\sin((n-1)(x/2))\ \sin((n+1)(x/2))|.$$

Hence

$$q(x) := 2|\sin(n-1)(x/2)\sin(n+1)(x/2)| \quad = \quad 2\left(\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos k(n-1)x}{4k^2-1}\right)\left(\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos k(n+1)x}{4k^2-1}\right).$$

Since for $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $m \neq n$,

$$\int_0^\pi \cos(mx)\cos(nx)dx = \frac{1}{2}\int_0^\pi \big(\cos(m-n)x + \cos(m+n)x\big)dx = 0,$$

we deduce from the fact that the Fourier series (39) is absolutely and uniformly convergent, that for $n \geq 2$

$$\begin{aligned} A_n = \int_0^\pi q(x)dx \quad &= \quad \frac{8}{\pi^2}\pi + \frac{32}{\pi^2}\sum_{j,k=1}^{\infty}\int_0^\pi \frac{\cos j(n-1)x\ \cos k(n+1)x}{(4k^2-1)(4j^2-1)}dx \\ &= \quad \frac{8}{\pi} + \frac{16\pi}{\pi^2}\sum_{\substack{j,k=1\\ j(n-1)=k(n+1)}}^{\infty}\frac{1}{(4k^2-1)(4j^2-1)}. \end{aligned}$$

Now $j(n-1) = k(n+1)$ if $j = r(n+1)m$ and $k = r(n-1)m$ for some $m \in \mathbb{N}$ and

$$r = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is odd} \end{cases}$$

In fact, if n is even, then the odd numbers $n-1$ and $n+1$ are relatively prime, since otherwise a joint divisor must also divide $2 = (n+1)-(n-1)$. Hence $k = m(n-1)$ and $j = m(n+1)$ for some $m \in \mathbb{N}$. And if n is odd, then $n-1$ and $n+1$ are even and 2 then is the gcd of $n-1$ and

$n+1$ since $(n+1)-(n-1)=2$. Now let $n \geq 3$. Due to $1 \leq 2(1/4)m^2(n\pm 1)^2 \leq 2r^2m^2(n+1)^2$, we finally conclude that

$$\begin{aligned}
\frac{8}{\pi} \leq A_n &= \frac{8}{\pi} + \frac{16\pi}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{(4r^2m^2(n-1)^2-1)(4r^2m^2(n+1)^2-1)} \\
&\leq \frac{8}{\pi} + \frac{16\pi}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{2r^2m^2(n+1)^2} = \frac{8}{\pi} + \frac{1}{(n+1)^2}\frac{16\pi}{\pi^2}\frac{1}{2r^2}\sum_{m=1}^{\infty}\frac{1}{m^2} \\
&\to \frac{8}{\pi}.
\end{aligned}$$

Remark. Using that for $a \notin \mathbb{Z}$, $\sum_{m=1}^{\infty} \frac{1}{m^2-a^2} = \frac{1-a\pi\cot(a\pi)}{2a^2}$, we obtain

$$\begin{aligned}
A_n &= \frac{8}{\pi} + \frac{16}{\pi}\frac{1}{16nr^2}\sum_{m=1}^{\infty}\left(\frac{1}{m^2-\frac{1}{4r^2(n-1)^2}} - \frac{1}{m^2-\frac{1}{4r^2(n+1)^2}}\right) \\
&= \frac{8}{\pi} + \frac{16}{\pi}\frac{1}{16nr^2}\left(\frac{1-\frac{\pi}{2r(n-1)}\cot(\frac{\pi}{2r(n-1)})}{\frac{2}{4r^2(n-1)^2}} - \frac{1-\frac{\pi}{2r(n+1)}\cot(\frac{\pi}{2r(n+1)})}{\frac{2}{4r^2(n+1)^2}}\right) \\
&= \frac{8}{\pi} + \frac{2}{\pi nr^2}\left(r^2(n-1)^2 - \frac{r(n-1)\pi}{2}\cot\frac{\pi}{2r(n-1)} - r^2(n+1)^2 + \frac{r(n+1)\pi}{2}\cot\frac{\pi}{2r(n+1)}\right) \\
&= -\frac{n-1}{nr}\cot\frac{\pi}{2r(n-1)} + \frac{n+1}{nr}\cot\frac{\pi}{2r(n+1)}.
\end{aligned}$$

For example,

$A_5 = \frac{4}{5}(3\sqrt{3}-2) \sim 2.5569219\ldots$

$A_4 = \frac{5}{4}\sqrt{5+2\sqrt{5}} - \frac{3\sqrt{3}}{4} \sim 2.54806631\ldots$

$A_3 = \frac{8}{3} \sim 2.6666666\ldots$

$A_2 = \frac{3\sqrt{3}}{2} \sim 2.59807621\ldots.$

It is now very easy to determine the limit directly (one may replace r above even by any $x \neq 0$): Let $a := \frac{\pi}{2x(n+1)}$ und $b := \frac{\pi}{2x(n-1)}$. Then

$$xA_n = (\cot a - \cot b) + \frac{1}{n}(\cot a + \cot b),$$

and so

$$xA_n = \frac{\sin(b-a)}{\sin a \sin b} + \frac{1}{n}\,\frac{\sin(a+b)}{\sin a \sin b}.$$

Note that

$$b-a = \frac{\pi}{x}\,\frac{1}{n^2-1} \text{ and } a+b = \frac{\pi}{x}\,\frac{n}{n^2-1}.$$

Since $\lim_{x\to 0}\frac{\sin x}{x} = 1$, xA_n has the same asymptotic as

$$\frac{\frac{\pi}{x}\,\frac{1}{n^2-1}}{\frac{\pi}{2x(n+1)}\,\frac{\pi}{2x(n-1)}} + \frac{1}{n}\,\frac{\frac{\pi}{x}\,\frac{n}{n^2-1}}{\frac{\pi}{2x(n+1)}\,\frac{\pi}{2x(n-1)}} = \frac{4x}{\pi} + \frac{4x}{\pi} = \frac{8x}{\pi}.$$

2193. *Proposed by Russell Gordon, Whitman College, Walla Walla, WA.*

Consider the following series:

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{8}}-\frac{1}{\sqrt{9}}-\frac{1}{\sqrt{10}}+\cdots$$

Prove that the sequence of partial sums is bounded and the series diverges.

Solution to problem 2193 Math. Mag. 97 (2) 2024, p. 223

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Let $\mathbb{N}=\{0,1,2,\dots\}$. We assume that the terms of the series $\sum_{j=1}^\infty a_j$ are regrouped as follows (this regrouping does not change the partial sums):

$$\frac{1}{\sqrt{1}}-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{6}}\right)-\left(\frac{1}{\sqrt{7}}+\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{10}}\right)+-\cdots=:\sum_{k=1}^\infty(-1)^{k-1}P_k,$$

where for $k\in\mathbb{N}$

$$P_k:=\sum_{j=A_k}^{A_{k+1}-1}\frac{1}{\sqrt{j}},$$

and where the number $A_k=\frac{k(k-1)}{2}+1$ is the j-index of the first summand in the k-th group. The sum P_k has k summands. We now estimate each P_k:

$$P_k\leq\sum_{j=A_k}^{A_{k+1}-1}\frac{1}{\sqrt{j}}\leq\frac{k}{\sqrt{A_k}}=\frac{k\sqrt{2}}{\sqrt{k(k-1)+2}}\leq 2\sqrt{2}. \tag{40}$$

$$P_k\geq\frac{k}{\sqrt{A_{k+1}-1}}=\frac{\sqrt{2}\,k}{\sqrt{k(k+1)}}\geq\frac{\sqrt{2}\,k}{k+1}\geq\frac{\sqrt{2}}{2}.$$

By Cauchy's criterion, the series diverges as the blocks P_k do not go to zero. Next we estimate the partial sums

$$\sum_{k=1}^N(-1)^{k-1}P_k.$$

To this end, we use the useful inequality

$$2(\sqrt{n+1}-\sqrt{n})\leq\frac{1}{\sqrt{n}}\leq 2(\sqrt{n}-\sqrt{n-1}).$$

This yields (via telescoping property)

$$P_k\leq 2(\sqrt{A_{k+1}-1}-\sqrt{A_k-1})=2\sqrt{2}\frac{\sqrt{k}}{\sqrt{k+1}+\sqrt{k-1}}=\sqrt{2}\left(1+\frac{1}{8}\,\frac{1}{k^2}\right)+\mathcal{O}(\frac{1}{k^2})$$

$$P_k\geq 2(\sqrt{A_{k+1}}-\sqrt{A_k})\geq 2\sqrt{2}\frac{k}{\sqrt{k(k+1)+2}+\sqrt{k(k-1)+2}}=\sqrt{2}\left(1-\frac{7}{8}\,\frac{1}{k^2}\right)+\mathcal{O}(\frac{1}{k^2}).$$

Numerical computations let us guess that the P_k are *increasing*. Anyway, the partial sums formed with full blocks write as

$$L_N:=\sum_{k=1}^N(-1)^{k-1}P_k=\begin{cases}(P_1-P_2)+(P_3-P_4)+\cdots+(P_{N-1}-P_N) & \text{if } N \text{ even}\\ (P_1-P_2)+(P_3-P_4)+\cdots+(P_{N-2}-P_{N-1})+P_N & \text{if } N \text{ odd,}\end{cases}$$

and the general one is given by

$$S_n := \sum_{j=1}^{n} a_j = \sum_{k=1}^{N} (-1)^{k-1} P_k + (-1)^N \sum_{j=A_{N+1}}^{n} \frac{1}{\sqrt{j}}, \tag{41}$$

where N is the unique number for which $A_{N+1} = \frac{N(N+1)}{2} + 1 \le n < \frac{(N+2)(N+1)}{2} = A_{N+2} - 1$. It remains to estimate the differences $|P_k - P_{k+1}|$.

$$\begin{aligned} P_k - P_{k+1} &\le \left(\sqrt{2} + \frac{\sqrt{2}}{8}\frac{1}{k^2} + o(\frac{1}{k^2})\right) - \left(\sqrt{2} - \frac{7\sqrt{2}}{8}\frac{1}{(k+1)^2} + o(\frac{1}{(k+1)^2})\right) \\ &\le \frac{\sqrt{2}}{k^2} + o(\frac{1}{k^2}) \end{aligned}$$

$$\begin{aligned} P_k - P_{k+1} &\ge \left(\sqrt{2} - \frac{7\sqrt{2}}{8}\frac{1}{k^2} + o(\frac{1}{k^2})\right) - \left(\sqrt{2} + \frac{\sqrt{2}}{8}\frac{1}{(k+1)^2} + o(\frac{1}{(k+1)^2})\right) \\ &\ge -\frac{\sqrt{2}}{k^2} + o(\frac{1}{k^2}). \end{aligned}$$

Hence, for all $k \ge k_0$,

$$|P_k - P_{k+1}| \le \frac{\sqrt{2}}{k^2} + o(\frac{1}{k^2}) \le \frac{\sqrt{2}+1}{k^2}$$

and so, for all k,

$$|P_k - P_{k+1}| \le Ck^2. \tag{42}$$

We deduce from (40) and (41) that for every n and N chosen as above

$$\begin{aligned} |S_n| &\le |L_N| + \sum_{j=A_{N+1}}^{n} \frac{1}{\sqrt{j}} \\ &\le \sum_{j=1}^{\lfloor N/2 \rfloor} |P_{2j-1} - P_{2j}| + \underbrace{P_N}_{\text{comes from the case } N \text{ odd}} + \sum_{j=A_{N+1}}^{n} \frac{1}{\sqrt{j}} \\ &\le \sum_{k=1}^{\infty} |P_k - P_{k+1}| + P_N + P_{N+1} \\ &\overset{(40)}{\underset{(42)}{\le}} C \sum_{k=1}^{\infty} \frac{1}{k^2} + 4\sqrt{2} =: \tilde{C}. \end{aligned}$$

Remark We also deduce that the associated parenthesized series converges:

$$\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{9}} - \frac{1}{\sqrt{10}}\right)$$
$$+ \left(\frac{1}{\sqrt{11}} + \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} - \frac{1}{\sqrt{16}} - \frac{1}{\sqrt{17}} - \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{19}} - \frac{1}{\sqrt{20}} - \frac{1}{\sqrt{21}}\right) + \cdots$$

2187. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.*

For $r > s \geq 0$, evaluate

$$\prod_{n=0}^{\infty}\left(1+\frac{\cosh 2^n s}{\cosh 2^n r}\right).$$

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We claim that for $r > s \geq 0$,

$$\boxed{P := \prod_{n=0}^{\infty}\left(1+\frac{\cosh(2^n s)}{\cosh(2^n r)}\right) = \frac{\sinh r}{\cosh r - \cosh s}}.$$

To this end we first show via induction that

$$\prod_{n=0}^{k}\cosh(2^n x) = \frac{\sinh(2^{k+1}x)}{2^{k+1}\sinh x}. \tag{43}$$

In fact, let $k = 0$. Then $\frac{\sinh 2x}{2\sinh x} = \cosh x$. If (43) is correct for some k, then

$$\begin{aligned}\prod_{n=0}^{k+1}\cosh(2^n x) &= \left(\prod_{n=0}^{k}\cosh(2^n x)\right)\cdot\cosh(2^{k+1}x) = \frac{\sinh(2^{k+1}x)}{2^{k+1}\sinh x}\cdot\cosh(2^{k+1}x)\\ &= \frac{\sinh(2^{k+2}x)}{2^{k+2}\sinh x}.\end{aligned}$$

A way to come up with such a formula, is to use the well-known funny formula

$$\prod_{n=0}^{k}(1+w^{2^n}) = \frac{1-w^{2^{k+1}}}{1-w}$$

for $w = e^{-x}$ and by writing $\cosh x = e^x(1+e^{-2x})/2$.

Now

$$1+\frac{\cosh u}{\cosh v} = \frac{\cosh u + \cosh v}{\cosh v} = 2\,\frac{\cosh(\frac{u+v}{2})\cosh(\frac{u-v}{2})}{\cosh v}.$$

Hence, with $u = 2^n s$ and $v = 2^n r$,

$$\begin{aligned}P_k := \prod_{n=0}^{k}\left(1+\frac{\cosh 2^n s}{\cosh 2^n r}\right) &= 2^{k+1}\prod_{n=0}^{k}\frac{\cosh(2^{n-1}(r+s))\cosh(2^{n-1}(r-s))}{\cosh 2^n r}\\ &= 2^{k+1}\frac{\cosh(\frac{r+s}{2})\,\cosh(\frac{r-s}{2})\,\prod_{j=0}^{k-1}\cosh 2^j(r+s)\prod_{j=0}^{k-1}\cosh 2^j(r-s)}{\prod_{n=0}^{k}\cosh 2^n r}\\ &= 2^{k+1}\cosh\left(\frac{r+s}{2}\right)\,\cosh\left(\frac{r-s}{2}\right)\,\frac{\sinh 2^k(r+s)}{2^k\sinh(r+s)}\,\frac{\sinh 2^k(r-s)}{2^k\sinh(r-s)}\,\frac{2^{k+1}\sinh r}{\sinh 2^{k+1}r}\\ &= 4\cosh\left(\frac{r+s}{2}\right)\,\cosh\left(\frac{r-s}{2}\right)\,\frac{\sinh r}{\sinh(r+s)\sinh(r-s)}\,\frac{\sinh 2^k(r+s)\sinh 2^k(r-s)}{\sinh 2^{k+1}r}\end{aligned}$$

Next we claim that for $r > s$,

$$\lim_{k\to\infty}\frac{\sinh 2^k(r+s)\sinh 2^k(r-s)}{\sinh 2^{k+1}r} = \frac{1}{2}.$$

In fact, using that $\sinh x = \frac{e^x}{2}(1 - e^{-2x})$ we obain

$$\begin{aligned} \frac{\sinh 2^k(r+s)\sinh 2^k(r-s)}{\sinh 2^{k+1}r} &= \frac{\frac{1}{4}e^{2^k(r+s)}e^{2^k(r-s)}(1-e^{-2^{k+1}(r+s)})(1-e^{-2^{k+1}(r-s)})}{\frac{e^{2^{k+1}r}}{2}(1-e^{2^{k+2}r})} \\ &= \frac{1}{2}\,\frac{(1-e^{-2^{k+1}(r+s)})(1-e^{-2^{k+1}(r-s)})}{(1-e^{2^{k+2}r})} \to \frac{1}{2}. \end{aligned}$$

Now note that $\cosh^2 r - \sinh^2 s = \sinh(r+s)\sinh(r-s)$, and so

$$\begin{aligned} 4\cosh\left(\frac{r+s}{2}\right)\cosh\left(\frac{r-s}{2}\right)\frac{\sinh r}{\sinh(r+s)\sinh(r-s)} &= 2\frac{\cosh r + \cosh s}{\sinh(r+s)\sinh(r-s)}\sinh r \\ &= \frac{2\sinh r}{\cosh r - \cosh s}. \end{aligned}$$

Thus

$$\lim_{k\to\infty} P_k = \frac{\sinh r}{\cosh r - \cosh s}.$$

2186. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Evaluate

$$\int_0^1 \frac{\operatorname{arctanh}\left(x\sqrt{2-x^2}\right)}{x}\,dx.$$

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A function "arctanh x" does not exist in the terminology we have learned (see R. Burckel, Classical Analysis in the plane, 2021, p. 135). It is either $\arctan x$ or $\operatorname{artanh} x$. Not knowing whether the letter c or the letter h in your statement is superfluous, we consider both cases. So we will prove the following:

$$(1)\quad \int_0^1 \frac{\operatorname{artanh}(x\sqrt{2-x^2})}{x}\,dx = \frac{3}{16}\pi^2 \sim 1.850550825204\cdots,$$

$$(2)\quad \int_0^1 \frac{\arctan(x\sqrt{2-x^2})}{x}\,dx = \frac{1}{2}C + \frac{\pi}{4}\log(\sqrt{2}+1) \sim 1.15021199360\cdots.$$

where C is the Catalan constant.

We need the following well-known integral:

Lemma 4. *Let $I_n := \int_0^{\pi/2}(\sin x)^n dx$. Then $I_0 = \pi/2$ and $I_1 = 1$. For $n \in \mathbb{N}^* := \{1,2,3\cdots\}$ we have*

$$I_{2n} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\frac{\pi}{2} = \frac{(2n)!}{4^n(n!)^2}\cdot\frac{\pi}{2} = \frac{\binom{2n}{n}}{4^n}\cdot\frac{\pi}{2}.$$

Proof. $I_{2n} = \frac{2n-1}{2n}I_{2n-2}$ for $n \in \mathbb{N}^*$ and $I_0 = \frac{\pi}{2}$, because

$$\begin{aligned} 2nI_{2n} - (2n-1)I_{2n-2} &= \int_0^{\pi/2} (\sin x)^{2n-2}\left(2n\sin^2 x - (2n-1)\right)dx \\ &= -\int_0^{\pi/2} (\sin x)^{2n-2}\left((2n-1)\cos^2 x - \sin^2 x\right)dx \\ &= -\left[(\sin x)^{2n-1}\cos x\right]_0^{\pi/2} = 0. \end{aligned}$$

□

Moreover, we will use that $\displaystyle\sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$ as well as $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2} = C$. Finally we need that for $|x| \le 1$

$$\arcsin x = \sum_{n=0}^\infty \frac{\binom{2n}{n}}{4^n}\frac{x^{2n+1}}{2n+1} \quad\text{and}\quad \operatorname{arsinh} x = \sum_{n=0}^\infty (-1)^n\frac{\binom{2n}{n}}{4^n}\frac{x^{2n+1}}{2n+1}.$$

Note that in view of Stirling's formula $\frac{\binom{2n}{n}}{4^n} \sim \frac{1}{\sqrt{\pi}\sqrt{n}}$, so the series converge absolutely for $x = \pm 1$.

(1) We make the substitution $x = \sqrt{2}\sin t$, $dx = \sqrt{2}\cos t\,dt$. Then, by using $\int\sum = \sum\int$ (all terms are positive),

$$\begin{aligned}
\int_0^1 \frac{\operatorname{artanh}(x\sqrt{2-x^2})}{x}\,dx &= \int_0^{\pi/4} \frac{\operatorname{artanh}(\sin(2t))}{\sin t}\,\cos t\,dt \\
&= \int_0^{\pi/4} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{(\sin(2t))^{2n+1}}{\sin t} \cos t\; dt \\
&\overset{\sin(2t)=2\sin t\cos t}{=} \int_0^{\pi/4} \sum_{n=0}^{\infty} \frac{1}{2n+1} (\sin(2t))^{2n} \underbrace{2\cos^2 t}_{=1+\cos(2t)}\; dt \\
&= \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\int_0^{\pi/4} (\sin(2t))^{2n} dt + \int_0^{\pi/4} (\sin(2t))^{2n} \cos(2t) dt \right) \\
&\overset{2t=u}{=} \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\int_0^{\pi/2} (\sin u)^{2n} du + \int_0^{\pi/2} (\sin u)^{2n} \cos u du \right) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{\binom{2n}{n}}{4^n} \cdot \frac{\pi}{2} + \frac{1}{2n+1} \right) \\
&= \frac{\pi}{4} \arcsin 1 + \frac{\pi^2}{16} = \frac{3}{16}\pi^2.
\end{aligned}$$

(2) In this case case, the factor $\frac{1}{2n+1}$ is replaced by $\frac{(-1)^n}{2n+1}$. Moreover, $\int\sum = \sum\int$, since

$$\left| \sum_{n=0}^{N} \frac{(-1)^n}{2n+1} (\sin u)^{2n} (1+\cos u) \right| \le \sum_{n=0}^{\infty} \frac{1}{2n+1} (\sin u)^{2n} (1+\cos u),$$

which is an integrable majorant by (1). Hence

$$\int_0^1 \frac{\arctan(x\sqrt{2-x^2})}{x}\,dx = \frac{\pi}{4}\operatorname{arsinh} 1 + \frac{1}{2}C.$$

Since $\operatorname{arsinh} x = \log(x + \sqrt{1+x^2})$, we are done.

2184. *Proposed by the Columbus State University Problem Solving Group, Columbus State University, Columbus, GA.*

Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve

$$y = \frac{x}{x^2 + 1}$$

in exactly one point. (Be careful!)

Solution to problem 2184 Math. Mag. 96 (5) 2023, p. 567

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We give two proofs (one geometric/intuitive informal one and one analytic one) of the following result:

Proposition *Let $f(x) = \frac{x}{1+x^2}$. Then the set of all those $(a, b) \in \mathbb{R}^2$ for which the line $y = ax + b$ cuts the graph $G := \{(x, f(x)) : x \in \mathbb{R}\}$ of f in exactly one point is given by the "exterior" E* [7] *of the closed Jordan curve (displayed in red below)*

$$\Gamma(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \frac{1-t^2}{(1+t^2)^2} \\ \frac{2t^3}{(1+t^2)^2} \end{pmatrix}, \text{ for } t \in \mathbb{R}, \quad \text{and} \quad \Gamma(\pm\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{44}$$

together with $\{(0,0)\} \cup \{(1,0)\} \cup \{(0, \pm\frac{1}{2})\} \cup \{(-\frac{1}{8}, \pm\frac{3\sqrt{3}}{8})\}$ *and deleted by the half lines*

$\{0\}\times\,]1/2, \infty[$ *and* $\{0\}\times] - \infty, -1/2[$.

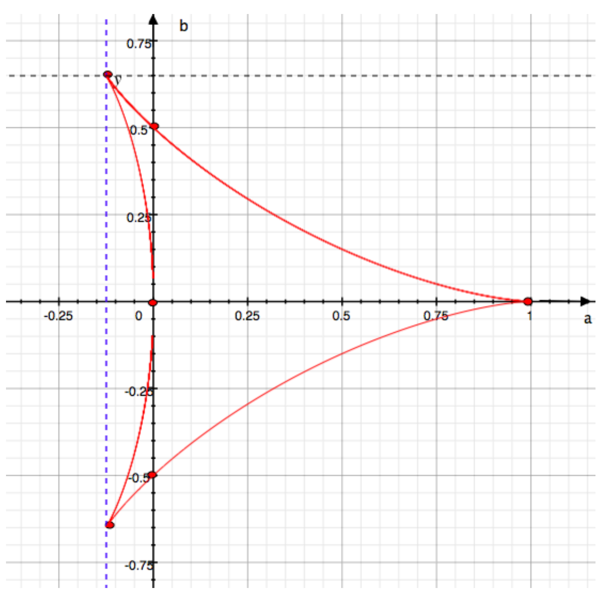

FIGURE 6. The red curve

[7] This is the unbounded component of the complement of the curve.

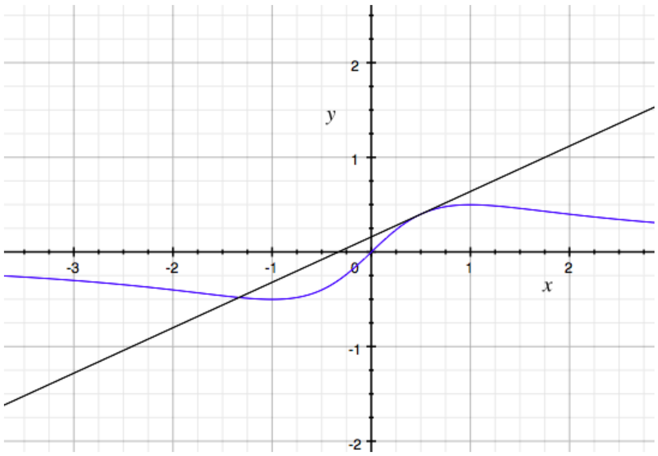

FIGURE 7. Graph of f and one tangent

Proof. We first discuss the geometry of the graph of f.

• Note that $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$. Hence the red curve Γ, excepted the point $(0,0)$, is the set of $(a,b) = (a(x), b(x))$ such that $s \mapsto as + b$ is a tangent to the graph of f at the point $(x, f(x))$ (since $a(x)x + b(x) = f(x)$ and $a(x) = f'(x)$).

Next, $f''(x) = \dfrac{2x(x^2-3)}{(1+x^2)^3}$. Since $f''(0) = f''(\pm\sqrt{3}) = 0$,

(45) $$\max f' = f(0) = 1 \text{ and } \min f' = f'(\pm\sqrt{3}) = -\tfrac{1}{8}.$$

Moreover $f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4} \sim \pm 0.433013\ldots$ and 0 and $\pm\sqrt{3}$ are inflection points for f and $\max f = f(1) = \frac{1}{2}$, respectively $\min f = f(-1) = -\frac{1}{2}$. If in (44) $t = \pm\sqrt{3}$, then $a(t) = -\frac{1}{8}$ and $b(t) = \pm\frac{3\sqrt{3}}{8} \sim \pm 0.64951\ldots$.

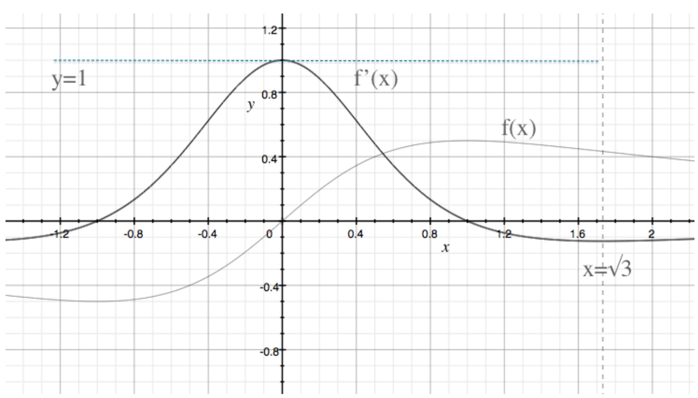

FIGURE 8. f and f'

• Observe that if $s \mapsto as + b$ cuts the graph of f in at least two different points, then $a \in [-1/8, 1]$. In fact, by the mean value theorem, if x_j are two intersection points, then

$$a = \frac{(ax_1 + b) - (ax_2 + b)}{x_1 - x_2} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(\eta).$$

Now (45) yields the assertion. Consequently, if $a \notin [-1/8, 1]$, then the line $s \mapsto as + b$ is either disjoint from the graph of f or cuts it in a single point.

• The only lines $s \mapsto as + b$ which do not intersect the graph of f are those that are parallel to the real axis (that is $a = 0$) and for which $|b| > 1/2$ (obviously clear by having a glimpse at the figure 8 of the graph G of f). In fact, any "oblique" line L (and any vertical line) has points in both domains determined by G and so the connectedness of the line implies that $L \cap G \neq \emptyset$. Moreover, if $a = 0$, then $b = 0 \cdot x + b = \frac{x}{1+x^2}$ is equivalent to $bx^2 - x + b = 0$. So no solution exists if and only if the discriminant $1 - 4b^2$ is negative; that is if $|b| > 1/2$.

• We will see below that the only tangents meeting the graph of f at a single point are the lines $y = \pm 1/2$ and $y = x$ and $y = \frac{-1}{8}x \pm \frac{3\sqrt{3}}{8}$ (those tangents associated with the extrema and the inflection points of f). All other tangents have another point of intersection: this is seen "geometrically" by looking at the graph and by considering the three cases (and of course the associated opposites) : $0 \leq x_0 \leq 1$, $1 < x_0 \leq \sqrt{3}$ and $x_0 > \sqrt{3}$, and by noticing that on the interior I_j of these three intervals the tangents are on one side of the graph $\{(x, f(x) : x \in I_j\}$, as we have no change of curvature (f is either convex or concave on I_j.) See figure 9.

• The behavior of the lines of the form $s \mapsto as + b$ with $a \in [-1/8, 1]$ and $b > b(x)$ or $b < b(x)$ can be intuitively guessed by looking at the graph of f (for a precise analytic proof, see next section). □

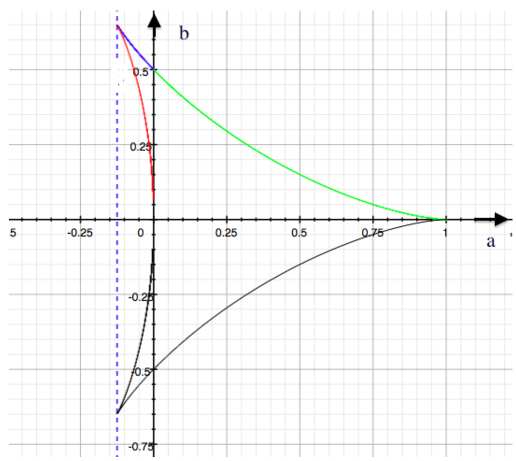

FIGURE 9. The $a(x)$ and $b(x)$'s for $x \in I_j$

2.1. **An analytic proof.** The intersection condition is equivalent to solving, for $a \neq 0$, the cubic polynomial equation

$$ax + b = \frac{x}{1+x^2} \iff ax^3 + bx^2 + (a-1)x + b = 0$$

and for $a = 0$ the quadratic equation

$$b = \frac{x}{1+x^2} \iff bx^2 - x + b = 0.$$

Put

$$p(z) := p_{a,b}(z) := az^3 + bz^2 + (a-1)z + b.$$

Then several cases occur when discussing the equation $p_{a,b}(z) = 0$, $a \neq 0$:

i) one real solution and two complex ones (which are conjugated),
ii) three distinct real solutions,
iii) one double real solution and a second real solution,
iv) a triple real solution.

A way to deal with this, is to use the discriminant. For $\boxed{a \neq 0}$, let

$$D := a^4(z_1 - z_2)^2(z_2 - z_3)^2(z_1 - z_3)^2$$

be the discriminant of this cubic equation. Here z_1, z_2, z_3 are the zeros. Then,

$$D = -4a^4 - 8a^2b^2 - 4b^4 + 12a^3 - 20ab^2 - 12a^2 + b^2 + 4a.$$

A lenghthier calculation (a posteriori verified by Maple and wolframalpha) gives

$$D = D(a,b) = -4\left[\left(b^2 + a^2 + \frac{5}{2}a - \frac{1}{8}\right)^2 - 8\left(a + \frac{1}{8}\right)^3\right].$$

It is well-known that the cubic equation has a multiple zero if and only if the discriminant is zero. In other words, if and only if

$$\left(b^2 + a^2 + \frac{5}{2}a - \frac{1}{8}\right)^2 = 8\left(a + \frac{1}{8}\right)^3.$$

Also, $D > 0$ if and only if the cubic equation (with real coefficients) has three distinct real zeros, and $D < 0$ if and only if there is a unique real zero. In our situation here, $D < 0$ if and only if

$$-4\left[\left(b^2 + a^2 + \frac{5}{2}a - \frac{1}{8}\right)^2 - 8\left(a + \frac{1}{8}\right)^3\right] < 0,$$

equivalently

$$\left(b^2 + a^2 + \frac{5}{2}a - \frac{1}{8}\right)^2 > 8\left(a + \frac{1}{8}\right)^3.$$

Now we have the following result:

Lemma 5. *Let $(a,b) \in \mathbb{R}^2$. The following assertions are equivalent:*

(1) *$D(a,b) = 0$ if $a \neq 0$ or $(a,b) = (0, \pm 1/2)$ if $a = 0$.*
(2) *$p_{a,b}(z) = az^3 + bz^2 + (a-1)z + b$ has a multiple zero.*
(3) *The line $L : s \mapsto as + b$ is tangent to the graph G of f at the point $(x, f(x))$ for some*[8] *$x \in \mathbb{R}$, and*

$$a = a(x) = \frac{1-x^2}{(1+x^2)^2} \text{ and } b = b(x) = \frac{2x^3}{(1+x^2)^2}.$$

Proof. (1) $\iff$ (2): Discussed above for the case $a \neq 0$. The case $a = 0$ follows since the discriminant of the quadratic $bz^2 - z + b$ is $1 - 4b^2$.

(2) $\Longrightarrow$ (3): Suppose that $x \in \mathbb{R}$ is a multiple zero of p. Recall that $p'(z) = 3az^2 + 2bz + (a-1)$. Then $p(x) = p'(x) = 0$ imply that $ax + b = \frac{x}{1+x^2}$ and

$$3ax^2 + 2x\left(\frac{x}{1+x^2} - ax\right) + (a-1) = 0.$$

Thus

$$a = \frac{1-x^2}{(1+x^2)^2}.$$

(In case $a = 0$, $x = \pm 1$). Consequently, $s \mapsto as + b$ is a tangent to the graph of f at x (since $ax + b = f(x)$ and $a = a(x) = f'(x)$). Moreover,

$$b = \frac{x}{1+x^2} - \frac{1-x^2}{(1+x^2)^2}x = \frac{2x^3}{(1+x^2)^2}.$$

(In case $a = 0$, $b = \pm 1/2$).

(3) $\Longrightarrow$ (1): Suppose that $s \mapsto as + b$ is a tangent at $(x, f(x))$ and that a and b have the form given in the assumption (3). If $a = a(x) \notin \{0, 1\}$, then x is (at least !) a double zero of $p_{a,b}$, since $p_{a,b}(x) = 0$ (equivalently $ax + b = f(x)$), and $p'_{a,b}(x) = 0$ because

$$3\frac{1-x^2}{(1+x^2)^2}x^2 + 2\frac{2x^3}{(1+x^2)^2}x + \frac{1-x^2}{(1+x^2)^2} - 1 \equiv 0.$$

Moreover, if $a = a(x) = 1$, then x is a triple zero of $p_{a,b}$ and $b = b(x) = 0$. Hence, as (2) $\Longrightarrow$ (1), $D(a(x), b(x)) = 0$. If $a = a(x) = 0$, then $x = \pm 1$ and $b = \pm 1/2$. Thus (1) holds. $\square$

[8] Later we shall see that x is uniquely determined; so a line L can be tangent to G at at most one point.

Conclusion: The set $(a,b)\in\mathbb{R}^2$ of points where $p_{a,b}$ has a multiple zero is in a one to one correspondance with those lines $s\mapsto as+b$ which are tangent to the graph of f. It coincides with

$$\{(a,b)\in\mathbb{R}^2\setminus(\{0\}\times\mathbb{R}) : D(a,b)=0\}\cup\Big\{(0,-\frac{1}{2}),(0,\frac{1}{2})\Big\},$$

and is the *Jordan arc* parametrized by

$$\Gamma(t)=\begin{pmatrix}a(t)\\ b(t)\end{pmatrix}=\begin{pmatrix}\frac{1-t^2}{(1+t^2)^2}\\ \frac{2t^3}{(1+t^2)^2}\end{pmatrix},\quad t\in\mathbb{R}.$$

To see that Γ is injective, suppose that there exists $(a,b)\in\mathbb{R}^2$ such that $a=a(t)=a(t')$ and $b=b(t)=b(t')$ for $t\neq t'$. By Lemma 5 (and its proof), the line $s\mapsto as+b$ is tangent to the graph of f at the points $(t,f(t))$ and $(t',f(t'))$, and so t and t' are (at least) double zeros of $p_{a,b}$. This would imply that the degree of $p_{a,b}$ is bigger than 4. A contradiction.

The two components determined by the closure $J:=\Gamma(\mathbb{R})\cup\{(0,0)\}$ of this Jordan arc (which is a closed Jordan curve) coincide with [9]

$$\tilde{D}(a,b)^{-1}(\,]0,\infty[\,)\text{ and }\tilde{D}(a,b)^{-1}(\,]-\infty,0[\,),$$

respectively, where [10]

$$\tilde{D}(a,b)=\begin{cases}D(a,b) & \text{if } a\neq 0\\ b^2(1-4b^2) & \text{if } a=0.\end{cases}$$

The following observations now will show that the exterior of this Jordan domain is the set where $\tilde{D}(a,b)<0$. Always have in mind figure 9. But attention: this is not yet the final set the problem is asking for.

Consider a tangent at x with $0<a(x)\leq 1$. Since $a(x)=f'(x)$ we deduce that $|x|<1<\sqrt{3}$. This implies that

$$b(x)^2+a(x)^2+\frac{5}{2}a(x)-\frac{1}{8}=\frac{1}{8}\,\frac{(3-x^2)^3}{(x^2+1)^3}>0.$$

Hence, if $b>b(x)$, we get

$$8\left(a(x)+\frac{1}{8}\right)^3=\left(b(x)^2+a(x)^2+\frac{5}{2}a(x)-\frac{1}{8}\right)^2<\left(b^2+a(x)^2+\frac{5}{2}a(x)-\frac{1}{8}\right)^2,$$

and so $D(a(x),b)<0$. This implies that there is a unique real zero of p and so the line $s\mapsto a(x)s+b$ cuts the graph of f at a single point.

Next, if $b=0$ and if $a\to-\infty$, then $D(a,b)\to-\infty$. So again $\tilde{D}<0$ in that part of the exterior of J that is contained in the left-hand plane.

Finally, if $a=0$, the discriminant $1-4b^2$ of bz^2-z+b is negative if and only if $p_{0,b}$ has no real zeros; so no intersection points of $ax+b$ exist whenever $a=0$ and $|b|>1/2$, but two if $0<|b|<1/2$ and one if $b=0$. Consequently, the exterior of the Jordan curve is the set where $\tilde{D}<0$.

To achieve the solution to the problem, a last case has to be investigated: for which (a,b) the polynomial $p_{a,b}$ has triple zero (as this yields tangents which cut the graph G of f at a single point).

So let $p_{a,b}(x)=p'_{a,b}(x)=p''_{a,b}(x)=0$. By Lemma 5, $s\mapsto as+b$ is tangent to the graph G of f. Hence $a=a(x)=\frac{1-x^2}{(1+x^2)^2}$ and $b=b(x)=\frac{2x}{(1+x^2)^2}$.

Now $p''_{a,b}(z)=6az+2b$. Hence

$$x=-\frac{b}{3a}=\frac{2x}{(1+x^2)^2}\Big/3\frac{1-x^2}{(1+x^2)^2}=-\frac{2}{3}\,\frac{x^3}{1-x^2}.$$

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[9] Take e.g. two points in the exterior complemented component of J, denoted by Ω. Join those with an arc inside Ω. Then $\tilde{D}$ must have the same sign at both points; otherwise this arc would meet the set where $\tilde{D}$ is zero. As this set coincides with the boundary of Ω, that is the Jordan curve J, we get a contradiction.

[10] In order to have continuity of $\tilde{D}$, we need to add the factor b^2.

Consequently, either $x = 0$ or $x = \pm\sqrt{3}$. This yields the values $(a, b) = (1, 0)$ and $(a, b) = (-\frac{1}{8}, \pm\frac{3\sqrt{3}}{8})$.

• We are now able to answer the question, which lines $s \mapsto as + b$ intersect the graph G of $f(x) = x/(1 + x^2)$ in a single point:

i) All points $(a, b) \in \mathbb{R}^2$ for which $a \neq 0$ and $D(a, b) < 0$.

ii) The 6 points $(a, b) \in \{(0, 0),\ (1, 0),\ (0, \pm\frac{1}{2}),\ (-\frac{1}{8}, \pm\frac{3\sqrt{3}}{8})\}$, which induce via the map $s \mapsto as + b$ tangents to G whenever $(a, b) \neq (0, 0)$.

1140. *Proposed by Raymond Mortini, Université du Luxembourg, Esch-sur-Alzette, Luxembourg and Rudolf Rupp, Technische Hochschule Nürnberg, Georg Simon Ohm, Nürnberg, Germany.*

Let m and n be nonnegative integers. Determine the value of

$$B(n,m) := \sum_{k=0}^{n} (-1)^k \binom{m+k}{k} \binom{m+n+1}{n-k}.$$

Quicky 1140 Math. Mag. 97 (2024) by
Raymond Mortini and Rudolf Rupp

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Submitted statement:

(a) Let $m,n \in \mathbb{N} = \{0,1,2,\dots\}$. Determine the value of

$$\boxed{B(n,m) := \sum_{k=0}^{n} (-1)^k \binom{m+k}{k} \binom{m+n+1}{n-k}.}$$

(b) Let $z,w \in \mathbb{C} \setminus \{-1,-2,-3\dots\}$. Suppose that $z-w \notin \{-1-2,-3\dots\}$. Using that for these parameters $\binom{z}{w} := \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$ is well defined, show that for $a,b \in \mathbb{C}$ with $\operatorname{Re} a > 0$, $\operatorname{Re} b > 1$, and $b \notin \mathbb{Z}$, the series

$$\boxed{S(a,b) := \sum_{k=0}^{\infty} (-1)^k \binom{a+k-1}{k} \binom{a+b-1}{b-k-1}.}$$

converges absolutely and that $S(a,b) = 1$.

Solution (a) Note that

$$\begin{aligned}\binom{m+k}{k}\binom{m+n+1}{n-k} &= \frac{(m+k)!}{m!k!} \frac{(m+n+1)!}{(m+k+1)!(n-k)!} \\ &= \frac{(m+n+1)!}{m!} \frac{1}{k!(m+k+1)(n-k)!} = \frac{(m+n+1)!}{m!n!}\binom{n}{k}\frac{1}{m+k+1}.\end{aligned}$$

Hence

$$\sum_{k=0}^{n} (-1)^k \binom{m+k}{k} \binom{m+n+1}{n-k} = \frac{(m+n+1)!}{m!n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{m+k+1}.$$

Put

$$f(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{m+k+1} \, x^{m+k+1}.$$

Then

$$f'(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{m+k} = x^m (1-x)^n.$$

Consequently, as $\int_0^1 f'(x)dx = f(1) - f(0)$ and $f(0) = 0$,

$$B(n,m) = \frac{(m+n+1)!}{m!n!} \int_0^1 x^m (1-x)^n \, dx.$$

The value of the integral is given by Euler's β function

$$\beta(m+1,n+1) = \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)} = \frac{m!\,n!}{(m+n+1)!}.$$

Hence

$$B(n,m) = \frac{(m+n+1)!}{m!n!}\ \frac{m!\,n!}{(m+n+1)!} = 1.$$

(b) First we note that under the conditions on a and b, the complex binomial coefficients are well defined. Since $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C} \setminus (-\mathbb{N})$, and since the Γ-function has no zeros, we have

$$\begin{aligned}\binom{a+k-1}{k}\binom{a+b-1}{b-k-1} &= \frac{\Gamma(a+k)}{\Gamma(a)\,\Gamma(k+1)}\ \frac{\Gamma(a+b)}{\Gamma(a+k+1)\Gamma(b-k)}\\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ \frac{\Gamma(b)}{(a+k)\Gamma(k+1)\Gamma(b-k)}\\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\binom{b-1}{k}\frac{1}{a+k}.\end{aligned}$$

Hence

$$S(a,b) = \sum_{k=0}^{\infty}(-1)^k\binom{a+k-1}{k}\binom{a+b-1}{b-k-1} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}\frac{1}{a+k}.$$

It is known that the binomial series $\displaystyle\sum_{k=0}^{\infty}\binom{b-1}{k}$ converges absolutely for Re $b > 1$ (see [42, p. 140]). Hence $S(a,b)$ converges. Now consider for $c \in \mathbb{C}$ and $0 < x \le 1$ the functions $x^c := \exp(c\log x)$, and

$$f(x) := \begin{cases} x^a\ \displaystyle\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}\frac{1}{a+k}x^k & \text{if } 0 < x \le 1\\ 0 & \text{if } x = 0,\end{cases}$$

which is continuous [11] on $[0,1]$. Using for $0 < x < 1$ the Newton-Abel formula for the binomial series with complex powers (see [42, p. 158]), we obtain

$$f'(x) = \sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}x^{a+k-1} = x^{a-1}\sum_{k=0}^{\infty}\binom{b-1}{k}(-x)^k = x^{a-1}(1-x)^{b-1}.$$

Consequently, as $\int_0^1 f'(x)dx = f(1) - f(0)$ and $f(0) = 0$,

$$S(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

This integral is the β-function. Note that $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$. Hence this integral is well defined and $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (see [43, p. 67 ff]. Consequently, $S(a,b) = 1$.

For this proposal, we were motivated by Problem 4862 Crux Math. 49 (7) 2023, 375. We hope that this sum has not been considered earlier.

[11] Note that Re $a > 0$ and so $0^a := 0$ is the correct value if one wants continuity: $|x^a| \le \exp(\operatorname{Re} a\ \log x) \to \exp(-\infty) = 0$ as $x \to 0^+$. Also, as usual in the realm of power series, $0^0 := 1$.

2185. *Proposed by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

Suppose n is a nonnegative integer. Let $P_n(x)$ be the nth degree polynomial defined by

$$P_n(x) = \frac{(-1)^n(1+x^2)^{n+1}}{n!}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Evaluate

$$\int_{-1}^{1} P_n(x)\,dx.$$

Solution to problem 2185 Math. Mag. 96 (5) 2023, p. 567

Raymond Mortini and Rudolf Rupp

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We show that the value of the integral $I_n := \int_{-1}^{1} P_n(x)dx$ is

$$\boxed{\frac{(-1)^n\, i^n}{n+2}\,\big(1+(-1)^n\big)\, 2^{\frac{n+2}{2}}\cos\left(n\frac{\pi}{4}\right).}$$

Another representation is

$$I_n = \varepsilon_n \frac{2^{\frac{n+4}{2}}}{n+2},$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv 0 \mod 8\\ 0 & \text{if } n \equiv 1 \mod 8\\ 0 & \text{if } n \equiv 2 \mod 8\\ 0 & \text{if } n \equiv 3 \mod 8\\ -1 & \text{if } n \equiv 4 \mod 8\\ 0 & \text{if } n \equiv 5 \mod 8\\ 0 & \text{if } n \equiv 6 \mod 8\\ 0 & \text{if } n \equiv 7 \mod 8. \end{cases}$$

A very strange result! In fact,

$$\frac{d^n}{dx^n}\left(\frac{1}{1+ix}\right) = \frac{i^n(-1)^n n!}{(1+ix)^{n+1}} \quad \text{and} \quad \frac{d^n}{dx^n}\left(\frac{1}{1-ix}\right) = \frac{i^n n!}{(1-ix)^{n+1}}.$$

Hence

$$\begin{aligned} \frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right) &= \frac{1}{2}\frac{d^n}{dx^n}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right)\\ &= \frac{i^n}{2}n!\,\frac{(1+ix)^{n+1} + (-1)^n(1-ix)^{n+1}}{(1+x^2)^{n+1}}. \end{aligned}$$

From this we get that P_n is a polynomial of degree n with $n+1$ as leading coefficient. We are now ready to calculate the integral:

$$\begin{aligned}
I_n = \int_{-1}^{1} P_n(x)dx &= \frac{(-1)^n}{2} i^n \int_{-1}^{1} \left((1+ix)^{n+1} + (-1)^n (1-ix)^{n+1}\right)\ dx \\
&= \frac{(-1)^n}{2} i^n \left(\frac{(1+i)^{n+2} - (1-i)^{n+2}}{i(n+2)} + (-1)^n \frac{(1-i)^{n+2} - (1+i)^{n+2}}{(-i)(n+2)} \right) \\
&= \frac{(-1)^n i^{n-1}}{2(n+2)}\ \left(1 + (-1)^n\right)\ \left((1+i)^{n+2} - (1-i)^{n+2}\right).
\end{aligned}$$

Since

$$\begin{aligned}
(1+i)^{n+2} - (1-i)^{n+2} &= \sqrt{2}^{\,n+1} \left(\left(\frac{1+i}{\sqrt{2}}\right)^{n+2} - \left(\frac{1-i}{\sqrt{2}}\right)^{n+2} \right) \\
&= 2^{\frac{n+1}{2}} \left(e^{(n+2)i\pi/4} - e^{-(n+2)i\pi/4}\right) \\
&= 2i\ 2^{\frac{n+1}{2}} \sin\left((n+2)\frac{\pi}{4}\right),
\end{aligned}$$

we conclude that

$$\begin{aligned}
I_n &= \frac{(-1)^n\, i^n}{n+2} \left(1 + (-1)^n\right) 2^{\frac{n+2}{2}} \sin\left((n+2)\frac{\pi}{4}\right) \\
&= \frac{(-1)^n\, i^n}{n+2} \left(1 + (-1)^n\right) 2^{\frac{n+2}{2}} \cos\left(n\frac{\pi}{4}\right).
\end{aligned}$$

2181. *Proposed by Raymond Mortini, Université de Lorraine (emeritus), Metz, France, Peter Pflug, Carl von Ossietzky Universität Oldenburg (emeritus), Oldenburg, Germany, and Rudolf Rupp, Technische Hochschule Nürnberg Georg Simon Ohm, Nürnberg, Germany.*

Evaluate

$$\lim_{x\to\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^{m+1}}{(2(m+1))!}\frac{1}{2m+2+k}x^{2m+2+k}.$$

Solution to problem 2181 Math. Mag. 96 (5) 2023, p. 566

Raymond Mortini, Peter Pflug and Rudolf Rupp

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a) The double series

$$S(x) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!}\frac{1}{[2(m+1)]!}\frac{1}{2m+2+k}x^{2m+2+k}$$

converges since for every $j\in\mathbb{N}$ the partial sums can be estimated as follows:

$$\begin{aligned}\sum_{n=0}^{j}\sum_{m=0}^{j}\frac{1}{k!}\frac{1}{[2(m+1)]!}\frac{1}{2m+2+k}x^{2m+2+k} \quad &\le \quad \sum_{n=0}^{j}\sum_{m=0}^{j}\frac{1}{k!}\frac{1}{[2(m+1)]!}x^k x^{2m+2}\\ &= \left(\sum_{n=0}^{j}\frac{1}{k!}\,x^k\right)\left(\sum_{m=0}^{j}\frac{1}{[2(m+1)]!}\,x^{2m+2}\right)\end{aligned}$$

Hence the series P converges absolutely (and so does any re-arrangement) locally uniformly to some finite value $P(x)$.

b) By the same reason the formal derivated series

$$H(x) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^{m+1}}{[2(m+1)]!}x^{2m+1+k}$$

converges absolutely and locally uniformly for $x\ge 0$. Hence $P' = H$. Thus

$$H(x) = \left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,x^k\right)\left(\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{[2(m+1)]!}\,x^{2m+1}\right) = e^{-x}\frac{\cos x - 1}{x}.$$

Consequently P is a primitive of $e^{-x}\frac{\cos x-1}{x}$ which vanishes at 0. Hence

$$J := \lim_{x\to\infty}P(x) = \int_0^\infty e^{-x}\frac{\cos x - 1}{x}\,dx.$$

Next we show that $J = -\frac{1}{2}\log 2$ by interpreting this integral as the Laplace transform $L(q)(s)$ of the function $q(x) = (\cos x - 1)/x$ evaluated at $s=1$. By a well-known formula, if $L(F(t))(s) = f(s)$, then

$$L(q)(s) = L(\frac{F(t)}{t})(s) = \int_s^\infty f(u)du,$$

where

$$f(s) = \int_0^\infty e^{-st}(\cos t - 1)\,dt = \frac{1}{s^3+s}.$$

Hence $L(q)(s) = -\frac{1}{2}\log(1+s^{-2})$ and so $J = L(q)(1) = -\frac{1}{2}\log 2$.

Remark A formal (but probably unjustifiable) way to calculate the value of J would be the following:

$$J := \int_0^\infty e^{-x}\frac{\cos x - 1}{x}dx = \int_0^\infty e^{-x}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n-1}dx$$

$$\stackrel{!}{=} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^{\infty} x^{2n-1} e^{-x} dx$$

Since for $m \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$,

$$\int_0^{\infty} x^m e^{-kx} dx = m!/k^{m+1},$$

we *would* obtain

$$J = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}(2n-1)! = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} = -\frac{1}{2}\log 2.$$

Note also that the softwares Wolframalpha/mathematica give the exact value of the integral, too.

The problem itself come to our mind when solving Problem number 12338 in Amer. Math. Soc..

2176. *Proposed by Elton Bojaxhiu, Eppstein am Taunus, Germany and Enkel Hysnelaj, Sydney, Australia.*

Show that

$$\int_0^1 \frac{\log\left(x^2+x+1\right)}{x^2+1}\,dx = \frac{1}{6}\pi\log\left(\sqrt{3}+2\right) - \frac{C}{3},$$

where $C = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots$ is the Catalan constant.

Solution to problem 2176 Math. Mag. 96 (3) 2023, p. 468

Raymond Mortini and Rudolf Rupp

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Let $I := \int_0^1 \frac{\log(1+x+x^2)}{1+x^2}\,dx$. We make the substitution $x = \tan u$, $dx/du = 1+\tan^2 u = \frac{1}{\cos^2 u}$, $x = 0 \to u = 0$ and $x = 1 \to u = \pi/4$. Then

$$\begin{aligned}
I &= \int_0^{\pi/4} \frac{\log(\tan^2 u + \tan u + 1)}{1+\tan^2 u}\,(1+\tan^2 u)du = \int_0^{\pi/4} \log\left(\frac{1+\cos u \sin u}{\cos^2 u}\right) du\\
&= \int_0^{\pi/4} \log\left(1+\frac{1}{2}\sin(2u)\right) du - 2\int_0^{\pi/4} \log(\cos u) du\\
&\underset{\text{Lem. 6}}{=} \frac{1}{2}\int_0^{\pi/2} \log\left(1+\frac{1}{2}\sin(v)\right) dv - 2\left(\frac{C}{2} - \frac{\pi}{4}\log 2\right)\\
&= \frac{1}{2}\int_0^{\pi/2} \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\frac{1}{2^k}(\sin x)^k dx + \frac{\pi}{2}\log 2 - C\\
&\underset{\substack{\text{unif.abs.conv.}\\ \int\sum=\sum\int}}{\overset{\text{Lem.4}}{=}} \frac{1}{2}\sum_{n=0}^\infty \frac{1}{2n+1}\frac{1}{2^{2n+1}}\frac{4^n}{(2n+1)\binom{2n}{n}} - \frac{1}{2}\sum_{n=1}^\infty \frac{1}{2n}\frac{1}{2^{2n}}\frac{\binom{2n}{n}}{4^n}\frac{\pi}{2} + \frac{\pi}{2}\log 2 - C\\
&= \frac{1}{4}\sum_{n=0}^\infty \frac{1}{(2n+1)^2\binom{2n}{n}} - \frac{\pi}{8}\sum_{n=1}^\infty \frac{1}{n}\binom{2n}{n}16^{-n} + \frac{\pi}{2}\log 2 - C\\
&\underset{\text{Lemm. 9}}{\overset{\text{Lemm. 8}}{=}} \frac{1}{4}\left(C - \frac{1}{8}\pi\log(2+\sqrt{3})\right)\frac{8}{3} - \frac{\pi}{8}\,2\log\left(\frac{1-\sqrt{1-4(1/16)}}{2(1/16)}\right) + \frac{\pi}{2}\log 2 - C\\
&= -\frac{1}{3}C + \frac{\pi}{6}\log(2+\sqrt{3}).
\end{aligned}$$

2.2. **Appendix.** Here we present for completeness the proofs of all those known results used above to derive the value of the integral.

Lemma 6. [44, formula (8)]

$$C = 2\int_0^{\pi/4} \log(2\cos x)dx.$$

Proof. Since on $]0,1]$ the integrable function $|\log x|$ dominates the modulus of the partial sums $\sum_{n=0}^N (-1)^n x^{2n}\log x$, we have

$$\int_0^1 \frac{\log x}{1+x^2}dx = \sum_{n=0}^\infty \int_0^1 (-1)^n x^{2n}\log x\,dx = \sum_{n=0}^\infty (-1)^n \frac{-1}{(n+1)^2} = -C.$$

Hence, with $x = \tan u$ and $dx = (1+\tan^2 u)du$,

$$C = -\int_0^{\pi/4} \log(\tan u)\,du = \int_0^{\pi/4} \log(\cos u)\,du - \int_0^{\pi/4} \log(\sin u)\,du =: L_c - L_s. \tag{46}$$

Now, using the standard result that $\int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2}\log 2$, we obtain

$$L_c + L_s + \frac{\pi}{4}\log 4 = \int_0^{\pi/4} \log(2\sin(2u))\, du = \int_0^{\pi/2} \log(2\sin x)\, dx = 0. \tag{47}$$

Adding (46) and (47), yields

$$C - \frac{\pi}{4}\log 4 = 2L_c.$$

In other words,

$$2\int_0^{\pi/4} \log(2\cos x) dx = C.$$

□

Lemma 7. *Let* $I_n := \int_0^{\pi/2} (\sin x)^n dx$. *Then* $I_0 = \pi/2$, $I_1 = 1$ *and for* $n \in \mathbb{N}^*$,

(1) $I_{2n} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\frac{\pi}{2} = \frac{(2n)!}{4^n(n!)^2}\cdot\frac{\pi}{2} = \frac{\binom{2n}{n}}{4^n}\cdot\frac{\pi}{2}$.

(2) $I_{2n+1} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1} = \frac{4^n(n!)^2}{(2n+1)!} = \frac{4^n}{(2n+1)\binom{2n}{n}}$.

Proof. (1) $I_{2n} = \frac{2n-1}{2n} I_{2n-2}$ for $n \in \mathbb{N}^*$ and $I_0 = \frac{\pi}{2}$, because

$$\begin{aligned} 2nI_{2n} - (2n-1)I_{2n-2} &= \int_0^{\pi/2} (\sin x)^{2n-2}\left(2n\sin^2 x - (2n-1)\right) dx \\ &= -\int_0^{\pi/2} (\sin x)^{2n-2}\left((2n-1)\cos^2 x - \sin^2 x\right) dx \\ &= -\left[(\sin x)^{2n-1}\cos x\right]_0^{\pi/2} = 0. \end{aligned}$$

(2) $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$ for $n \in \mathbb{N}^*$ and $I_1 = 1$, because

$$\begin{aligned} (2n+1)I_{2n+1} - 2nI_{2n-1} &= \int_0^{\pi/2} (\sin x)^{2n-1}\left((2n+1)\sin^2 x - 2n\right) dx \\ &= -\int_0^{\pi/2} (\sin x)^{2n-1}\left(2n\cos^2 x - \sin^2 x\right) dx \\ &= -\left[(\sin x)^{2n}\cos x\right]_0^{\pi/2} = 0. \end{aligned}$$

□

Lemma 8. [45]

(1) $\displaystyle\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$ *for* $|x| < 1/4$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n}{n} x^n = 2\log\left(\frac{1-\sqrt{1-4x}}{2x}\right)$ *for* $|x| < 1/4$.

Proof. (1) Note that

$$\begin{aligned} \binom{-1/2}{n} &= \frac{(-\frac{1}{2})\cdot(-\frac{3}{2})\cdot\cdots\cdot(-\frac{1}{2}-n+1)}{n!} = (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} \\ &= (-1)^n \frac{(2n)!}{(2^n n!)^2} = (-1)^n \frac{\binom{2n}{n}}{4^n}. \end{aligned}$$

Hence, by Newton's binomial theorem

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} (4x)^n = (1-4x)^{-1/2}.$$

(2) Let $f(x) := \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n}{n}x^n$. Then, for $0 < x < 1/4$,

$$f'(x) = \frac{1}{x}\sum_{n=1}^{\infty}\binom{2n}{n}x^n = \frac{1}{x\sqrt{1-4x}} - \frac{1}{x}.$$

To calculate the primitive, we make the transformation $u := \sqrt{1-4x}$, or equivalently $x = \frac{1-u^2}{4}$. Since

$$\int \frac{4}{1-u^2}\frac{1}{u}\left(-\frac{u}{2}\right)du = \log\left(\frac{1-u}{1+u}\right),$$

we deduce that

$$\begin{aligned} f(x) &= \log\left(\frac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}\right) - \log x = \log\left(\frac{(1-\sqrt{1-4x})^2}{4x}\right) - \log x \\ &= \log\left(\frac{(1-\sqrt{1-4x})^2}{(2x)^2}\right) = 2\log\left(\frac{1-\sqrt{1-4x}}{2x}\right). \end{aligned}$$

The following formula is due to Ramanujan.

Lemma 9. [44, formula (62)]

$$C = \frac{1}{8}\pi\log(2+\sqrt{3}) + \frac{3}{8}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}},$$

equivalently

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}} = \frac{8}{3}C + \frac{1}{3}\pi\log(2-\sqrt{3}).$$

Proof. First we note that

$$\operatorname{artanh} z = \frac{1}{2}\log\frac{1+z}{1-z} = \sum_{n=0}^{\infty}\frac{1}{2n+1}z^{2n+1}, \ \ |z|<1. \tag{48}$$

Hence, by Lemma 7

$$\begin{aligned} J := \int_0^{\pi/2}\log\left(\frac{1+\frac{1}{2}\sin x}{1-\frac{1}{2}\sin x}\right)dx &= \int_0^{\pi/2} 2\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}\frac{(\sin x)^{2n+1}}{2n+1}\,dx \underset{\int\sum=\sum\int}{\overset{\text{unif.abs.conv.}}{=}} \sum_{n=0}^{\infty}\frac{1}{2^{2n}}\frac{4^n}{(2n+1)^2\binom{2n}{n}} \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}}. \end{aligned}$$

To calculate this integral we combine calculations done in [46] and [48], where it is also shown that

$$J = \int_0^1 \frac{\operatorname{artanh}\sqrt{u(1-u)}}{\sqrt{u(1-u)}}du$$

(just put $u = \sin^2 x$). See [49], too. Let us introduce the parametric integral

$$I(a) := \int_0^{\pi/2}\log\left(\frac{1+\sin a\sin x}{1-\sin a\sin x}\right)dx.$$

Now $\frac{d}{da}\int = \int\frac{d}{da}$ (as all functions considered here are continuously differentiable). Hence

$$\begin{aligned} I'(a) &= \int_0^{\pi/2}\left(\frac{\cos a\sin x}{1+\sin a\sin x} + \frac{\cos a\sin x}{1-\sin a\sin x}\right)dx = \int_0^{\pi/2}\frac{2\cos a\sin x}{1-\sin^2 a\sin^2 x}\,dx \\ &= \frac{2\cos a}{\sin^2 a}\int_0^{\pi/2}\frac{\sin x}{\frac{1}{\sin^2 a}+\cos^2 x - 1}\,dx = \frac{2\cos a}{\sin^2 a}\int_0^{\pi/2}\frac{\sin x}{\cot^2 a + \cos^2 x}\,dx \\ &= -\frac{2}{\sin a}\arctan(\cos x\tan a)\Big|_0^{\pi/2} = \frac{2a}{\sin a}. \end{aligned}$$

Thus, using partial integration, and the fact that $\tan(x/2) = \frac{\sin x}{1+\cos x}$,

$$\begin{aligned} J &= I(\pi/6) = I(\pi/6) - I(0) = \int_0^{\pi/6} I'(a)da = \int_0^{\pi/6} \frac{2a}{\sin a}\, da \\ &= 2\int_0^{\pi/6} x\left(\log\left(\tan\frac{x}{2}\right)\right)'\, dx = 2x\log\left(\tan\frac{x}{2}\right)\Big|_0^{\pi/6} - 2\int_0^{\pi/6}\log\left(\tan\frac{x}{2}\right)\, dx \\ &\underset{\frac{x}{2}=t}{=} \frac{\pi}{3}\log(2-\sqrt{3}) - 4\int_0^{\pi/12}\log(\tan t)dt. \end{aligned}$$

Now we follow [48] [12]. Recall that on $]0,\pi[$ the Fourier series for $-\log\tan(t/2)$ is

$$2\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos(2n+1)t.$$

Since the Fourier series converges in the L^2-norm, hence L^1-norm on $]0,\pi[$, we have $\sum\int = \int\sum$. Hence

$$\begin{aligned} -\int_0^{\pi/12}\log(\tan x)dx &\underset{x=t/2}{=} \int_0^{\pi/6}\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos(2n+1)t\, dt = \sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\pi/6}\cos(2n+1)t\, dt \\ &= \underbrace{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\left(\frac{\pi}{6}(2n+1)\right)}_{:=S} \overset{!}{=} \frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}(-1)^n = \frac{2}{3}\, C. \end{aligned}$$

To show the penultimate identity, we follow [47]. To this end we first note that

$$\sin\left(\frac{\pi}{6}(2k+1)\right) = \begin{cases} \frac{1}{2} & \text{if } k\equiv 0 \mod 6 \\ 1 & \text{if } k\equiv 1 \mod 6 \\ \frac{1}{2} & \text{if } k\equiv 2 \mod 6 \\ -\frac{1}{2} & \text{if } k\equiv 3 \mod 6 \\ -1 & \text{if } k\equiv 4 \mod 6 \\ -\frac{1}{2} & \text{if } k\equiv 5 \mod 6. \end{cases}$$

Hence

$$\begin{aligned} S &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+1)^2} + \sum_{n=0}^{\infty}\frac{1}{(12n+3)^2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+5)^2} \\ &\quad -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+7)^2} - \sum_{n=0}^{\infty}\frac{1}{(12n+9)^2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+11)^2} \\ &= \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{1}{(4n+1)^2} - \frac{1}{(4n+3)^2}\right) + \frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{1}{11^2}+\cdots\right) \\ &= \frac{1}{9}\, C + \frac{1}{2}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots\right) + \frac{1}{2}\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}-\cdots\right) \\ &= \frac{1}{9}\, C + \frac{1}{2}C + \frac{1}{2}\cdot\frac{1}{3^2}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right) \\ &= \frac{1}{9}\, C + \frac{1}{2}C + \frac{1}{18}\, C = \frac{12}{18}\, C = \frac{2}{3}\, C. \end{aligned}$$

We conclude that

$$J = \frac{\pi}{3}\log(2-\sqrt{3}) + \frac{8}{3}\, C = \frac{8}{3}\, C - \frac{\pi}{3}\log(2+\sqrt{3}).$$

□

Here is a second proof to calculate the value of $\int_0^{\pi/12}\log(\tan t)dt$.

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[12] We thank Roberto Tauraso for providing us this link.

Proof. We follow [47]. Consider for $a > 0$ the integral

$$Q(a) = -\int_0^{\pi/12} \operatorname{artanh}\left(\frac{2\cos 2x}{a + a^{-1}}\right) dx = -\int_0^{\pi/12} \operatorname{artanh}\left(\frac{2a\cos 2x}{a^2+1}\right) dx.$$

(Note that $a + 1/a \geq 2$, so this is well defined). Again $\frac{d}{da}\int = \int \frac{d}{da}$. Using that $(\operatorname{artanh} z)' = \dfrac{1}{1-z^2}$, and that

$$\frac{d}{da}\left(\frac{1}{a+a^{-1}}\right) = \frac{1-a^2}{(a^2+1)^2},$$

$$\begin{aligned} Q'(a) &= -\int_0^{\pi/12} \frac{\frac{1-a^2}{(1+a^2)^2}\, 2\cos 2x}{1 - \frac{4\cos^2(2x)}{(a+a^{-1})^2}}\, dx = \int_0^{\pi/12} \frac{(a^2-1)\, 2\cos 2x}{(a^2+1)^2 - 4a^2\cos^2 2x}\, dx \\ &= -\frac{1}{2a} \arctan\left(\frac{2a\sin 2x}{1-a^2}\right)\Big|_0^{\pi/12} = \frac{\arctan \frac{a}{a^2-1}}{2a}. \end{aligned}$$

Hence, by using that $Q(0) = 0$, and that $\arctan u + \arctan v = \arctan(\frac{u+v}{1-uv})$,

$$\log(\tan x) = -\frac{1}{2}\log\left(\frac{1+\cos 2x}{1-\cos 2x}\right) = -\operatorname{artanh}(\cos 2x).$$

Consequently, as $C = -\int_0^1 \frac{\arctan x}{x} dx$ (use the power series for $\arctan x$)

$$\begin{aligned} \int_0^{\pi/12} \log(\tan x) dx &= Q(1) = \int_0^1 Q'(a) da = \int_0^1 \frac{\arctan \frac{a}{a^2-1}}{2a} da \\ &= -\int_0^1 \left(\frac{\arctan a}{2a} + \frac{\arctan a^3}{2a}\right) da \\ &\underset{a^3 \to b}{=\!=} -\left(\frac{1}{2} + \frac{1}{6}\right) \int_0^1 \frac{\arctan a}{a}\, da = -\frac{2}{3} C. \end{aligned}$$

We conclude that

$$J = \frac{\pi}{3}\log(2-\sqrt{3}) + \frac{8}{3}\, C = \frac{8}{3}\, C - \frac{\pi}{3}\log(2+\sqrt{3}).$$

□

2.3. **Remarks.** (1) The integral $L := \int_0^\infty \frac{\log(1+x+x^2)}{1+x^2}\, dx$ is mentioned on wikipedia [51] (without a source) under the form

$$C = \frac{3}{4}\, L - \frac{\pi}{4} \operatorname{arcosh} 2.$$

We notice that $L = 2I + 2C$. In fact,

$$\begin{aligned} I &\underset{u=1/x}{=} \int_1^\infty \frac{\log(u^2+u+1) - \log(u^2)}{1+u^2}\, du \\ &= \int_1^\infty \frac{\log(u^2+u+1)}{u^2+1} - 2C. \end{aligned}$$

Hence

$$2I = I + I = \int_0^\infty \frac{\log(u^2+u+1)}{u^2+1} - 2C.$$

Using the assertion of the problem dealt with here,

$$L = 2\left(\frac{\pi}{6}\log(\sqrt{3}+2) - \frac{C}{3}\right) + 2C,$$

and so

$$C = \frac{3}{4} L - \frac{\pi}{4}\log(\sqrt{3}+2).$$

Now just note that $\operatorname{arcosh} 2 = \log(\sqrt{3}+2)$. □

This integral L is also calculated in [50].

2171. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Evaluate the following sums in closed form.

(a) $\sum_{n=0}^{\infty}\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!}\right)$

(b) $\sum_{n=0}^{\infty}\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots (-1)^{n-1}\frac{x^{2n+1}}{(2n+1)!}\right)$

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For logical reasons, we think that the exponent of (-1) in the two statements above has to be $n+1$, since one starts with $-(-1)^n$, where $n=0$. Since the double series is absolutely convergent, we may arrange as we wish.

(a) Let

$$C(x) := \sum_{n=0}^{\infty}\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n+1}\frac{x^{2n}}{(2n)!}\right)$$

and let T_n be the $2n$-th Taylor polynomial for $\cos x$, which is given by

$$T_n(x) = \sum_{j=0}^{n}(-1)^j\frac{x^{2j}}{(2j)!}.$$

Then

$$C(x) = \sum_{n=0}^{\infty}(\cos x - T_n(x)).$$

Hence

$$\begin{aligned}
C(x) &= \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} = \sum_{k=1}^{\infty}\sum_{n=0}^{k-1}(-1)^k\frac{x^{2k}}{(2k)!} \\
&= \sum_{k=1}^{\infty}k(-1)^{k+1}\frac{x^{2k}}{(2k)!} = \frac{1}{2}\sum_{k=1}^{\infty}2k(-1)^k\frac{x^{2k}}{(2k)!} \\
&= \frac{x}{2}\sum_{k=1}^{\infty}(-1)\frac{x^{2k-1}}{(2k-1)!} \\
&= -\frac{1}{2}x\sin x.
\end{aligned}$$

For (b) we give two solutions.

Similarily to (a) , let

$$S(x) := \sum_{n=0}^{\infty}\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\right).$$

Then

$$\begin{aligned}
S(x) &= \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} = \sum_{k=1}^{\infty}\sum_{n=0}^{k-1}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \\
&= \sum_{k=1}^{\infty}k(-1)^k\frac{x^{2k+1}}{(2k+1)!} = \frac{1}{2}\sum_{k=1}^{\infty}2k(-1)^k\frac{x^{2k+1}}{(2k+1)!} \\
&= \frac{1}{2}\sum_{k=1}^{\infty}(2k+1)(-1)^k\frac{x^{2k+1}}{(2k+1)!} - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \\
&= \frac{x}{2}\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} - \frac{1}{2}(\sin x - x) \\
&= \frac{x}{2}(\cos x - 1) - \frac{1}{2}(\sin x - x) \\
&= \frac{x\cos x - \sin x}{2}
\end{aligned}$$

The second method is to integrate termwise and then to interchange the sum with the integral (uniform convergence on compacta).

$$\begin{aligned}
S(x) &= \sum_{n=0}^{\infty}\int_0^x\left(\cos t - 1 + \frac{t^2}{2!} - \frac{t^4}{4!} + \cdots + (-1)^{n+1}\frac{t^{2n}}{(2n)!}\right)\, dt \\
&= \int_0^x\sum_{n=0}^{\infty}\left(\cos t - 1 + \frac{t^2}{2!} - \frac{t^4}{4!} + \cdots + (-1)^{n+1}\frac{t^{2n}}{(2n)!}\right)\, dt \\
&\underset{(a)}{=} -\frac{1}{2}\int_0^x(t\sin t)\, dt \\
&= \frac{x\cos x - \sin x}{2}.
\end{aligned}$$

2167. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

Prove that

$$\lim_{n\to\infty} e^{n/2}\prod_{i=2}^{n} e^{i^2}\left(1-\frac{1}{i^2}\right)^{i^4} = \pi \exp\left(-\frac{5}{4}+\frac{3\zeta(3)}{\pi^2}\right),$$

where $\zeta(3)=\sum_{n=1}^{\infty} 1/n^3$.

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Let

$$L := e^{n/2}\prod_{j=2}^{n} e^{j^2}\ \prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)^{j^4}.$$

Then, by using that for $|x|<1$, $-\log(1-x)=\sum_{k=1}^{\infty}\frac{1}{k}x^k$,

$$\begin{aligned}
\log L &= \frac{n}{2}+\sum_{j=2}^{n} j^2+\sum_{j=2}^{n} j^4\log\left(1-\frac{1}{j^2}\right)\\
&= \frac{n}{2}+\sum_{j=2}^{n} j^2-\sum_{j=2}^{n}\sum_{k=1}^{\infty}\frac{j^4}{k}\frac{1}{j^{2k}}\\
&= \frac{n}{2}+\sum_{j=2}^{n} j^2-\sum_{j=2}^{n} j^2-\frac{1}{2}\sum_{j=2}^{n}1-\sum_{k=3}^{\infty}\frac{1}{k}\sum_{j=2}^{n}\frac{1}{(j^2)^{k-2}}\\
&\overset{m:=k-2}{\underset{n\to\infty}{\longrightarrow}} \frac{1}{2}-\sum_{m=1}^{\infty}\frac{\zeta(2m)-1}{m+2}.
\end{aligned}$$

So we need to show that

$$\boxed{\sum_{m=1}^{\infty}\frac{\zeta(2m)-1}{m+2}=\frac{7}{4}-\log\pi-\frac{3}{\pi^2}\zeta(3)\sim 0.239888629\cdots,} \tag{49}$$

from which we conclude that

$$L=\pi\ \exp\left(\frac{3}{\pi^2}\zeta(3)-\frac{5}{4}\right)\sim 1.2970745345\cdots$$

To achieve our goal we use the partial fraction decomposition of

$$\pi z\cot(\pi z)=1+\sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2},\ z\in\mathbb{C}\setminus\mathbb{Z},$$

a formula which implies (see [52, p. 182]) that

$$\sum_{n=1}^{\infty}(\zeta(2n)-1)x^{2n}=\frac{1}{2}\big(1-\pi x\cot(\pi x)\big)-\frac{x^2}{1-x^2}=-\sum_{n=2}^{\infty}\frac{x^2}{x^2-n^2}.$$

Since $z=0$ and $z=\pm1$ are removable singularities for $\frac{1}{2}\big(1-\pi z\cot(\pi z)\big)-\frac{z^2}{1-z^2}$, the holomorphy in $|z|<2$ implies that we have uniform convergence of $\sum_{n=1}^{\infty}(\zeta(2n)-1)x^{2n}$ on $[0,1]$. Also, for $x\in\,]0,1[$,

$$\sum_{n=1}^{\infty}(\zeta(2n)-1)x^{n+1}=\frac{x}{2}-\frac{\pi}{2}x^{3/2}\cot(\pi\sqrt{x})-\frac{x^2}{1-x}.$$

A primitive on $]0,1[$ is then given by

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)-1}{n+2} x^{n+2} = \frac{x^2}{4} - \int \left(\frac{\pi}{2} x^{3/2} \cot(\pi\sqrt{x}) + \frac{x^2}{1-x} \right) dx.$$

Using again uniform convergence on $[0,1]$, the substitution $s = \sqrt{x}$ and integration between 0 and 1 yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\zeta(2n)-1}{n+2} &= \frac{1}{4} - \int_0^1 \left(\frac{\pi}{2} s^3 \cot(\pi s) + \frac{s^4}{1-s^2} \right) 2s ds \\ &= \frac{1}{4} - \int_0^1 \left(\pi s^4 \cot(\pi s) + \frac{2s^5}{1-s^2} \right) ds\end{aligned}$$

Let

$$\boxed{I := \int_0^1 \left(\pi\, t^4 \cot(\pi t) + \frac{2t^5}{1-t^2} \right) dt.}$$

We claim that

(50) $$I = \frac{3}{\pi^2}\zeta(3) - \frac{3}{2} + \log\pi \sim 0.0101113705\cdots$$

To determine the value of I, we first calculate a primitive of

$$f(x) := \pi\, x^4 \cot(\pi x)$$

on $]0,1[$. This is done by using partial integration and the 1-periodic Fourier series

$$\sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k} = -\log(2\sin(\pi x)) = -\log 2 - \log(\sin(\pi x)),$$

where the convergence is considered in the L^2-norm on $]0,1[$, which also guarantees that $\int \sum = \sum \int$ below.

$$\begin{aligned}\int f(x)dx &= x^4 \log(\sin(\pi x)) - 4\int x^3 \log(\sin(\pi x))\, dx \\ &= x^4 \log(\sin(\pi x)) + 4\int \sum_{k=1}^{\infty} x^3 \frac{\cos(2k\pi x)}{k}\, dx + 4\int x^3 \log 2\, dx \\ &= x^4 \log(\sin(\pi x)) + 4\sum_{k=1}^{\infty} \int x^3 \frac{\cos(2k\pi x)}{k}\, dx + 4\int x^3 \log 2\, dx\end{aligned}$$

Before evaluating at the boundary points, we need to add

$$\frac{2x^5}{1-x^2} = -2x^3 - 2x - \frac{1}{1+x} + \frac{1}{1-x},$$

since the integral $\int_0^1 f(x)dx$ is divergent (at 1). Defining the symbol $[h(x)]_0^1$ below as

$$[h(x)]_0^1 := \lim_{x\to 1-} h(x) - \lim_{x\to 0} h(x),$$

we obtain

$$\begin{aligned}I = \int_0^1 \left(\pi\, x^4 \cot(\pi x) + \frac{2x^5}{1-x^2} \right) dx &= \left[x^4 \log(\sin(\pi x)) - \log(1-x) - \frac{x^4}{2} - x^2 - \log(1+x) \right]_0^1 \\ &\quad +4\sum_{k=1}^{\infty} \int_0^1 x^3 \frac{\cos(2k\pi x)}{k}\, dx + \log 2.\end{aligned}$$

Note that

$$x^4 \log(\sin(\pi x)) - \log(1-x) = x^4 \ \log\frac{\sin(\pi x)}{1-x} + (x^4-1)\log(1-x) \to \log\pi \ \text{ as } x \to 1.$$

Also, three times partial integration yields

$$\int_0^1 x^3 \frac{\cos(2k\pi x)}{k}\, dx = \frac{3}{4k^3\pi^2}.$$

Hence

$$\begin{aligned} I &= \log\pi - \frac{3}{2} - \log 2 + 4\sum_{k=1}^{\infty} \frac{3}{4k^3\pi^2} + \log 2 \\ &= \log\pi - \frac{3}{2} + \frac{3}{\pi^2}\zeta(3), \end{aligned}$$

yielding (50). We conclude that (49) is satisfied, that is

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)-1}{n+2} = \frac{1}{4} - I = \frac{7}{4} - \log\pi - \frac{3}{\pi^2}\zeta(3) \sim 0.23988862\cdots$$

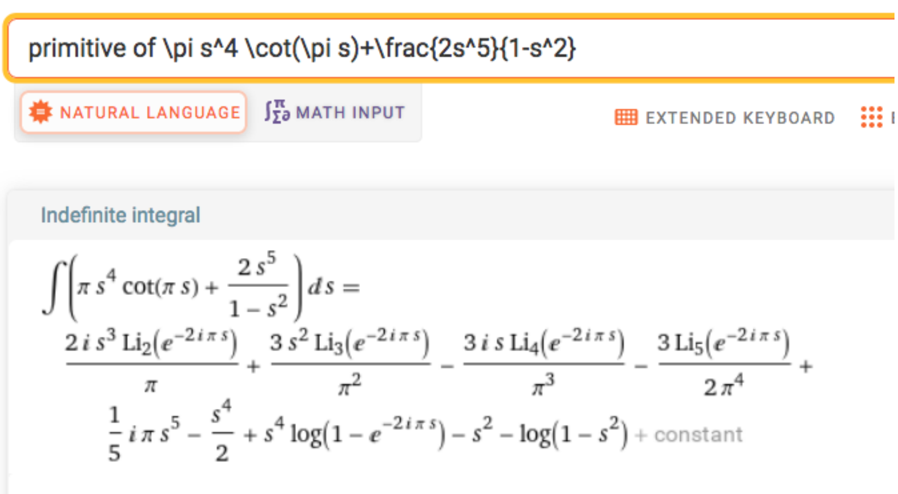
Remarks (1) The value for I is also given directly by Maple

$$int\left(\mathrm{Pi}\cdot s^4\cdot\cot(\mathrm{Pi}\cdot s) + \frac{2\cdot s^5}{1-s^2}, s = 0..1\right);$$

$$\frac{2\pi^2\ln(\pi) - 3\pi^2 + 6\zeta(3)}{2\pi^2}$$

(2) Using Wolframalpha's representation below of a primitive of $\pi\ t^4\cot(\pi t) + \frac{2t^5}{1-t^2}$ and evaluating at the boundary points, we also obtain the value of I. Just note that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$:

$$\begin{aligned} I &= 2i\frac{\zeta(2)}{\pi} + 3\frac{\zeta(3)}{\pi^2} - 3i\frac{\zeta(4)}{\pi^3} - 3\frac{\zeta(5)}{2\pi^4} + \frac{i\pi}{5} - \frac{1}{2} - 1 + \lim_{s\to 1}\left(s^4\log(1-e^{-2\pi i s}) - \log(1-s^2)\right) \\ &\quad +3\frac{\zeta(5)}{2\pi^4} - \lim_{s\to 0}\left(s^4\log(1-e^{-2\pi i s}) - \log(1-s^2)\right) \\ &= 3\frac{\zeta(3)}{\pi^2} + i\left(\frac{1}{3} - \frac{1}{30}\right)\pi + \frac{i\pi}{5} - \frac{3}{2} + \left(-i\frac{\pi}{2} + \log\pi\right) - (0) \\ &= 3\frac{\zeta(3)}{\pi^2} - \frac{3}{2} + \log\pi. \end{aligned}$$



(3) Generalizations of formula 49 are given in [7].

2147. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

Evaluate

$$\prod_{n=2}^{\infty} \frac{n^4+4}{n^4-1}.$$

Solution to problem 2147 Math. Mag. 95 (2) 2022, p. 242

Raymond Mortini and Rudolf Rupp

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We show that, in accordance with WolframAlpha,

$$P := \prod_{n=2}^{\infty} \frac{n^4+4}{n^4-1} = \frac{2\sinh\pi}{5\pi}.$$

Due to $\sin(iz) = i\sinh z$, we have

$$P(z) := \prod_{n=1}^{\infty}\left(1-\frac{z^4}{n^4}\right) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right) = \frac{\sin\pi z\ \sinh\pi z}{\pi^2 z^2}$$

and so

$$Q(z) := \prod_{n=2}^{\infty}\left(1-\frac{z^4}{n^4}\right) = \frac{P(z)}{1-z^4}$$

Note that

$$P = \prod_{n=2}^{\infty} \frac{1+\frac{4}{n^4}}{1-\frac{1}{n^4}}.$$

We put either $z=1$ or $z=1+i$. Note that $(1+i)^4 = -4$ and that

$$\lim_{z\to 1} Q(z) = \lim_{z\to 1}\frac{\sin\pi z}{1-z}\ \lim_{z\to 1}\frac{1}{(1+z)(1+z^2)}\ \frac{\sinh\pi}{\pi^2} = \frac{1}{4}\ \frac{\sinh\pi}{\pi}.$$

Hence

$$P = \frac{1}{1-(1+i)^4}\frac{\sin(\pi(1+i))\ \sinh(\pi(1+i))}{\pi^2(1+i)^2}\Bigg/\frac{1}{4}\ \frac{\sinh\pi}{\pi} = \frac{2}{5}\frac{\sinh\pi}{\pi}$$

2141. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.*

Evaluate

$$\mathrm{I}:=\int_0^\infty \ln\left(1+2x^{-2}\cos\varphi+x^{-4}\right)\,dx.$$

Solution to problem 2141 Math. Mag. 95 (2) 2022, p. 157

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The value of the integral I is $2\pi\cos(\varphi/2)$.

First we note that $(u+e^{i\varphi})(u+e^{-i\varphi})=1+2u\cos\varphi+u^2$.
Case 1 $\cos\varphi\neq 0$ (or equivantly $\varphi\notin\{\pi+2k\pi : k\in\mathbb{Z}\}$).
Let $\log z=\log|z|+i\arg z$ be the main branch of the complex logarithm (that is $-\pi<\arg z<\pi$). Put $H:=\mathbb{C}\setminus\,]-\infty,0]$. Note that for $z\in S:=\mathbb{C}\setminus\{it : |t|>1\}$ we have

$$\arctan z =: \frac{1}{2i}\log\frac{1+iz}{1-iz} \tag{51}$$

is a primitive of $1/(1+z^2)$.
Now for $x\in\mathbb{R}$, $1/x^2+e^{\pm i\varphi}\in H$ and so $f(x):=\log(1/x^2+e^{\pm i\varphi})$ is well defined. A primitive is given by

$$F(x)=x\log(1/x^2+e^{\pm i\varphi})-\int x\frac{d}{dx}\log(1/x^2+e^{\pm i\varphi})=x\log(1/x^2+e^{\pm i\varphi})+\int\frac{2}{1+x^2e^{\pm i\varphi}}dx.$$

Since $\varphi\neq\pm\pi$, $z:=xe^{\pm i\varphi/2}\in S$ and so, by using (51) and the fact that $\frac{1+iz}{1-iz}$ maps the right-half plane onto the upper-half-plane,

$$\int\frac{2}{1+x^2e^{i\varphi}}dx=2e^{-i\varphi/2}\arctan(e^{i\varphi/2}x)\underset{x\to\infty}{\longrightarrow}2e^{-i\varphi/2}\pi/2$$

Now $\arg z+\arg\overline{z}=0$ and so $\log z+\log\overline{z}=\log|z|^2$, $z\in H$. Hence, with $z=1/x^2+e^{i\varphi}\in H$,

$$x\log(1/x^2+e^{i\varphi})+x\log(1/x^2+e^{-i\varphi})=x\log(1+2x^{-2}\cos\varphi+x^{-4})\overset{x\to\infty}{\underset{x\to 0}{\longrightarrow}}0$$

Hence $\lim_{x\to\infty}I(x)=0+\pi(e^{-i\varphi/2}+e^{i\varphi/2})=2\pi\cos\varphi/2$.
Case 2 $\cos\varphi=0$. In other words $I=\int_0^\infty\log((1-1/x^2)^2)dx$, which is improper at 0 and 1. In this case $I=0$. In fact, for $x>0$ and $x\neq 1$,

$$\begin{aligned}h_1(x)\quad &:=\quad x\log\left(\frac{1}{x^2}-1\right)^2+\log\left(\frac{1+x}{1-x}\right)^2\\ &=\quad (x-1)\log(x-1)^2+2(x+1)\log(1+x)-4x\log x\end{aligned}$$

is a primitive of $\log((1-1/x^2)^2)$. Hence

$$\int_0^1\log((1-1/x^2)^2)dx=\log 16$$

and

$$\int_1^\infty\log((1-1/x^2)^2)dx=\log(1/16).$$

A related method (for $\cos\varphi \neq 0$) is to apply partial integration directly to $B(x) := \log(1 + 2x^{-2}\cos\varphi + x^{-4})$ and which gives

$$\int B(x)dx = xB(x) - \int xB'(x)dx$$

with

$$xB'(x) = 4\frac{1 + \cos\varphi\, x^2}{x^4 + 2\cos\varphi\, x^2 + 1} =: 4R(x).$$

This rational function writes as

$$R(x) = \frac{A}{x^2 + e^{i\varphi}} + \frac{B}{x^2 + e^{-i\varphi}}$$

with

$$A = \frac{e^{i\varphi}\cos\varphi - 1}{2i\sin\varphi},\ B = \overline{A}.$$

By using (51), we obtain

$$\int_0^\infty \frac{dx}{x^2 + b^2} = \lim_{x\to\infty} \frac{1}{b}\arctan(x/b) = \begin{cases} \frac{\pi}{2b} & \text{if Re } b > 0 \\ -\frac{\pi}{2b} & \text{if Re } b < 0. \end{cases}$$

Since $\lim\limits_{\substack{x\to 0\\ x\to\infty}} xB(x) = 0$, we deduce (with $b := e^{i\varphi/2}$) that $\int_0^\infty B(x)dx = 2\pi\cos(\varphi/2)$.

2128. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $0 < a < b < 1$ and $\epsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1-b^m)^n < \epsilon$ and $(1-a^m)^n > 1-\epsilon$.

Solution to problem 2128 Math. Mag. 94 (2021), p. 308

Raymond Mortini

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We first show that

$$\lim_{m\to\infty}(1-b^m)^{\frac{1}{ma^m}} = 0 \text{ and } \lim_{m\to\infty}(1-a^m)^{\frac{1}{ma^m}} = 1.$$

In fact, taking logartithms, this is equivalent to show that

$$L := \lim_{m\to\infty}\frac{\log(1-b^m)}{ma^m} = -\infty \quad \text{and } R := \lim_{m\to\infty}\frac{\log(1-a^m)}{ma^m} = 0.$$

Since $\sum_m ma^m$ converges for $|a|<1$, $ma^m \to 0$. Hence we have an indeterminate form $0/0$ and may use l'Hospital's rule. Using that for $x>1$, we have $\lim x^m/m = \infty$, we obtain

$$\begin{aligned} L &= \lim_{m\to\infty}\frac{\frac{1}{1-b^m}(-b^m)\log b}{a^m(1+m\log a)} = -\log b \lim_{m\to\infty}\left(\frac{b}{a}\right)^m \frac{1}{1+m\log a} \\ &= -\frac{\log b}{\log a}\lim_{m\to\infty}\left(\frac{b}{a}\right)^m\frac{1}{m} = -\infty. \end{aligned}$$

Moreover

$$R = \lim_{m\to\infty}\frac{\frac{1}{1-a^m}(-a^m)\log a}{a^m(1+m\log a)} = -\log a \lim_{m\to\infty}\frac{1}{m\log a} = 0.$$

Next we use that for $s_m := \frac{1}{ma^m}$ and $x_m := \log(1-b^m) \to 0$ the inequalities

$$(s_m - 1)x_m < \lfloor s_m \rfloor x_m \le s_m x_m$$

imply that the limits R and L do not change if we replace s_m by $\lfloor s_m \rfloor$.

Consequently,

$$\lim_{m\to\infty}(1-b^m)^{\lfloor\frac{1}{ma^m}\rfloor} = 0 \text{ and } \lim_{m\to\infty}(1-a^m)^{\lfloor\frac{1}{ma^m}\rfloor} = 1$$

Hence, for each $\varepsilon \in]0,1[$ we obtain $m, n \in \mathbb{N}$ such that

$$(1-b^m)^n < \varepsilon \text{ and } (1-a^m)^n > 1-\varepsilon.$$

2118. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

It is well known that the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

converges. Does the series

$$\sum_{k=1}^{\infty} e^{-\lfloor \ln k \rfloor} \sin k$$

converge or diverge?

Solution to problem 2118 Math. Mag. 94 (2021), p. 150

Raymond Mortini

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The series converges. This is an immediate consequence to Abel's theorem telling us that if (a_n) is a sequence of positive numbers with $a_n \searrow 0$, then the trigonometric series $S(t) := \sum_{n=0}^{\infty} a_n e^{int}$ converges for all $t \notin \{2k\pi : k \in \mathbb{Z}\}$ (see i.e. Appendix 4 in my encyclopedic monograph: R. Mortini, R. Rupp, Extension Problems and Stable Ranks, A Space Odyssey, Birkhäuser 2021, ca 2150 pages):

Just take $a_n = e^{-\lfloor \log n \rfloor}$, $t = 1$, and the imaginary part of $S(t)$. The proof is based on the Abel-Dirichlet rule, telling us that with $b_n = e^{int}$, and

$$|b_0 + b_1 + \cdots + b_m| = |1 + e^{it} + \cdots + e^{imt}| =$$
$$= \left| \frac{1 - e^{(m+1)it}}{1 - e^{it}} \right| \text{ if } e^{it} \neq 1.$$

we obtain for $t \notin 2\pi\mathbb{Z}$ that

$$|b_0 + b_1 + \cdots + b_m| \leq \frac{2}{|1 - e^{it}|} =: M. \tag{52}$$

Hence the series $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

2117. *Proposed by Ahmad Sabihi, Isfahan, Iran.*

Find all positive integer solutions to the equation

$$(m+1)^n = m! + 1.$$

Solution to problem 2117 in Math. Mag. 94 (2021), p. 150

Raymond Mortini, Rudolf Rupp and Amol Sasane

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There are only the three solutions $(n, m) \in \{(1, 1), (1, 2), (2, 4)\}$.
It is easy to check that these are solutions.
Now suppose that $n \geq m \geq 2$. Then (n, m) cannot be a solution since

$$(m+1)^n \geq (m+1)^m > m^m > m!, \text{ so } (m+1)^n > m! + 1.$$

Now, if $2 = n < m$, then

$$(m+1)^2 = m! + 1 \iff m + 2 = (m-1)!$$

which is obviously only satisfied for $m = 4$.

Next let $2 < n < m$. Then we see that if (n, m) is a solution to $(m+1)^n = m! + 1$, then m must be even. (Actually, by Wilson's theorem, $m+1$ divides $m! + 1$ if and only $m+1$ is prime; but we do not need this result). In particular, $m \geq 4$. Note that the equation $(m+1)^n - 1 = m!$ under discussion is equivalent to

$$\sum_{k=0}^{n-1} (m+1)^k = (m-1)!. \tag{53}$$

<u>1°</u> $m = 4$. Then, due to (53), $6 = 3! = 1 + 5 + \cdots$ implying that $n = 2$. A contradiction to the assumption $2 < n < m$..

<u>2°</u> $m \geq 6$. Then $2 < m/2 < m - 1$. Hence the integer $m/2$ divides $(m-1)!$. Since $m/2 > 2$, additionally the number 2 divides $(m-1)!$. Thus $m = 2 \cdot (m/2)$ divides $(m-1)!$.

Now, (53) yields $n \equiv 0, \mod m$. That is, m divides n and so $m \leq n$. This is again a contradiction to the assumption $2 < n < m$.

2116. *Proposed by Fook Sung Wong, Temasek Polytechnic, Singapore.*

Evaluate

$$\int_0^\infty \frac{e^{\cos x}\cos(\alpha x+\sin x)}{x^2+\beta^2}\,dx,$$

where α and β are positive real numbers.

Solution to problem 2116 Math. Mag. 94 (2021), 150

Raymond Mortini, Rudolf Rupp

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We show that

$$\boxed{\int_0^\infty \frac{e^{\cos(x)}\cos(\alpha x+\sin(x))}{x^2+\beta^2}dx=\frac{\pi}{2\beta}e^{e^{-\beta}-\alpha\beta}}.$$

We use the Residue theorem for the meromorphic function f, given by $f(z):=\frac{e^{e^{iz}+i\alpha z}}{z^2+\beta^2}$,

and the positively oriented contour $\Gamma_R:=S_R+H_R$, where S_R denotes the segment $[-R,R]$ and H_R the half-circle connecting $R, iR, -R$ for some $R>\beta>0$. Thus the simple pole $z_0:=i\beta$ is surrounded once in positive direction. The Residue theorem tells us that

(*) $$\oint_{\Gamma_R} f(z)dz=2\pi i\operatorname{Res}(f,z_0)=2\pi i\frac{e^{e^{-\beta}-\alpha\beta}}{2i\beta}=\frac{\pi}{\beta}e^{e^{-\beta}-\alpha\beta}.$$

From $f(x)=\frac{e^{e^{ix}+i\alpha x}}{x^2+\beta^2}=\frac{e^{\cos(x)+i\sin(x)+i\alpha x}}{x^2+\beta^2}=\frac{e^{\cos(x)}\left[\cos(\alpha x+\sin(x))+i\sin(\alpha x+\sin(x))\right]}{x^2+\beta^2}$, we find that $f(-x)=\overline{f(x)}$ for all $x\ge 0$, hence

$$\int_{-R}^{R} f(x)dx=\int_0^R (f(x)+\overline{f(x)})dx=2\int_0^R \frac{e^{\cos(x)}\cos(\alpha x+\sin(x))}{x^2+\beta^2}dx.$$

For $z\in H_R$, i.e. $z=\mathrm{Re}^{it}, 0\le t\le\pi$ we estimate

$$|f(z)|\le\frac{e^{\mathrm{Re}(e^{iR\cos(t)-R\sin(t)})+i\alpha R\cos(t)-\alpha R\sin(t))}}{R^2-\beta^2}=\frac{e^{e^{-R\sin(t)}}\cos(R\cos(t))-\alpha R\sin(t)}{R^2-\beta^2}\le\frac{e}{R^2-\beta^2}.$$

The standard estimate for integrals now shows that

$$\left|\int_{H_R} f(z)dz\right| \le \pi R \cdot \frac{e}{R^2-\beta^2} \to 0 \text{ for } R \to +\infty .$$

Passing to the limit in (*) then gives the value of the integral in question:

1075. *Proposed by Raymond Mortini, Université de Lorraine and IECL, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Assume that there exist $a, b \in \mathbb{R}$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in \mathbb{R}$. Is it true that, for every $d > 0$, there exists a horizontal segment of length d with endpoints on the graph of f?

Solution to Quicky 1075 in Math. Mag. 90 (2017), 384

Raymond Mortini

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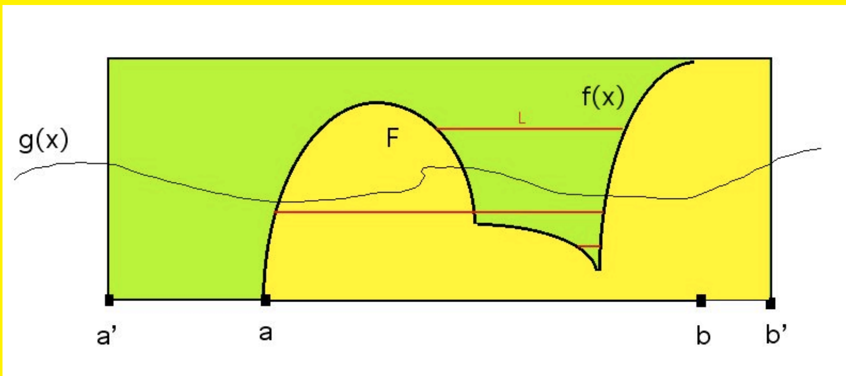

FIGURE 10. Intersecting curves

Yes. We have to show that for every $d \in \mathbb{R}$, there is $x_0 \in \mathbb{R}$ such that $f(x_0) = f(x_0 - d)$.

1) Non-elementary geometric approach.

Put $g(x) := f(x-d)$ and choose $a, b \in \mathbb{R}$ such that $m = f(a)$, $M = f(b)$ and $m < f(x) < M$ for $x \in]a, b[$. We may assume that $a < b$. Of course, $m \leq g(x) \leq M$. Let $a' < a$ and $b > b'$. Then $x \mapsto (x, g(x)), a' \leq x \leq b'$ is a curve in the rectangle $R := [a', b'] \times [m, M]$ starting at the left of the graph $F := \{(x, f(x)) : a \leq x \leq b\}$ of f and ending at the right (here we need that the Jordan arc F is a cross-cut of R). Thus this curve meets the graph: that is there is $a' \leq x_0 \leq b'$ such that $(x_0, g(x_0)) \in F$. Hence, there is $a \leq x_1 \leq b$ such that $(x_0, g(x_0)) = (x_1, f(x_1))$. Consequently, $x_0 = x_1$ and so $f(x_0) = f(x_0 - d)$.

2) Analytic approach. Let $H := f - g$. Then $H(a) = m - g(a) \leq 0$ and $H(b) = M - g(b) \geq 0$. If $g(a) = m$ or $g(b) = M$, then we are done. So we may assume that $H(a) < 0$ and $H(b) > 0$. Hence, by the intermediate value theorem, there is $x_0 \in]a, b[$ such that $H(x_0) = 0$. We conclude that $f(x_0) = g(x_0) = f(x_0 - d)$.

Let us point out that the assertion does not hold whenever merely $\inf_{\mathbb{R}} f$ and $\sup_{\mathbb{R}} f$ exist: just look at $f(x) = \arctan x$. Motivation for the problem came from the paper: Peter Horak, Partitioning $\mathbb{R}^n$ into connected components. Am. Math. Mon. 122, No. 3, 280-283 (2015), where periodic functions were considered.

1947. *Proposed by Raymond Mortini and Jérôme Noël, Université de Lorraine, Metz, France.*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n}{2}.$$

Solution to problem 1947 Math. Mag. 87 (2014), 230

Raymond Mortini, Jérôme Noël

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$$\sum_{k=0}^{n} |\cos k| \geq 1 + \sum_{k=1}^{n} (\cos k)^2 = 1 + \sum_{k=1}^{n} \frac{\cos(2k)+1}{2} = 1 + \frac{n}{2} + \frac{1}{2}\mathrm{Re}\left(\sum_{k=1}^{n} e^{2ik}\right).$$

Now

$$\sum_{k=1}^{n} e^{2ik} = e^{2i}\frac{1-e^{2in}}{1-e^{2i}} = e^{2i}\frac{e^{in}}{e^{i}}\frac{\sin n}{\sin 1} = e^{i(n+1)}\frac{\sin(n)}{\sin 1}.$$

Hence

$$\sum_{k=0}^{n} |\cos k| \geq 1 + \frac{n}{2} + \frac{\cos(n+1)\sin n}{2\sin 1} \geq 1 + \frac{n}{2} - \frac{1}{2\sin 1}$$
$$= \frac{n}{2} + \underbrace{\left(1 - \frac{1}{2\sin 1}\right)}_{>0} \geq \frac{n}{2},$$

because $2\sin 1 > 1$ (note that $\pi/4 < 1 < \pi/3$ implies $1 < \sqrt{2} < 2\sin 1 < \sqrt{3}$).

Let us remark that in the very first step it was important to begin the sum at $k = 1$ in order to have the summand 1. Otherwise we would have obtained

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n+1}{2} + \frac{\cos n \sin(n+1)}{2\sin 1} \geq \frac{n+1}{2} - \frac{1}{2\sin 1}$$
$$= \frac{n}{2} + \frac{1}{2}\left(1 - \frac{1}{\sin 1}\right),$$

an estimate that is less than $n/2$.

1871. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ.*

Let f, g be two differentiable real functions such that $g(x) \neq 0$ for all real numbers x. Suppose that c is a real number such that

$$f(c)\int_a^b g(x)\,dx \neq g(c)\int_a^b f(x)\,dx,$$

for all pairwise distinct real numbers a and b. Prove that $(f/g)'(c) = 0$.

Solution to problem 1871 Math. Mag. 84 (2011), p. 229

Raymond Mortini

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The solution is a based on the following Lemma:

Lemma 10. *Let F be a continuous, real-valued function on $\mathbb{R}\times\mathbb{R}$. Suppose that F is not zero outside the diagonal D and not constant 0 on D. Then either $F \geq 0$ or $F \leq 0$ everywhere.*

Proof. Let $P^+ = \{(x,y) \in \mathbb{R}^2; x < y\}$ and $P^- = \{(x,y) \in \mathbb{R}^2; x > y\}$.
Case 1: if $F(x_0,y_0) < 0$ and $F(x_1,y_1) > 0$ for some points $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$ in P^+, then F must have a zero on the segment S joining P_0 and P_1 in P^+ (since the image of S under F is an interval).

Case 2: if $F(P_0) < 0$ and $F(P_1) > 0$ for some $P_0 \in P^+$ and $P_1 \in P^-$, then we may choose an arc A (piecewise parallel to the axis) such that $F \neq 0$ on $A \cap D$, which is a singleton. By the the intermediate value theorem, there is a zero of F on the arc A, but outside D.

Case 3: if $F(Q_0) < 0$ and $F(Q_1) > 0$ for some $Q_0, Q_1 \in D$, then there are $P_0 \in P^+$ and $P_1 \in P^-$ such that $F(P_0) < 0$ and $F(P_1) > 0$. Hence we are in the second case.

Thus, all cases yield a contradiction to the assumption. Hence, in the image space, 0 is a global extremum. □

Solution to the problem Without loss of generality, we may assume that $g > 0$. Let

$$H(a,b) = \begin{cases} \dfrac{\int_a^b f(x)dx}{\int_a^b g(x)dx} & \text{if } a \neq b \\[2ex] \dfrac{f(a)}{g(a)} & \text{if } a = b. \end{cases}$$

We claim that H is continuous on $\mathbb{R}\times\mathbb{R}$. In fact, it suffices to prove continuity at the diagonal. So let $(a_0,a_0) \in D$. Then, for $(a,b) \in \mathbb{R}^2 \setminus D$, there is $\xi \in \,]a,b[$ such that $\int_a^b f(x)dx/(b-a) = f(\xi) \to f(a_0)$ if $(a,b) \to (a_0,a_0)$. Thus $\lim H(a,b) = H(a_0)$.

By assumption, $H(a,b) \neq f(c)/g(c)$ whenever (a,b) is outside the diagonal in $\mathbb{R}^2$.

Case 1: $H \equiv f(c)/g(c)$ on the diagonal D. Then the function $x \mapsto f(x)/g(x)$ has derivative 0 everywhere, and so satisfies the assertion of the problem.

Case 2: H not constant $f(c)/g(c)$ on D. Then, by Lemma 10 applied to $F = H - f(c)/g(c)$, we see that $H \geq f(c)/g(c)$ on $\mathbb{R}\times\mathbb{R}$ or $H \leq f(c)/g(c)$ on $\mathbb{R}\times\mathbb{R}$. In particular, c is an extrema of the function $x \mapsto f(x)/g(x)$ and so the differentiability of f/g implies that $(f/g)'(c) = 0$.

1867. *Proposed by Ángel Plaza and César Rodríguez, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(t)\,dt = 1$ and n a positive integer. Show that

1. there are distinct $c_1, c_2, \ldots, c_n$ in $(0,1)$ such that

$$f(c_1) + f(c_2) + \cdots + f(c_n) = n,$$

2. there are distinct $c_1, c_2, \ldots, c_n$ in $(0,1)$ such that

$$\frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \cdots + \frac{1}{f(c_n)} = n.$$

Solution to problem 1867 Math. Mag. 84 (2011), p. 150

Raymond Mortini

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If $f \equiv 1$, then the assertions are trivially true (just take any n points in $]0,1[$). If $f \not\equiv 1$, then there exist points at which f is strictly less than 1 and points where f is strictly bigger than one (note that this is the only occasion where we have used the hypothesis that $\int_0^1 f(t)dt = 1$). Hence, due to intermediate value theorem, there is at least one point at which f takes the value 1. In particular, if $h = f$ or $h = 1/f$, and noticing that the image of $[0,1]$ under f is an interval containing the point 1 in its interior, there exist $b \in [0,1]$ with $M := h(b) > 1$ and a sequence (a_i) with $h(a_i) < 1$ and $\lim h(a_i) = 1$. By compactness, we may assume that (a_i) is converging to some $a \in [0,1]$. Hence $h(a) = 1$ and

$$m(\delta) := \min\{h(x) : x \in [a-\delta, a+\delta] \cap [0,1]\} \to 1 \text{ if } \delta \to 0.$$

For later purposes, we note that $m(\delta) < 1$. Choose δ so small that

$$(n-1)(1 - m(\delta)) \le M - 1.$$

Then

$$n - M = (n-1) - (M-1) \le (n-1)m(\delta).$$

Now choose $n-1$ distinct points $x_1, \ldots, x_{n-1}$ in $[a-\delta, a+\delta] \cap]0,1[$ such that

$$m(\delta) < h(x_j) < 1.$$

Then $A := \sum_{j=1}^{n-1} h(x_j)$ satisfies

$$(n-1)m(\delta) \le A < n-1.$$

Thus $n - M \le A$ and so $1 < n - A \le M$. Again, by the intermediate value theorem, there is $x_n \in]0,1[$ such that $h(x_n) = n - A$. Hence

$$\sum_{j=1}^{n} h(x_j) = n.$$

Note that $x_n \notin \{x_1, \ldots, x_{n-1}\}$.

Alternate proof concerning the existence of the c_j

Let $F(x) = \int_0^x f(t)dt$ be the primitive of f vanishing at the origin. Let $x_j = j/n, j = 0, 1, \ldots, n$. Then, by the mean-value theorem of differential calculus, there exist $c_j \in]x_{j-1}, x_j[\subseteq]0,1[$ such that

$$1 = F(1) - F(0) = \sum_{j=1}^{n}(F(x_j) - F(x_{j-1})) = \sum_{j=1}^{n} F'(c_j)(x_j - x_{j-1}) = \frac{1}{n}\sum_{j=1}^{n} f(c_j).$$

1863. *Proposed by Duong Viet Thong, Department of Economics and Mathematics, National Economics University, Hanoi, Vietnam.*

Let f be a continuously differentiable function on $[a,b]$ such that $\int_a^b f(x)\,dx = 0$. Prove that

$$\left|\int_a^b xf(x)\,dx\right| \leq \frac{(b-a)^3}{12}\max\{|f'(x)| : x \in [a,b]\}.$$

Solution to problem 1863, Math. Mag. 84 (2011), 64.
Raymond Mortini

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We use Carathéodory's definition of differentiability: A function $f : I \to \mathbb{R}$ is differentiable at a point $x_0 \in I$, $I \subseteq \mathbb{R}$ an interval, if there exists a function $g = g_{x_0} : I \to \mathbb{R}$ continuous at x_0 such that

$$f(x) = f(x_0) + (x - x_0)g(x);$$

just define $g_{x_0}(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x \neq x_0 \end{cases}.$

Now if $f \in C^1[a,b]$, then g_{x_0} is continuous and, by Rolle's theorem, $g_{x_0}(x) = f'(\xi)$ for some $\xi \in]a,b[$, ξ depending on x_0 and x. Hence

$$\sup_{a\leq s\leq b}|g_{x_0}(s)| \leq \max_{a\leq t\leq b}|f'(t)| =: M.$$

Let $c = (a+b)/2$. Then, using the hypotheses that $\int_a^b f(x)dx = 0$ and the fact that $\int_a^b (x-c)dx = 0$ we obtain the following equalities:

$$J := \int_a^b xf(x)dx = \int_a^b (x-c)f(x)dx =$$

$$\int_a^b (x-c)(f(x)-f(c))dx = \int_a^b (x-c)^2 g_c(x)dx.$$

Thus

$$|J| \leq \int_a^b (x-c)^2 M dx = \frac{1}{3}\left[(x-c)^3\right]_a^b M =$$

$$\frac{2}{3}\left(\frac{b-a}{2}\right)^3 M = \frac{1}{12}(b-a)^3 M.$$

If $f(x) = x$ and $a = -1, b = 1$ then $\int_{-1}^1 f(x)dx = 0$ and $\int_{-1}^1 xf(x)dx = 1/3 = (b-a)^3/12$.

1860. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bârlad, Romania.*

Let α be a complex number such that $|\alpha| > 1$ and let n be an integer such that $n > 2$. Prove that at least $n-2$ roots of the equation $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ have norm equal to 1.

Solution to problem 1860, Math. Mag. 83 (2010), 392

Raymond Mortini

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We use the Schwarz-Pick Lemma telling us that holomorphic selfmaps of the unit disk are contractions with respect to the (pseudo)-hyperbolic metric ρ and that $\rho(f(z), f(w)) = \rho(z, w)$ for some pair $(z, w) \in \mathbb{D}^2$, $z \neq w$ implies that f is a conformal selfmap of $\mathbb{D}$ (hence of the form $e^{i\theta} \frac{b-z}{1-\overline{b}z}$) and so a (pseudo)-hyperbolic isometry.

Note that $z^n + \alpha z^{n-1} + \overline{a} z + 1 = 0$ for some $z \in \mathbb{D}$ if and only if $z^{n-1} = -\frac{\overline{a}z+1}{a+z}$. Now suppose that there are two solutions z, w in $\mathbb{D}$. Let $f(z) = -\frac{\overline{a}z+1}{a+z}$. Then

$$\rho(z, w) = \rho(f(z), f(w)) = \rho(z^{n-1}, w^{n-1}).$$

But this would imply that z^{n-1} is a bijection of $\mathbb{D}$ onto itself; a contradiction since $n \geq 3$.

Thus the equation $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ has at most one solution in $\mathbb{D}$. Since z is a solution if and only if $\frac{1}{\overline{z}}$ is a solution, we see that this polynomial of degree n must have at least $n-2$ solutions (multiplicities counting) on the unit circle.

Next we note that $u \in \mathbb{T}$ s a solution of modulus one of $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ if and only if u is a fixed point on $\mathbb{T}$ of the selfmap $\varphi(z) = f^{-1}(z^{n-1})$ of $\mathbb{D}$. Since the derivative of φ does not vanish at boundary fixed points, we conclude that there are at least $n-2$ *distinct* solutions of unit modulus.

3. College Math. J.

1310. *Proposed by Eugen J. Ionaşcu, Columbus State University, Columbus, GA.*

Prove, for each $n \in \mathbb{N}$, that

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{2k+1} > \frac{1}{2n+1}.$$

Solution to problem 1310 College Math. J. 56 (2025), 327

Raymond Mortini, Rudolf Rupp

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1309. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate the following expressions and prove your solutions are correct.

(a) $L = \lim_{n\to\infty} \int_0^1 \left(\frac{n}{n+\sqrt[n]{x}}\right)^n dx$

(b) $\lim_{n\to\infty} n \left(\int_0^1 \left(\frac{n}{n+\sqrt[n]{x}}\right)^n dx - L\right)$

Solution to problem 1309 College Math. J. 56 (2025), 327

Raymond Mortini, Rudolf Rupp

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1306. *Proposed by Raymond Mortini, Universit é du Luxembourg, Esch-sur-Alzette, Luxembourg and Rudolf Rupp, Technische Hochschule Nürnberg, Nürnberg, Germany.*

Consider for $x \in (0, 1]$ and for $n \in \mathbb{N} := \{0, 1, 2, \dots\}$ the equation

$$\frac{(1-x)^{n-1}}{x^{2n-1}} = 2. \qquad (*)$$

Prove the following:

(1) For each n there exists a unique $x_n \in (0, 1)$ solving equation $(*)$.
(2) Prove that $L := \lim_{n\to\infty} x_n$ exists.
(3) Determine L.

Solution to problem 1306 College Math. J. 56 (2025), 326

Raymond Mortini, Rudolf Rupp

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1302. *Proposed by Adán Medrano Martín del Campo, Radix Trading, Chicago, IL.*

Find all strictly decreasing functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy that for all integers x, y,

$$f(x)f(y) = -f(xy)$$
$$f(xf(y)) + yf(x) = xf(y) + f(yf(x)).$$

Solution to problem 1302 College Math. J. 56 (2025), 244

Raymond Mortini, Rudolf Rupp

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1301. *Proposed by Zach Chroman, Jane Street, New York, NY and Mihir Singhal, UC Berkeley, Berkeley, CA.*

Find all $m, n \in \mathbb{Z}$ for which

$$48mn = 16 + (m - n - 2)^3.$$

Solution to problem 1301 College Math. J. 56 (2025), 244

Raymond Mortini, Rudolf Rupp

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1298. *Proposed by Cezar Lupu, BIMSA, Tsinghua University, Bejing, China.*
Let $f : [0, 1] \to [0, \infty)$ be continuous. Show that

$$\int_0^1 f^5(x)\,dx \geq 6\left(\int_0^1 x^3 f^2(x) dx\right)\left(x^2 f^3(x)\,dx\right),$$

where $f^n(x) = (f(x))^n$ when $n \geq 1$.

Solution to problem 1298 College Math. J. 56 (2025), 159

Raymond Mortini

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1296. *Proposed by Joe Santmyer (unaffiliated), Las Cruces, NM.*

Let $n \geq 1$ be an integer, and let $\theta \neq \frac{(2m+1)\pi}{2n}$ for any integer m. Prove that

$$n \sec(n\theta) = \sum_{k=1}^{n} \frac{(-1)^{k+1} \sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos(\theta) - \cos\left(\frac{(2k-1)\pi}{2n}\right)}.$$

Solution to problem 1296 College Math. J. 56 (2025), 159

Raymond Mortini, Rudolf Rupp

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1295. *Proposed by Narendra Bhandari (unaffiliated), Bajura, Nepal.*

Prove that $\int_0^\infty \left(\frac{\coth(\pi x)}{x} - \frac{1}{\pi x^2}\right)^2 dx = \frac{4\zeta(3)}{\pi}$, where $\zeta(3) = \sum_{n=1}^\infty \frac{1}{n^3}$.

Solution to problem 1295 College Math. J. 56 (2025), 69

Raymond Mortini, Rudolf Rupp

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1294. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.*

Evaluate $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2k}{n^2}\right)^n$, with proof.

Solution to problem 1294 College Math. J. 56 (2025), 69

Raymond Mortini, Rudolf Rupp

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1293. *Proposed by Moubinool Omarjee, Lycée Henri IV High School, Paris, France.*

Let $f: [0, 1] \to [0, \infty)$ be a continuous function with $\int_0^1 x f(x)\, dx = 1$. Prove that $\int_0^1 f(x)\, dx + \int_0^1 \sqrt{f(x)}\, dx + \int_0^1 f^2(x)\sqrt{f(x)}\, dx \geq \frac{19}{5}$.

Solution to problem 1293 College Math. J. 56 (2025), 69

Raymond Mortini, Peter Pflug

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We think that the statement was not carefully thought out, as already the third term is bigger than 19/5: in fact, by Hölder's inequality

$$1 = \int_0^1 xf(x)dx \leq \left(\int_0^1 x^{5/3}dx\right)^{3/5} \left(\int_0^1 f(x)^{5/2}dx\right)^{2/5}$$

Thus

$$\int_0^1 f(x)^{5/2}dx \geq \frac{1}{\left((8/3)^{-3/5}\right)^{5/2}} = \left(\frac{8}{3}\right)^{3/2} \sim 4.3546\cdots > 19/5 = 3.80.$$

- -

Just for curiosity:

$$0 \leq \int_0^1 (\sqrt{f} - x)^2 \sqrt{f} dx = \int_0^1 (f\sqrt{f} + x^2\sqrt{f} - 2xf)dx \leq \int_0^1 (f\sqrt{f} + \sqrt{f})dx - 2$$

Hence

$$\int f + \int \sqrt{f} + \int f\sqrt{f} \geq 3.$$

Since

$$1 = \int xf dx \leq \left(\int x^3 dx\right)^{1/3} \left(\int f^{3/2}dx\right)^{2/3}$$

we also have that

$$\int f^{3/2}dx \geq \frac{1}{\left(4^{-1/3}\right)^{3/2}} = 2.$$

1283. *Proposed by Moubinool Ormajee, Paris, France (unaffiliated).*
Let $f : [0,1] \to \mathbb{R}$ be continuous, with $\int_0^1 x(x-1)^2 f(x)\,dx = 0$. Prove that there is a real number $c \in (0,1)$ such that $\int_0^c x^2 f(x)\,dx = c\int_0^c x f(x)\,dx$.

Solution to problem 1283 College Math. J. 55 (2024), 353

Raymond Mortini, Peter Pflug

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We shall use the mean value theorem (MVT) for integrals. Let

$$h(x) := x\int_0^x tf(t)dt - \int_0^x t^2 f(t)dt.$$

Then this $c \in]0,1[$ with $h(c) = 0$ exists once we can show that $I := \int_0^1 h(x)dx = 0$. This is true, though, in view of the assumption

$$\int_0^1 (1-t^2)tf(t) = 0.$$

In fact, using Fubini's theorem, and noticing that $0 \le t \le x \le 1$,

$$\begin{aligned}
\int_0^1 h(x)dx &= \int_0^1\int_0^x \big(xtf(t) - t^2 f(t)\big)\,dtdx = \int_0^1\int_t^1 \big(xtf(t) - t^2 f(t)\big)\,dxdt \\
&= \int_0^1\int_t^1 (x-t)tf(t)dxdt = \int_0^1 \left(\frac{1-t^2}{2} - (1-t)t\right) tf(t)\,dt \\
&= \frac{1}{2}\int_0^1 (1-t)^2 tf(t)\,dt = 0.
\end{aligned}$$

4. Elemente der Mathematik

Aufgabe 1462: Berechne den Wert des Integrals

$$I(a) = \int_0^{\pi} \frac{x}{1 + a\sin(x)}\,dx, \quad a > -1.$$

Michael Vowe, Therwil, CH

Solution to problem 1462 Elem. Math. 80 (2025), 132

Raymond Mortini, Rudolf Rupp

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Aufgabe 1456: Seien a, b, c Vektoren des $\mathbb{R}^3$ mit $a'b \neq -1$ sowie

$$a \times (b \times c) = a + b + c.$$

Dabei bezeichne $a' = (a_1, a_2, a_3)$ den zu a transponierten Vektor und „×" das Kreuzprodukt in $\mathbb{R}^3$. Man weise nach, dass c eine eindeutig bestimmte Linearkombination von a und b ist.

Götz Trenkler, Dortmund, D und Dietrich Trenkler, Osnabrück, D

Solution to problem 1456 Elem. Math. 80 (2025), 38

Raymond Mortini, Rudolf Rupp

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Aufgabe 1455: Beweise, dass

$$I = \int_0^1 \frac{\arctan(x)\log(x)}{1+x^2}\,dx = \frac{7}{16}\zeta(3) - \frac{\pi}{4}C,$$

wobei $C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ die Catalansche Konstante ist und leite daraus den Wert her des Integrals

$$J = \int_0^1 \frac{(\arctan(x))^2}{x}\,dx.$$

Raymond Mortini, Metz, F und Rudolf Rupp, Nürnberg, D

Solution to problem 1455 Elem. Math. 79 (2024), 176

Raymond Mortini, Rudolf Rupp

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Aufgabe 1453: Für reelle $x > 1$ und natürliche Zahlen n beweise man die Ungleichung

$$\left(1-\frac{1}{x}\right)^n \le 1-\frac{n}{x+n-1}.$$

Yagub Aliyev, Baku, AZ

Solution to problem 1453 Elem. Math. 79 (2024), 176

Raymond Mortini, Rudolf Rupp

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This is entirely trivial. First we have that

$$\left(1-\frac{1}{x}\right)^n \le 1-\frac{n}{x+n-1} = \frac{x-1}{x+n-1} = x\,\frac{1-\frac{1}{x}}{x+n-1} \quad\iff\quad \boxed{\left(1-\frac{1}{x}\right)^{n-1} \le \frac{x}{x+n-1}}.$$

Now the right inequality is shown by induction ($x > 1$):

$n = 0$: $\left(1-\frac{1}{x}\right)^{-1} = \frac{x}{x-1}$.
$n \to n+1$:

$$\begin{aligned}\left(1-\frac{1}{x}\right)^n &\overset{\text{ind.hyp}}{\le} \frac{x}{x+n-1}\left(1-\frac{1}{x}\right) = \frac{x}{x+n-1}\,\frac{x-1}{x} = \frac{x+n-1-n}{x+n-1}\\ &= 1-\frac{n}{x+n-1} \le 1-\frac{n}{x+n} = \frac{x}{x+n}.\end{aligned}$$

Aufgabe 1449: Für $n = 0, 1, 2, \dots$ zeige man

$$\sum_{k=1}^{2n+1} (-1)^{k-1} \binom{2n+1}{k} \binom{k-1}{2} 2^{k-3} = n^2.$$

Raymond Mortini, Metz, F

Solution to problem 1449 Elem. Math. 79 (2024), 131

Raymond Mortini

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Es sei $f(x) = \dfrac{(x+1)^{2n+1}}{x}$. Dann gilt

$$f''(x) = \frac{(x+1)^{2n-1}\Big(2n(2n-1)x^2 - (4n-2)x + 2\Big)}{x^3}$$

sowie

$$f''(-2) = 2n^2 - \frac{1}{4}$$

Nun berechnen wir die Laurentreihe und deren zweite Ableitung:

$$f(x) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} x^{k-1}.$$

$$\begin{aligned} f''(x) &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} (k-1)(k-2) x^{k-3} \\ &= 2 \sum_{k=0}^{2n+1} \binom{2n+1}{k} \binom{k-1}{2} x^{k-3} \end{aligned}$$

Auswertung an $x = -2$ ergibt:

$$\begin{aligned} f''(-2) &= 2 \sum_{k=0}^{2n+1} (-1)^{k-1} \binom{2n+1}{k} \binom{k-1}{2} 2^{k-3} \\ &= 2(-1)^1 \binom{2n+1}{0} \binom{-1}{2} 2^{-3} + 2A \\ &= -\frac{1}{4} + 2A. \end{aligned}$$

Folglich gilt $A = n^2$.

This was a very special case of problems treated in [53, p.8].

Aufgabe 1447: In memoriam Šefket Arslanagić.[1] Man bestimme alle Zahlen, die sowohl in der Folge $(20 + 24\sqrt{2})^n$, $n \geq 0$, als auch in der Folge $(24 + 20\sqrt{2})^n$, $n \geq 0$, vorkommen.

Walther Janous, Innsbruck, A

Solution to problem 1447 Elem. Math. 79 (2024), 84

Raymond Mortini, Rudolf Rupp

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Zu untersuchen ist für welche n, m die Gleichung $(20 + 24\sqrt{2})^n = (24 + 20\sqrt{2})^m$ gilt. Wir zeigen, dass nur die 1 als gemeinsame Zahl vorkommt (dies für $n = m = 0$). In der Tat, bezeichnet $N(a+b\sqrt{2}) := a^2 - 2b^2$ die sogenannte Norm des Elements $a+b\sqrt{2}$ im quadratischen Zahlkörper $\mathbb{Q}[\sqrt{2}]$, so erhalten wir wegen deren Multiplikativität $N(xy) = N(x)N(y)$ folgende Bedingung:

$$N(4^n(5 + 6\sqrt{2})^n) = N(4^m(6 + 5\sqrt{2})^m).$$

Äquivalent:

$$4^{2n}(25 - 72)^n = 4^{2m}(36 - 50)^m$$

also

$$16^n(-47)^n = 16^m(-14)^m.$$

Da 47 eine Primzahl ist, die rechte Seite aber nicht durch diese Zahl teilbar ist falls $n, m \geq 1$, bleibt nur $n = m = 0$ übrig.

Aufgabe 1443: Man zeige, dass der Wert des Integrals

$$I = \frac{1}{2}\int_1^{\sqrt{2}} \frac{\log(\frac{x+1}{x-1})}{\sqrt{2-x^2}}\,dx$$

mit der Catalanschen Konstante C übereinstimmt.

Raymond Mortini, Metz, F

Solution to problem 1443 Elem. Math. 79 (2024), 38

Raymond Mortini

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We use that (see [54, p. 453])

$$\int_0^{\pi/4} \log\left(\frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}\right) dx = 2C.$$

Now put $v := \sqrt{2}\cos x$, $1 \le v \le \sqrt{2}$,

$$dv = -\sqrt{2}\sin x\; dx = -\sqrt{2}\sqrt{1-\cos^2 x}\; dx = -\sqrt{2}(\sqrt{1-v^2/2}\; dx = -\sqrt{2-v^2}\; dx.$$

Hence

$$2C = \int_1^{\sqrt{2}} \frac{\log\left(\frac{v+1}{v-1}\right)}{\sqrt{2-v^2}}\; dv.$$

Aufgabe 1441: Sei c die Kurve gegeben durch die Parameterdarstellung

$$x(t) = t - \frac{A}{t}, \quad y(t) = t^2 + \frac{B}{t} \quad \text{für } t \in \mathbb{R} \setminus \{0\}.$$

Für welche ganzzahligen Werte von A und B besitzt c eine Selbstüberschneidung mit senkrechtem Schnittwinkel?

Gregory Dresden, Lexington VA, USA

Solution to problem 1441 Elem. Math. 78 (2023), 180

Raymond Mortini, Rudolf Rupp

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Diese Aufgabe ist unseren Ermessens nach nicht korrekt formuliert, da per Definition eine Kurve das stetige Bild eines *Intervals* in einen topologischen Raum ist. Deshalb liegen hier zwei Kurven vor und wir werden folgendes zeigen [13]:

4.1. Nichtnegative Parameter.

Proposition 11. *Es seien $a, b \in \mathbb{R}$, $a, b \geq 0$ und $\Gamma^\pm$ die Kurven, welche gegeben sind durch die Parameterdarstellung*

$$\Gamma^+ : z(t) = (x(t), y(t)) = \left(t - \frac{a}{t}, t^2 + \frac{b}{t}\right), t > 0,$$

beziehungsweise

$$\Gamma^- : z(t) = (x(t), y(t)) = \left(t - \frac{a}{t}, t^2 + \frac{b}{t}\right), t < 0.$$

Dann gilt:

1. $\Gamma^\pm$ *besitzen keine Selbstüberschneidungen; sind also Jordanbögen.*
2. *Für $a > 0$ schneidet Γ^+ die Kurve Γ^- in genau einem Punkt.*
3. *Für $a = 0$ liegt der Graph einer Funktion auf $\mathbb{R} \setminus \{0\}$ vor, und folglich sind keine Selbstüberschneidungen vorhanden.*
4. Γ^+ *schneidet Γ^- für $a, b \in \mathbb{N} = \{0, 1, 2 \ldots\}$ mit senkrechten Schnittwinkel genau dann wenn $(a, b) = (2r^2 + 1, r(2r^2 + 1))$ für ein $r \in \mathbb{N}$.*

Lösung (1) (2) Zu betrachten ist das folgende System von Gleichungen für $s, t \in \mathbb{R} \setminus \{0\}$:

$$\begin{cases} t - a/t &= s - a/s \\ t^2 + b/t &= s^2 + b/s. \end{cases} \iff \begin{cases} t - s &= a\left(\frac{1}{t} - \frac{1}{s}\right) = a(s-t)\frac{1}{st} \\ t^2 - s^2 &= b\left(\frac{1}{s} - \frac{1}{t}\right) = b(t-s)\frac{1}{st}. \end{cases}$$

Für $s \neq t$ und $a \neq 0$ ist dies äquivalent zu

$$\begin{cases} st &= -a \\ st(t+s) &= b \end{cases} \iff \begin{cases} st &= -a \\ t+s &= -\frac{b}{a}. \end{cases}$$

Dies führt auf die Lösung der quadratischen Gleichung

$$0 = x^2 - (s+t)x + st = x^2 + \frac{b}{a}x - a.$$

Die Lösungen hierzu sind

$$s = \frac{-b + \sqrt{b^2 + 4a^3}}{2a} \text{ und } t = \frac{-b - \sqrt{b^2 + 4a^3}}{2a}. \tag{54}$$

Es liegen also ein negativer und ein positiver Wert der Kurven-Parameter s und t vor. Folglich schneidet Γ^- die Kurve Γ^+ in genau einem Punkt und es liegen keine Selbstüberschneidungen vor.

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[13] Die Schreibweise mit $-A$ und B, anstatt A und B, lässt uns vermuten, dass bloss nichtnegative Zahlen gemeint waren mit der Bezeichnung "ganzzahlig". Im zweiten Abschnitt betrachten wir auch den Fall wo a, b beliebig sind.

(3) ist klar.

(4) Es sei $M := (z(s), z(t))$ dieser eindeutige Schnittpunkt mit $s < 0$ und $t > 0$. Diese Kurven Γ^- und Γ^+ schneiden sich nun senkrecht in M genau dann wenn gilt

$$\begin{aligned} 0 &= \langle \dot{z}(s), \dot{z}(t) \rangle = \dot{x}(s)\,\dot{x}(t) + \dot{y}(s)\,\dot{y}(t) \\ \Longleftrightarrow & \left(1 + \frac{a}{s^2}\right)\left(1 + \frac{a}{t^2}\right) = -\left(2s - \frac{b}{s^2}\right)\left(2t - \frac{b}{t^2}\right). \end{aligned}$$

Einsetzen von $st = -a$ und $st(s+t) = b$ ergibt

$$-\left(1 - \frac{t}{s}\right)\left(1 - \frac{s}{t}\right) = \left(2s - \frac{t(t+s)}{s}\right)\left(2t - \frac{s(t+s)}{t}\right).$$

Multiplikation mit st liefert

$$\begin{aligned} -(s-t)(t-s) &= \big(2s^2 - t(t+s)\big)\big(2t^2 - s(t+s)\big) \\ &= (s-t)(t+2s)\,(t-s)(s+2t) \\ &= (s-t)(t-s)(t+2s)(s+2t). \end{aligned}$$

Da $s \neq t$, erhält man schliesslich

$$(t+2s)(s+2t) = -1. \tag{55}$$

Umformen ergibt

$$-1 = ts + 2s^2 + 2t^2 + 4st = 5st + 2\big((s+t)^2 - 2st\big) = st + 2(s+t)^2.$$

Durch Einsetzen von $st = -a$ und $s + t = -b/a$ ergibt das

$$-1 = -a + 2\left(\frac{b}{a}\right)^2 \Longleftrightarrow -a^2 = -a^3 + 2b^2 \Longleftrightarrow 2b^2 + a^2 - a^3 = 0. \tag{56}$$

Wir müssen nun alle Lösungspaare $(a, b) \in \mathbb{N}^2$ dieser diophantischen Gleichung bestimmen. Ein Umschreiben ergibt

$$2\frac{b^2}{a^2} - a = -1. \tag{57}$$

Es sei $r := \frac{b}{a}$. Gemäss der Voraussetzung ist $r \in \mathbb{Q}$. Also hat a die Form $a = 2r^2 + 1$ und $b = ra = r(1 + 2r^2)$. Dies ist jedoch nur möglich wenn r selbst in $\mathbb{N}$ liegt, was man wie folgt einsehen kann. Ist $r = 0$ so ist das evident. Sei also $r \neq 0$. Die Voraussetzungen $a \in \mathbb{N}$ und $a = 2r^2 + 1$ implizieren $m := 2r^2 \in \mathbb{N}$. Wir zeigen dass m von der Form $m = 2n^2$ ist für ein $n \in \mathbb{N}$ und damit ist $r \in \mathbb{N}$. Es sei $r = p/q$, mit ggt $(p, q) = 1$. Sodann $mq^2 = 2p^2$. Ist $m = 2i$ gerade, so erhalten wir $iq^2 = p^2$. Da jeder Primfaktor von q nun p^2 teilt, also auch p, muss wegen ggt $(p, q) = 1$ nun $q = 1$ sein. D.h. m hat die gewünschte Form. Ist $m = 2i+1$ ungerade, so muss wegen $2p^2 = mq^2$ die Zahl q auch gerade sein. Sagen wir $q = 2^j u$, wobei $j \in \mathbb{N}, j \neq 0$, und u ungerade. Folglich ist $p^2 = (mu^2)2^{2j-1}$. Weil 2 kein gemeinsamer Faktor von p und q ist, erhalten wir den Widerspruch, da die linke Seite von $p^2 = (mu^2)2^{2j-1}$ ungerade ist, die rechte aber gerade.

Damit haben alle Lösungen von (57) notwendigerweise die Form $(a, b) = (1+2r^2, r(1+2r^2))$, $r \in \mathbb{N}$. Umgekehrt, ist auch jedes solche Paar Loŝung der Gleichung (57):

$$2r^2 - (1 + 2r^2) = -1.$$

Beispiele für (a, b): $(1, 0)$, $(3, 3)$, $(9, 18)$, $(19, 57)$, $(33, 132)$.

Damit hat man mit $(a, b) = (1, 0)$ den einzigen Schnittpunkt $M = (0, 1)$ von Γ^+ mit Γ^- im 90 Grad Winkel für $(s, t) = (1, -1)$ bei

$$\Gamma^+(s) = \left(s - \frac{1}{s}, s^2\right), s > 0, \quad \Gamma^-(t) = \left(t - \frac{1}{t}, t^2\right), t < 0,$$

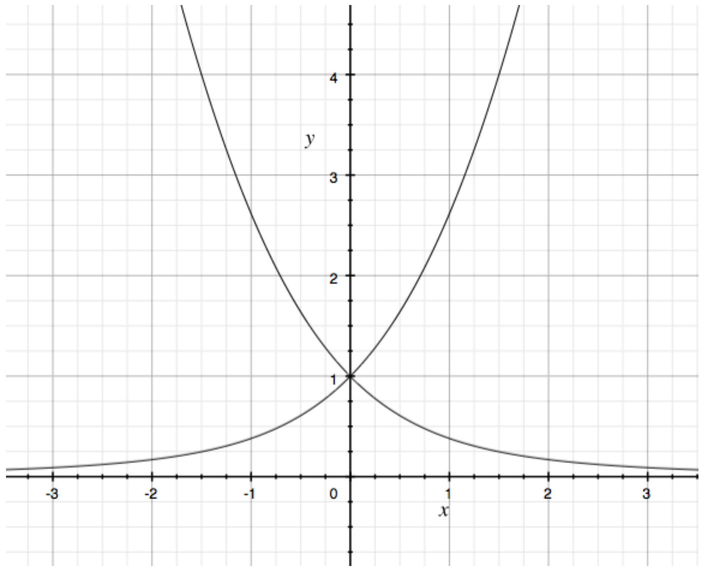

FIGURE 11. $r = 0$, $a = 1$, $b = 0$

oder mit $(a, b) = (3, 3)$ den Schnittpunkt $M = (-1, 4)$ von Γ^+ mit Γ^- im 90 Grad Winkel für $(s, t) = (\frac{-1+\sqrt{13}}{2}, \frac{-1-\sqrt{13}}{2})$ bei

$$\Gamma^+(s) = \left(s - \frac{3}{s}, s^2 + \frac{3}{s}\right), s > 0, \quad \Gamma^-(t) = \left(t - \frac{3}{t}, t^2 + \frac{3}{t}\right), t < 0,$$

oder mit $(a, b) = (9, 18)$ den Schnittpunkt $M = (-2, 13)$ von Γ^+ mit Γ^- im 90 Grad Winkel für $(s, t) = (\sqrt{10} - 1, -1 - \sqrt{10})$ bei

$$\Gamma^+(s) = \left(s - \frac{9}{s}, s^2 + \frac{18}{s}\right), s > 0, \quad \Gamma^-(t) = \left(t - \frac{9}{t}, t^2 + \frac{18}{t}\right), t < 0,$$

Das Überraschendste für uns bei dieser proposition: der Schnittpunkt M hat auch ganzzahlige Komponenten:

$$M = (-r, 1 + 3r^2).$$

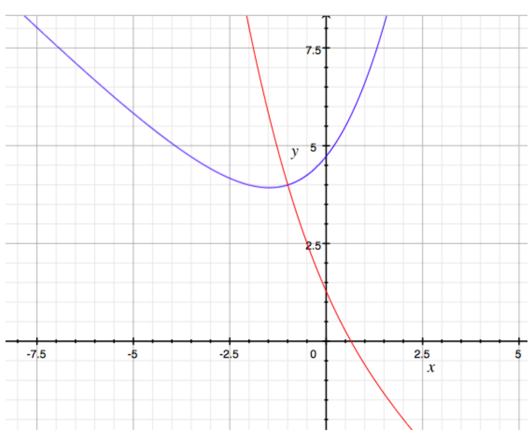

FIGURE 12. $r = 1$, $a = 3$, $b = 3$

4.2. **Die restlichen Fälle.** *Fall 2.1*: $b^2 + 4a^3 < 0$, äquivalent $a < -\left(\frac{b^2}{4}\right)^{1/3}$. In diesem Fall hat die quadratische Gleichung (54) keine reellen Lösungen, und folglich sind Γ^+ und Γ^- Jordanbögen die sich nicht schneiden (siehe Grafik 13).

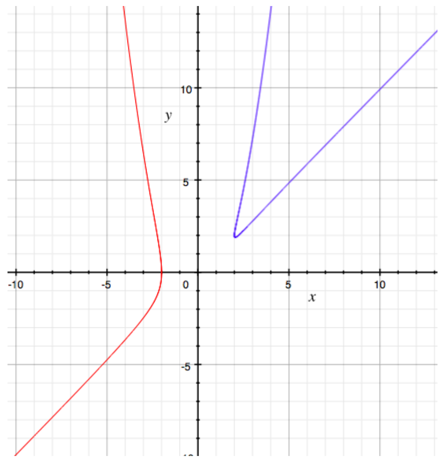

FIGURE 13. $a = -1$, $b = 1$

Fall 2.2: $-\left(\frac{b^2}{4}\right)^{1/3} < a < 0$. In diesem Fall ist $b^2 + 4a^3 > 0$ und es liegen zwei verschiedene Lösungen s, t der quadratische Gleichung (54) vor, welche aber wegen $st = -a > 0$ dasselbe Vorzeichen haben. Damit schneiden sich die Kurven Γ^+ und Γ^- nicht, aber genau eine von denen hat einen Selbstüberschneidungspunkt (siehe Grafik 14). Nämlich Γ^+ falls $b > 0$ und Γ^- falls $b < 0$. Anmerken möchten wir noch, dass die Gleichung (56), $2b^2 + a^2 - a^3 = -1$, keine Lösung hat falls $a < 0$. Folglich ist diese Selbstüberschneidung nie senkrecht.

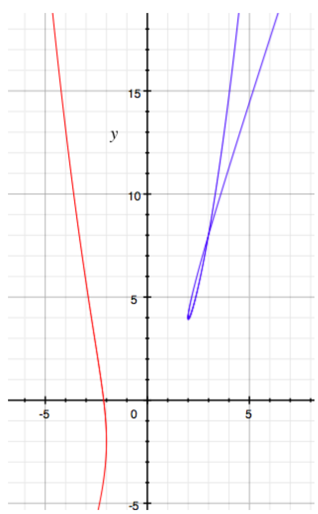

FIGURE 14. $a = -1$, $b = 3$

Fall 2.3: $b^2 + 4a^3 = 0$.
Auch hier schneiden sich die Kurven Γ^+ und Γ^- nicht, und beide sind wieder Jordanbögen. Siehe Grafik (15) zum Beispiel

$$\Gamma^+(s) = \left(s + \frac{1}{s}, s^2 + \frac{2}{s}\right), s > 0, \quad \Gamma^-(t) = \left(t + \frac{1}{t}, t^2 + \frac{2}{t}\right), t < 0.$$

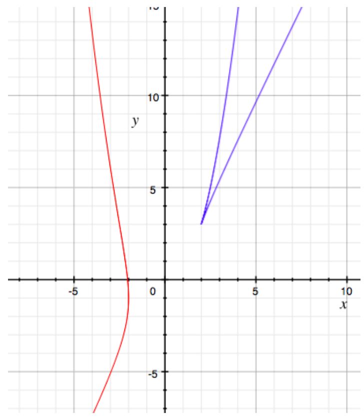

FIGURE 15. $a = -1$, $b = 2$

Fall 3: $a > 0$, $b \leq 0$. Da ist prinzipiell kein Unterschied zum Fall $a > 0, b \geq 0$; es liegt nur eine Spiegelung der Kurven an der y-Achse vor. Z.B. hat für $b < 0$ die Kurve

$$(x(t), y(t)) = \left(t - \frac{a}{t}, t^2 + \frac{b}{t}\right), t > 0,$$

mit der Transformation $t \to -t$ auch die Parameterdarstellung

$$\left(-(t - \frac{a}{t}), t^2 - \frac{b}{t}\right), t < 0.$$

Die Spiegelung an der y Achse ist dann gegeben durch

$$\left(t - \frac{a}{t}, t^2 + \frac{(-b)}{t}\right), t < 0$$

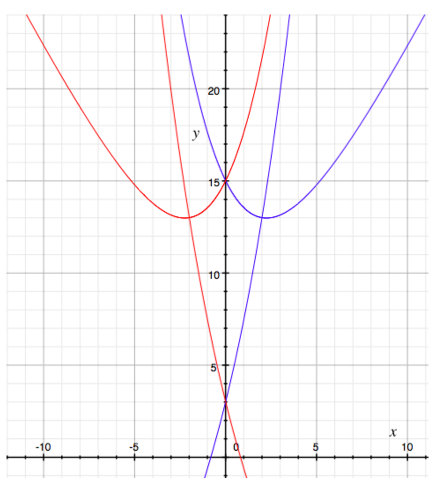

FIGURE 16. $a = 9$, $b = 18$ blau, $b = -18$ rot

Fazit: Die Kurve $\Gamma^+ = \Gamma^+(a,b)$ schneidet für $a, b \in \mathbb{Z}$ die Kurve $\Gamma^-(a,b)$ in einem rechten Winkel genau dann wenn

$$(a,b) = \big((1+2r^2), \pm r(1+2r^2)\big) \text{ mit } r \in \mathbb{N} = \{0,1,2,3\dots\}.$$

Oder in der Formulierung der proposition: Die unstetige "Kurve" $c = c(a,b)$ besitzt eine "Selbstüberschneidung" mit senkrechtem Schmittwinkel genau dann wenn

$$(a,b) = \big((1+2r^2), r(1+2r^2)\big) \text{ mit } r \in \mathbb{Z}.$$

Aufgabe 1442 (Die einfache dritte Aufgabe): Es sei

$$f(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{2}{z}$$

mit positiven x, y, z in $\mathbb{R}$. Man bestimme

$$\min_{x^2+y^2+z^2=1} f(x,y,z).$$

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Let $f(x,y,z) := \frac{1}{x} + \frac{1}{y} + \frac{2}{z}$. Using Lagrange multipliers, we show that

$$\inf_{\substack{x^2+y^2+z^2=1\\ x>0,y>0,z>0}} f(x,y,z) = \min_{\substack{x^2+y^2+z^2=1\\ x,y,z\geq \frac{1}{4\sqrt{3}}}} f(x,y,z) = (2+2^{2/3})^{3/2} \sim 6.794693902\cdots$$

First we recall the version of Lagrange's theorem, we will use:

Theorem 12. *Let $G := \{\xi = (x,y,z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$ be the first octant and $g(x,y,z) := x^2 + y^2 + z^2 - 1$. Then f and g belong to $C^1(G)$. If $\zeta \in G$ is a local extremum of f on the set $N := \{(x,y,z) \in G : g(x,y,z) = 0\}$ for which $\frac{\partial}{\partial z} g(\zeta) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that (ζ, λ) is a stationary point of Lagrange's function*

$$L(x,y,z,\lambda) := f(x,y,z) + \lambda g(x,y,z).$$

Next, we prove the existence of such a local extremum. Let $\eta := (3^{-1/2}, 3^{-1/2}, 3^{-1/2})$. Then $\eta \in N$. Moreover, $f(\eta) = 4\sqrt{3}$. Also, if $(x,y,z) \in N$ is such that at least one of its coordinates is strictly bigger than $(4\sqrt{3})^{-1}$, then $f(x,y,z) > 4\sqrt{3}$. Hence

$$\inf_N f = \min\left\{ f(x,y,z) : x^2+y^2+z^2 = 1,\ x,y,z \geq \frac{1}{4\sqrt{3}} \right\}.$$

Finally we solve Lagrange's equations for $(x,y,z) \in G$ and $\lambda \in \mathbb{R}$:

$$(1) \qquad \frac{\partial}{\partial x} L(x,y,z,\lambda) = -\frac{1}{x^2} + \lambda(2x) \overset{!}{=} 0$$

$$(2) \qquad \frac{\partial}{\partial y} L(x,y,z,\lambda) = -\frac{1}{y^2} + \lambda(2y) \overset{!}{=} 0$$

$$(3) \qquad \frac{\partial}{\partial z} L(x,y,z,\lambda) = -\frac{2}{z^2} + \lambda(2z) \overset{!}{=} 0$$

$$(4) \qquad \frac{\partial}{\partial \lambda} L(x,y,z,\lambda) = x^2 + y^2 + z^2 - 1 \overset{!}{=} 0$$

(1) and (2) yield that $x = y$ and (2) and (3) yield that $\frac{2}{z^3} = \frac{1}{y^3}$, equivalently $z = 2^{1/3} y$. Due to (4),

$$1 = x^2 + x^2 + 2^{2/3} x^2,$$

hence

$$x = y = (2 + 2^{2/3})^{-1/2}, \qquad z = 2^{1/3}(2 + 2^{2/3})^{-1/2}.$$

Consequently, the unique stationary point of L on $G \times \mathbb{R}$ is

$$P = \left(\frac{1}{\sqrt{2+2^{2/3}}}, \frac{1}{\sqrt{2+2^{2/3}}}, \frac{2^{1/3}}{\sqrt{2+2^{2/3}}}, \frac{(2+2^{2/3})^{3/2}}{2} \right)$$

Let ζ be the point formed with the first three coordinates of P, which are of course bigger than $(4\sqrt{3})^{-1}$. Then

$$f(\zeta) = 2\sqrt{2 + 2^{2/3}} + 2\,\frac{\sqrt{2 + 2^{2/3}}}{2^{1/3}} = (2 + 2^{2/3})\,\sqrt{2 + 2^{2/3}} = (2 + 2^{2/3})^{3/2}.$$

Of course, this point ζ must now be that unique point on N where $\inf_N$ is taken (note that $\sup f_N = \infty$).

Aufgabe 1438: Es seien $z_j = r_j e^{it_j}$ zwei Punkte in der komplexen Ebene $\mathbb{C}$ mit $0 < r_j < 1$ und $|t_j| < \pi/2$. Weiterhin sei $\Delta = \langle z_1, z_2, 1\rangle$ das durch die Punkte z_1, z_2, 1 bestimmte abgeschlossene Dreieck. Man zeige: Für alle $z = re^{it} \in \Delta$ mit $|t| < \pi/2$ gilt

$$\frac{|1-z|}{1-|z|} \leq \max\left\{\frac{|1-z_1|}{1-|z_1|}, \frac{|1-z_2|}{1-|z_2|}\right\} \quad \text{und} \quad \frac{|t|}{1-r} \leq \max\left\{\frac{|t_1|}{1-r_1}, \frac{|t_2|}{1-r_2}\right\}.$$

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Three solutions for (1) in [16, Appendix 27], one for (2) in [16, Appendix 31].

(1) Man beachte zunächst

$$\Delta = \{t_1 z_1 + t_2 z_2 + t_3 : 0 \leq t_j \leq 1, t_1 + t_2 + t_3 = 1\}.$$

Ist nun $z \in \Delta$, so gilt

$$z = t_1 z_1 + t_2 z_2 + (1 - t_1 - t_2) = t_1(z_1 - 1) + t_2(z_2 - 1) + 1.$$

Daher

$$|1 - z| = |t_1(z_1 - 1) + t_2(z_2 - 1)| \leq t_1|z_1 - 1| + t_2|z_2 - 1|.$$

Andererseits

$$|z| \leq t_1|z_1| + t_2|z_2| + 1 - t_1 - t_2.$$

Somit

$$1 - |z| \geq t_1(1 - |z_1|) + t_2(1 - |z_2|).$$

Folglich gilt

$$\frac{|1-z|}{1-|z|} \leq \frac{t_1|z_1 - 1| + t_2|z_2 - 1|}{t_1(1 - |z_1|) + t_2(1 - |z_2|)} \leq \max\{Z_1, Z_2\} := \kappa,$$

wobei

$$Z_j = \frac{|1 - z_j|}{1 - |z_j|}.$$

(2) Ist für $\kappa > 0$, $D_\kappa^* = \{re^{it} \in \mathbf{D} : |t| \leq \kappa(1 - r)\}$, so ist der Teil der Randkurve welche im 1-ten Quadraten liegt, in Polarkoordinaten gegeben durch

$$r(t) = 1 - \frac{t}{\kappa}, 0 \leq t \leq \min\{\pi/2, \kappa\}.$$

Diese Kurve ist Graph einer konkaven Funktion (Bilder und Details wie im Buch [16, Appendix 31] unter Beachtung dass dort alles auch für $0 < \kappa < 1$ gilt). Folglich ist die Menge

$$\{(x, y) : 0 \leq x = x(t) \leq 1, 0 \leq y \leq y(t)\}$$

konvex, und somit auch $S(\kappa) := D_\kappa^* \cap \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. Unter Beachtung dass $D_{\kappa_1}^* \subseteq D_{\kappa_2}^*$ für $0 < \kappa_1 \leq \kappa_2$, schliessen wir, dass mit $z_1, z_2, 1 \in S(\kappa)$ auch Δ in $S(\kappa)$ liegt falls

$$\kappa = \max\left\{\frac{|t_1|}{1-r_1}, \frac{|t_2|}{1-r_2}\right\}.$$

Aufgabe 1437: Die Funktion

$$f(x) = \frac{x}{\ln(1-x)}, \quad x \neq 0, \quad f(0) = -1,$$

lässt sich in eine Potenzreihe entwickeln, etwa $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < 1$.

(a) Man beweise $\frac{1}{3k^2} \leq a_k \leq \frac{2}{k}$ für $k \geq 1$.

(b) Konvergieren die Reihen $\sum_{k=0}^{\infty} (-1)^k a_k$ und $\sum_{k=0}^{\infty} a_k$ und falls ja, gegen welchen Grenzwert?

Frieder Grupp, Bergrheinfeld, D

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This is well known since "hundreds of years" (see [56]).

(a) First note that $f(x) = -\int_0^1 (1-x)^s\, ds$. In fact

$$\begin{aligned} -\int_0^1 (1-x)^s\, ds &= -\int_0^1 e^{s\log(1-x)}\, ds = -\left[\frac{e^{s\log(1-x)}}{\log(1-x)}\right]_0^1 \\ &= -\left(\frac{1-x}{\log(1-x)} - \frac{1}{\log(1-x)}\right) = \frac{x}{\log(1-x)}. \end{aligned}$$

Now

$$(1-x)^s = \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} x^k.$$

Since $\int \sum = \sum \int$ (due to uniform convergence in s for every fixed $x \in]-1,1[$; note that $|(-1)^k \binom{s}{k}| \leq 1$), we obtain

$$f(x) = \sum_{k=0}^{\infty} \left[(-1)^{k+1} \int_0^1 \binom{s}{k}\, ds\right] x^k.$$

Hence

$$\begin{aligned} a_k &= (-1)^{k+1} \int_0^1 \binom{s}{k}\, ds = (-1)^{k+1} \int_0^1 \frac{s(s-1)(s-2)\cdots(s-k+1)}{k!}\, ds \\ &= \int_0^1 \frac{s(1-s)(2-s)\cdots(k-1-s)}{k!}\, ds \\ &= \frac{1}{k} \int_0^1 s\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{k-1}\right)\, ds. \end{aligned}$$

Since every factor is less than 1, we obtain

$$0 \leq a_k \leq \frac{1}{k}.$$

Moreover, as $0 \leq s \leq 1$, and for $k \geq 2$,

$$a_k \geq \int_0^1 s(1-s)\frac{1\cdot 2\cdot 3\cdot 4\ldots(k-2)}{k!}\, ds = \left(\frac{1}{2} - \frac{1}{3}\right)\frac{1}{k(k-1)} = \frac{1}{6k(k-1)}.$$

The right-hand side, though, is smaller than $(3k^2)^{-1}$. So we need a more careful estimate. Instead of "breaking" after the second factor, we brake after the fifth factor. Noticing that

$$\int_0^1 s(1-s)(2-s)(3-s)(4-s)\, ds = \frac{9}{4},$$

we obtain for $k \geq 5$,

$$a_k \geq \frac{9}{4}\,\frac{4\ldots5\ldots(k-2)}{k!} = \frac{9}{4}\frac{1}{3!}\frac{1}{k(k-1)} = \frac{3}{8}\,\frac{1}{k^2} \geq \frac{1}{3}\,\frac{1}{k^2}.$$

The estimate $a_k \geq \frac{1}{3k^2}$ now holds also for $k = 1, \ldots, 4$, due to the following explicit representation of the Taylor sums for $f(x)$:

$$-1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{24}x^3 + \frac{19}{720}x^4 + \frac{3}{160}x^5 + \frac{863}{60480}x^6 + \frac{275}{24192}x^7 + \frac{33953}{3628800}x^8$$

(b) We first show that (a_k) is decreasing (to 0):

$$\begin{aligned} a_{k+1} &= \frac{1}{k+1}\int_0^1 s\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{k-1}\right)\underbrace{\left(1-\frac{s}{k}\right)}_{\leq 1}\,ds \\ &\leq \frac{k}{1+k}a_k \leq a_k. \end{aligned}$$

The alternating series test of Leibniz now yields the convergence of $\sum_{k=0}^{\infty}(-1)^k a_k$. Finally, by Abel's rule (see [16, p. 1415]),

$$\sum_{k=0}^{\infty}(-1)^k a_k = \lim_{x\to -1^+} f(x) = -\frac{1}{\log 2}.$$

Next we show that $ka_k \to 0$. In fact, due to $1 - x \leq e^{-x}$ for $x \geq 0$,

$$\begin{aligned} ka_k &= \int_0^1 s\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{k-1}\right)\,ds \\ &\leq \int_0^1 \exp\left(-s\sum_{j=1}^{k-1}\frac{1}{j}\right)\,ds = -\left[\frac{\exp\left(-s\sum_{j=1}^{k-1}\frac{1}{j}\right)}{\sum_{j=1}^{k-1}\frac{1}{j}}\right]_0^1 \\ &\leq \frac{1}{\sum_{j=1}^{k-1}\frac{1}{j}} \leq \frac{1}{\int_1^k dx/x} = \frac{1}{\log k}. \end{aligned}$$

Note that $\lim_{x\to 1^-} f(x) = 0$. Hence, by Tauber's Theorem ([58, p. 52]), $F := \sum_{k=0}^{\infty} a_k$ is convergent and $\sum_{k=0}^{\infty} a_k = 0$. Very funny! By the way, F is called Fontana's series (Gregorio Fontana 1735–1803), see [57, (formula 20)], and the a_k are the (moduli) of the Gregory coefficients (James Gregory 1638–1675), see [57] and [56].

Aufgabe 1434: Berechne

$$S_1 = \sum_{k=0}^{\infty} \frac{1}{(3k+1)^3}, \quad S_2 = \sum_{k=0}^{\infty} \frac{1}{(3k+2)^3},$$
$$T_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)^3}, \quad T_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)^3}.$$

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Let

$$S_3 := \sum_{n=0}^{\infty} \frac{1}{(3n+3)^3} \text{ and } T_3 := \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+3)^3}.$$

Then

$$S_3 = \frac{1}{27}\zeta(3) \text{ and } T_3 = \frac{1}{27}\eta(3)$$

where $\eta(3)$ is the Dirichlet η-function. It is well-known that $\eta(3) = (3/4)\zeta(3)$. Thus

$$S_1 + S_2 + S_3 = \zeta(3) \text{ and } T_1 - T_2 + T_3 = \eta(3)$$

imply that

$$S_1 + S_2 = \tfrac{26}{27}\zeta(3) \text{ and } T_1 - T_2 = \tfrac{13}{18}\zeta(3).$$

On the other hand,

$$\begin{aligned} S_1 - T_1 &= \sum_{n=0}^{\infty} \frac{1-(-1)^n}{(3n+1)^3} \underset{n=2k+1}{=} 2\sum_{k=0}^{\infty} \frac{2}{(3(2k+1)+1)^3} \\ &= 2\sum_{k=0}^{\infty} \frac{1}{(6k+4)^3} = \frac{1}{4}\sum_{k=0}^{\infty} \frac{1}{(3k+2)^3} \\ &= \frac{1}{4}S_2. \end{aligned}$$

Moreover,

$$\begin{aligned} S_2 + T_2 &= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{(3n+2)^3} \underset{n=2k}{=} 2\sum_{k=0}^{\infty} \frac{1}{(6k+2)^3} \\ &= = \frac{1}{4}\sum_{k=0}^{\infty} \frac{1}{(3k+1)^3} \\ &= \frac{1}{4}S_1. \end{aligned}$$

This yields the linear system

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -\frac{1}{4} & -1 & 0 \\ -\frac{1}{4} & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \frac{26}{27}\zeta(3) \\ 0 \\ 0 \\ \frac{13}{18}\zeta(3) \end{pmatrix}.$$

The determinant is, unfortunately, zero. The null-space is the one dimensional vector space generated by $(4, -4, 5, 5)^{\perp}$.

We may write the system as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -\frac{1}{4} & -1 & 0 \\ -\frac{1}{4} & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \frac{26}{27}\zeta(3) \\ 0 \\ 0 \\ \frac{13}{18}\zeta(3) + 2T_2 \end{pmatrix}.$$

In other words,

$$\begin{pmatrix} S_1 \\ S_2 \\ T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{2}{5} & \frac{-2}{5} & \frac{2}{5} \\ \frac{1}{2} & \frac{-2}{5} & \frac{2}{5} & \frac{-2}{5} \\ \frac{3}{8} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-3}{8} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{26}{27}\zeta(3) \\ 0 \\ 0 \\ \frac{13}{18}\zeta(3) + 2T_2 \end{pmatrix}.$$

That is

(58)
$$\begin{cases} S_1 = & R_1 + \frac{4}{5}T_2 \\ S_2 = & R_2 - \frac{4}{5}T_2 \\ T_1 = & R_3 + T_2 \\ T_2 = & T_2 \end{cases}$$

where the R_j are rational multiples of $\eta(3)$. More precisely,

$$R_1 = \frac{1}{2} \cdot \frac{26}{27}\zeta(3) + \frac{2}{5} \cdot \frac{13}{18}\zeta(3) = \frac{104}{5 \cdot 27} \cdot \zeta(3),$$
$$R_2 = \frac{1}{2} \cdot \frac{26}{27}\zeta(3) - \frac{2}{5} \cdot \frac{13}{18}\zeta(3) = \frac{26}{5 \cdot 27} \cdot \zeta(3),$$
$$R_3 = \frac{3}{8} \cdot \frac{26}{27}\zeta(3) + \frac{1}{2} \cdot \frac{13}{18}\zeta(3) = \frac{13}{18} \cdot \zeta(3).$$

Next we use that for $0 < a < 2\pi$ (see below)

(59)
$$h(a) := \sum_{n=0}^{\infty} \frac{\sin(n+1)a}{(n+1)^3} = \frac{a^3 - 3\pi a^2 + 2\pi^2 a}{12},$$

and put $a = 2\pi/3$. Then, since $\sin(2\pi/3) = \sqrt{3}/2$,

$$S_1 - S_2 = \frac{2}{\sqrt{3}}\, h(2\pi/3) = \frac{2}{\sqrt{3}} \cdot \frac{2\pi^3}{81} = \frac{4\pi^3}{81\sqrt{3}}.$$

By the formula (58) above,

$$S_1 - S_2 = \frac{4}{5} \cdot \frac{13}{18}\zeta(3) + \frac{8}{5}T_2.$$

Hence

$$T_2 = \frac{5}{8}\left(\frac{4\pi^3}{81\sqrt{3}} - \frac{4}{5} \cdot \frac{13}{18}\zeta(3)\right) = \boxed{\frac{5\pi^3}{2 \cdot 81\sqrt{3}} - \frac{13}{36}\zeta(3)} \sim 0.11843 \cdots .$$

Finally, by (58) again,

$$\boxed{S_1 = \frac{13}{27}\,\zeta(3) + \frac{2\pi^3}{81\sqrt{3}}} = 3^{-5}(117\,\zeta(3) + 2\sqrt{3}\,\pi^3) \sim 1.02078 \cdots ,$$

$$\boxed{S_2 = \frac{13}{27}\,\zeta(3) - \frac{2\pi^3}{81\sqrt{3}}} = 3^{-5}(117\,\zeta(3) - 2\sqrt{3}\,\pi^3) \sim 0.13675 \cdots ,$$

$$\boxed{T_1 = \frac{13}{36}\,\zeta(3) + \frac{5\pi^3}{2 \cdot 81\sqrt{3}}} = \frac{1}{1944}\left(702\,\zeta(3) + 20\sqrt{3}\,\pi^3\right) \sim 0.98659 \cdots .$$

Addendum

The value in (59) for $h(a)$ is given as follows (we had developed this in solving the problem 12388 in AMM). Note that

$$h'(a) = \sum_{n=0}^{\infty} \frac{\cos(n+1)a}{(n+1)^2}.$$

Since $\frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is the Fourier series of the function $(x-\pi)^2$, $0 \le x < 2\pi$, extended 2π-periodically, we see that for $0 < a < 2\pi$,

$$h'(a) = \frac{(a-\pi)^2}{4} - \frac{\pi^2}{12}.$$

As $h(0) = 0$, we deduce that for $0 < a < 2\pi$,

$$h(a) = \frac{(a-\pi)^3}{12} - \frac{\pi^2}{12}\,a + \frac{\pi^3}{12} = \frac{a^3 - 3\pi a^2 + 2\pi^2 a}{12}.$$

Aufgabe 1435: Man beweise für $x, y \geq 0$, $x + y = 2$ die Ungleichung

$$\sqrt{x^2+3} + \sqrt{y^2+3} + \sqrt{xy+3} \geq 6.$$

Šefket Arslanagić, Sarajevo, BIH

Solution to problem 1435 Elem. Math. 77 (2022), 85

Raymond Mortini, Rudolf Rupp

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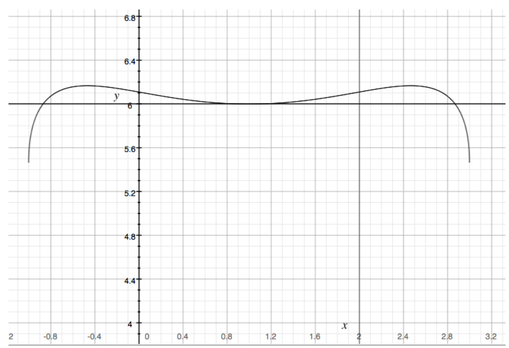
Let

$$A(x,y) := \sqrt{x^2+3} + \sqrt{y^2+3} + \sqrt{xy+3}.$$

Since by assumption $x + y = 2$, we just have to prove that 6 is the minimal value of the function

$$f(x) := \sqrt{x^2+3} + \sqrt{(2-x)^2+3} + \sqrt{x(2-x)+3}, \quad 0 \leq x \leq 2,$$

which is attained at $x = 1$.



First we note that f is symmetric with respect to $x = 1$; that is

$$f(1+x) = f(1-x) \text{ for } 0 \leq x \leq 1.$$

We show that f decreases on $[0,1]$. Note that $f(0) = 2\sqrt{3} + \sqrt{7} > 6 = f(1)$. It is sufficient to prove that $f' \leq 0$ on $[0,1]$:

$$f'(x) = \frac{x}{\sqrt{x^2+3}} - \frac{2-x}{\sqrt{(2-x)^2+3}} + \frac{1}{2}\frac{2-2x}{\sqrt{x(2-x)+3.}}$$

But $f' \leq 0$ for $0 \leq x \leq 1$ if and only if

(60) $$L(x) := \frac{x}{\sqrt{x^2+3}} + \frac{1-x}{\sqrt{x(2-x)+3}} \leq \frac{2-x}{\sqrt{(2-x)^2+3}}.$$

Next note that, due to $2 - x \geq x$,

$$\frac{1-x}{\sqrt{x(2-x)+3}} \leq \frac{1-x}{\sqrt{x^2+3}}.$$

Hence $L(x) \leq \frac{1}{\sqrt{x^2+3}}$. Thus (60) holds for $0 \leq x \leq 1$ if

(61) $$\frac{1}{\sqrt{x^2+3}} \leq \frac{2-x}{\sqrt{(2-x)^2+3}},$$

or equivalently

(62) $$\frac{\sqrt{(2-x)^2+3}}{\sqrt{x^2+3}} \leq 2-x.$$

This holds, though, due to the following equivalences for $0 \le x \le 1$:

$$(2-x)^2 + 3 \le (2-x)^2(x^2+3) \iff 3 \le (2-x)^2(x^2+2) =: R(x).$$

The latter is true, since $\min_{0\le x\le 1} R(x) = R(1) = 3$ (note that the derivative of R equals $R'(x) = -4(2-x)(x^2-x+1)$, so $R' \le 0$ on $[0,1]$.)

Aufgabe 1431: Man bestimme den Wert der Doppelreihe

$$\sum_{k,n=0}^{\infty} \frac{(-1)^{k+n}}{(2n+1)(2k+1)(2n+2k+3)^2} \binom{-1/2}{n}.$$

Raymond Mortini, Metz, F und Rudolf Rupp, Nürnberg, D

Partial solution to problem 1431, Elem. Math. 78 (2023), 44

Raymond Mortini, Rudolf Rupp

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Let

$$S := \sum_{k,n=0}^{\infty} \frac{(-1)^{k+n+1}}{(2n+1)(2k+1)(2n+2k+3)^2} \binom{-1/2}{n}.$$

Then S is the value of the integral

$$\int_0^1 \arcsin x \arctan x \log x \, dx.$$

Just take the series

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \binom{-1/2}{n} x^{2n+1}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

and use that

$$\int_0^1 x^m \log x \, dx = -(m+1)^{-2}.$$

For a solution see [59].

Aufgabe 1428: Es seien a, b und c positive reelle Zahlen. Man bestimme die grösste Zahl $k_1 > 0$ und die kleinste Zahl $k_2 > 0$ derart, dass die folgende Ungleichung gilt:

$$k_1 \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{ab+bc+ca}} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq k_2 \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Péter Ivády, Budapest, H

Partial solution to problem 1428 Elem. Math. 77 (2022), 196

Raymond Mortini, Rudolf Rupp

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Let

$$f(a,b,c) := \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

Then, by using that $\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2}$,

$$\begin{aligned}
f(a,b,c) &= \frac{1}{a^2+b^2+c^2}\left(a\,\frac{bc}{b+c} + b\,\frac{ca}{c+a} + c\,\frac{ab}{a+b}\right) + 1\\
&\leq \frac{1}{a^2+b^2+c^2}\left(a\,\frac{b+c}{4} + b\,\frac{c+a}{4} + c\,\frac{a+b}{4}\right) + 1\\
&= \frac{1}{a^2+b^2+c^2}\,\frac{2ab+2bc+2ca}{4} + 1\\
&= \frac{1}{a^2+b^2+c^2}\,\frac{(a+b+c)^2-(a^2+b^2+c^2)}{4} + 1\\
&\leq \frac{1}{a^2+b^2+c^2}\,\frac{(a^2+b^2+c^2)(1+1+1)-(a^2+b^2+c^2)}{4} + 1\\
&\leq \frac{3}{2}.
\end{aligned}$$

If we let $a=b=c$, then $f(a,a,a) = 3/2$ and so $\boxed{k_2 = 3/2}$. To determine k_1, let

$$g(a,b,c) := \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2 \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

Then, by using two of the estimates above, namley $f \geq 1$, and Cauchy-Schwarz,

$$g(a,b,c) = \frac{a^2+b^2+c^2}{ab+bc+ca}\, f(a,b,c)^2 > \frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1.$$

We guess $k_1 = \sqrt{2}$. In fact, we may restrict to triples $(x,1,c)$ (homogeniety). Then it remains to prove that

$$f_c(x) := \left(\frac{1}{x+c} + \frac{x}{c+1} + \frac{c}{1+x}\right)^2 \left(\frac{x+cx+c}{1+x^2+c^2}\right) \geq 2.$$

Now

$$\lim_{c\to 0} f_c(x) = x + \frac{1}{x} = f_0(x) \geq 2.$$

Graphical evidence seems to indicate that $m_c := \min_{x>0} f_c(x) \geq 2$ and $\lim_{c\to 0} m_c = 2$.

As it is customn with this type of questions, the infimum of the two-variable function $f(x,c) := f_c(x)$ is taken on the boundary of the first quadrant; that is when $c = 0$. We have no proof though of this last claim.

Aufgabe 1383:

a) Man zeige, dass für $\alpha \in (0,1]$ die in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ holomorphe Funktion $f(z) = (1-z)^\alpha$ Hölder–Lipschitz stetig ist zur Ordnung α, d.h.

$$\sigma_\alpha = \sup_{\substack{z,w \in \mathbb{D} \\ z \neq w}} \frac{|(1-z)^\alpha - (1-w)^\alpha|}{|z-w|^\alpha} < \infty.$$

Hierbei ist, wie üblich, $(1-z)^\alpha = \exp(\alpha \log(1-z))$, wobei der Hauptzweig des Logarithmus in der rechten Halbebene genommen wird.

b) Man bestimme σ_α explizit.

Raymond Mortini, Metz, F und Rudolf Rupp, Nürnberg, D

problem 1383 in Elem. Math 74 (2019), 38, by

Raymond Mortini and Rudolf Rupp

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Theorem 13. *Let $0 < \alpha \leq 1$. Then*

$$\begin{aligned}\sigma(\alpha) &:= \sup\left\{\frac{|(1-z)^\alpha - (1-w)^\alpha|}{|z-w|^\alpha} : |z|, |w| \leq 1, z \neq w\right\} \\ &= \max\{1, 2^{1-\alpha}\sin(\alpha\pi/2)\} \\ &= \begin{cases} 1 & \text{if } 0 < \alpha \leq 1/2 \\ 2^{1-\alpha}\sin(\alpha\pi/2) & \text{if } 1/2 \leq \alpha \leq 1. \end{cases}\end{aligned}$$

Moreover,

$$\max_{0<\alpha\leq 1} \log \sigma(\alpha) = \left(1 - \frac{2}{\pi}\arctan\left(\frac{\pi}{2\log 2}\right)\right)\log 2 + \log\left(\frac{\pi}{\sqrt{\pi^2 + 4(\log 2)^2}}\right).$$

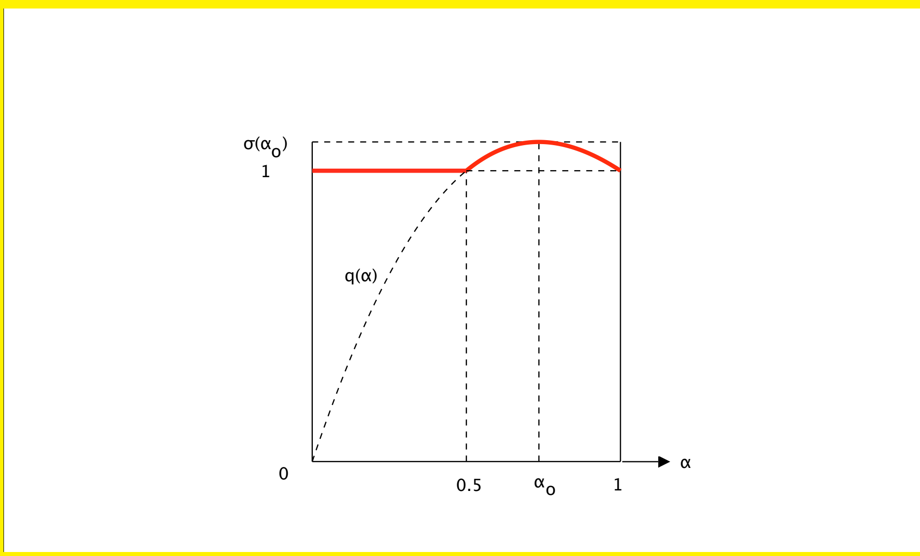

FIGURE 17. The Hölder-Lipschitz constant $\sigma(\alpha)$

See [10] for this best Hölder-Lipschitz constant associated with $(1-z)^\alpha$.

Aufgabe 1350: Für $p > 3$ sei

$$f(x) = x^{p-2}(2 - x^p).$$

Man zeige, dass $0 \le f(x) \le 1$ gilt, wann immer $0 \le x \le \cos(\frac{\pi}{p-1})$ ist.

Raymond Mortini, Metz, F

Solution to problem 1350 in Elem. Math 71 (2016), 84

Raymond Mortini

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Dies folgt aus Lemma 2 in [1], welches besagt dass für $0 \le t \le \pi/2 - \pi(p-1)$ die Ungleichung $\cos t + (\sin t)^p \le 1$ gilt, indem man folgende Transformationen benutzt: $x = \sin t$, $0 \le x \le \cos(\frac{\pi}{p-1})$,

$$\sqrt{1-x^2} + x^p \le 1 \iff 1 - x^2 \le (1-x^p)^2 \iff 0 \le x^2 + x^{2p} - 2x^p$$
$$\iff 0 \le 1 + x^{2p-2} - 2x^{p-2} \iff x^{p-2}(2 - x^p) \le 1.$$

Einen davon unabhängigen Beweis würde ich gerne sehen. Ist mir aber unbekannt.

References

[1] Mortini, R; Rhin, G.: Sums of holomorphic selfmaps of the unit disk II, Comp. Meth. Function Theory 11 (2011), 135-142.

Aufgabe 1339: Beweise die Produktdarstellung

$$(1+\sqrt{2})^{\sqrt{2}} = e \cdot \sqrt[2]{e^{1/3}} \cdot \sqrt[4]{e^{1/5}} \cdot \sqrt[8]{e^{1/7}} \cdot \sqrt[16]{e^{1/9}} \cdot \ldots$$

Horst Alzer, Waldbröl, D

Solution to problem 1339 Elem. Math. 70 (2015), 82.

Raymond Mortini

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Man betrachte die für $|x|<1$ konvergente Reihe

$$R(x) := \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} = x + \frac{x}{3} + \cdots .$$

Deren Wert ist leicht zu bestimmen durch Ableitung:

$$R'(x) = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}.$$

Somit gilt wegen der gleichmässigen Konvergenz auf $[0,r]$, $0<r<1$, und $\int_0^s \sum = \sum \int_0^s$, dass $R(x)$ die Stammfunktion von $(1-x^2)^{-1}$ ist, welche in $x=0$ verschwindet; also

$$R(x) = \frac{1}{2}\left(\log(1+x) - \log(1-x)\right) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Ersetzt man x durch $1/\sqrt{2}$, so erhält man:

$$R\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{2n+1}\,\frac{1}{2^n}.$$

Desweiteren gilt

$$R\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\log\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) = \log(1+\sqrt{2}).$$

Daher

$$\sum_{n=0}^{\infty} \frac{1}{2n+1}\,\frac{1}{2^n} = \sqrt{2}\log(1+\sqrt{2}).$$

Durch Übergang zur Exponentialfunktion unter Verwendung der für konvergente Reihen geltenden Formel $e^{\sum a_n} = \prod e^{a_n}$, erhält man die gewünschte Gleichung:

$$(1+\sqrt{2})^{\sqrt{2}} = \prod_{n=0}^{\infty} e^{\frac{1}{2^n\,(2n+1)}} = \prod_{n=0}^{\infty} \sqrt[2^n]{e^{1/(2n+1)}}.$$

Aufgabe 1281: Man bestimme den Wert der Reihen

$$S = \sum_{n=2}^{\infty} \left(2 + n \log\left(1 - \frac{2}{n+1}\right)\right)$$

und

$$S^* = \sum_{n=1}^{\infty} \left(1 - \left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right)\right).$$

Raymond Mortini, Metz, F

Solution to problem 1281 Elem. Math. 65 (2010), 127, by

Raymond Mortini, Jérôme Noël

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a) Exponentiating, we have to calculate the value of the infinite product

$$P = \prod_{n=2}^{\infty} \left(e^2 \left(\frac{n-1}{n+1}\right)^n\right).$$

We claim that $P = \frac{4\pi}{e^3}$; so $S = \log(4\pi) - 3$.

Let $P_N = \prod_{n=2}^{N} \left(e^2 \left(\frac{n-1}{n+1}\right)^n\right)$. Then by Stirlings formula, telling us that $n! \sim e^{-n} n^n \sqrt{2\pi n}$, we obtain

$$P_N = \frac{1}{e^2} e^{2N} \frac{\prod_{n=2}^{N}(n-1)^n}{\prod_{n=2}^{N}(n+1)^n} =$$

$$\frac{1}{e^2} e^{2N} \frac{\prod_{n=2}^{N}(n-1)^n}{\prod_{\mathbf{n=2}}^{\mathbf{N}} \mathbf{n^{n+1}}} \; \frac{\prod_{\mathbf{n=2}}^{\mathbf{N}} \mathbf{n^{n+1}}}{\prod_{n=2}^{N}(n+1)^n} =$$

$$\frac{2}{e^2} \frac{e^{2N}\,(N!)^2}{N^{N+1}(N+1)^N} = \frac{2}{e^2} \left(\frac{e^N N!}{N^N}\right)^2 \frac{N^N}{(N+1)^N} \; \frac{1}{N} \sim$$

$$\frac{2}{e^2} \frac{(\sqrt{2\pi N})^2}{N} \frac{1}{(1+\frac{1}{N})^N} \to \frac{4\pi}{e^3}.$$

b) To determine S^*, we use the same method and calculate the value of

$$P^* = \prod_{n=1}^{\infty} e \left(\frac{n}{n+1}\right)^{n+\frac{1}{2}}$$

We claim that $P^* = \frac{\sqrt{2\pi}}{e}$ and so $S^* = \frac{1}{2}\log(2\pi) - 1$.

In fact

$$P_N^* = \prod_{n=1}^{N} e \left(\frac{n}{n+1}\right)^{n+\frac{1}{2}} = \frac{e^N N!}{(N+1)^N} \; \frac{1}{\sqrt{N+1}}.$$

Using Stirling's formula we obtain

$$P_N \sim \frac{N^N}{(N+1)^N} \sqrt{2\pi N} \; \frac{1}{\sqrt{N+1}} = \sqrt{2\pi} \frac{1}{(1+\frac{1}{N})^N} \; \frac{\sqrt{N}}{\sqrt{N+1}} \to \frac{\sqrt{2\pi}}{e}.$$

Aufgabe 901. Die Funktion $f:\{z\in\mathbf{C}\mid |z|<1\}\to\{z\in\mathbf{C}\mid |z|\le 1\}$ sei holomorph und es sei $f(0)=0$. Dann trifft genau eine der beiden folgenden Aussagen zu:

$$\left|\int_{-1}^{1} f(x)\,dx\right| < 2/3. \tag{1}$$

Es gibt eine Konstante $a\in\mathbf{C}$ mit $|a|=1$ derart, dass

$$f(z)=a\,z^2. \tag{2}$$

Dies ist zu zeigen.

P. von Siebenthal, Zürich

Solution to problem 901 Elem. Math. 38 (1983), 128.

Raymond Mortini

El. Math., Vol. 39, 1984 131

Lösung: Es sei $\mathbf{D}$ die offene Einheitskreisscheibe, also $\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$. Zunächst zerlegen wir die Funktion f in den geraden Anteil w und den ungeraden Anteil v, also

$$f(z)=w(z)+v(z), \tag{3}$$

mit

$$w(z)=\frac{1}{2}\Big(f(z)+f(-z)\Big) \quad \text{und} \quad v(z)=\frac{1}{2}\Big(f(z)-f(-z)\Big).$$

Aus den Voraussetzungen über f folgt, dass w die Form

$$w(z)=z^2 g(z)$$

hat, wobei die Funktion g holomorph in $\mathbf{D}$ und $|g(z)|\leqslant 1$ für jedes $z\in\mathbf{D}$ ist.

1. Fall. $|g(z)|<1$ für jedes $z\in\mathbf{D}$.

Da die Funktion v ungerade ist, ist das Integral $\int_{-1}^{+1} v(x)\,dx=0$. Daher ergibt sich sofort die gewünschte Ungleichung:

$$\left|\int_{-1}^{+1} f(x)\,dx\right| = \left|\int_{-1}^{+1} w(x)\,dx\right| \leqslant \int_{-1}^{+1} |x^2 g(x)|\,dx < \int_{-1}^{+1} x^2\,dx = \frac{2}{3}.$$

2. Fall. $|g(z_0)|=1$ für ein $z_0\in\mathbf{D}$.

Nach dem Maximumprinzip ist demnach $|g(z)|=1$ für jedes $z\in\mathbf{D}$, also $g(z)\equiv\text{const}=\alpha$ mit $|\alpha|=1$.

Die Funktion f lässt sich daher nach (3) in der Form

$$f(z)=\alpha z^2+v(z)$$

darstellen.

Beachtet man, dass v ungerade ist, so gilt für jedes $z\in\mathbf{D}$.

$$|\alpha z^2+v(z)|=|f(z)|\leqslant 1$$
$$|\alpha z^2-v(z)|=|f(-z)|\leqslant 1.$$

Quadrieren und Addition ergibt wegen $|\alpha|=1$:

$$|z|^4+|v(z)|^2\leqslant 1 \qquad (z\in\mathbf{D}). \tag{4}$$

Die Ungleichung (4) impliziert jedoch, dass $v(z)$ gleichmässig gegen Null geht, falls z gegen den Rand von $\mathbf{D}$ strebt. Nach dem Maximumprinzip ist also v identisch Null. Somit hat f die Gestalt

$$f(z)=\alpha z^2,$$

mit $|\alpha|=1$.

R. Mortini, Karlsruhe, BRD

4988. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty}\left[(2n-1)\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\right)-2\right].$$

Solution to problem 4988 Crux Math. 50 (9) 2024, 466

Raymond Mortini, Rudolf Rupp

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We show that

$$S:=\sum_{n=1}^{\infty}\left[(2n-1)\sum_{j=0}^{\infty}\frac{1}{(j+n)^2}-2\right]=-\frac{1}{2}.$$

The series $\displaystyle\sum_{j=0}^{\infty}\frac{1}{(j+x)^2}$, denoted by $\psi^{(1)}(x)$ or $\zeta(2,x)$, is sometimes called the trigamma function. It is a special Hurwitz-zeta function. It can be represented as an integral: if $x>0$, then

$$\psi^{(1)}(x)=\int_0^{\infty}\frac{te^{(1-x)t}}{e^t-1}\,dt.$$

Now, via induction (thanks to the software wolframalpha; attention: that software says that the series diverges, sic!)

$$\sum_{n=1}^{N}[(2n-1)\psi^{(1)}(n)-2]=-N+N^2\psi^{(1)}(N+1).$$

Thus it remains to show that

$$I_N:=\left(-N+N^2\int_0^{\infty}\frac{t}{e^t-1}\,e^{-Nt}dt\right)\to-\frac{1}{2}.$$

This is done using twice partial integration:

$$\begin{aligned}
I_N &= -N+N^2\left[-\frac{e^{-Nt}}{N}\,\frac{t}{e^t-1}\Big|_0^{\infty}+\frac{1}{N}\int_0^{\infty}e^{-Nt}\,\frac{e^t-1-te^t}{(e^t-1)^2}\,dt\right]\\
&= -N+N^2\left[\frac{1}{N}+\frac{1}{N}\int_0^{\infty}e^{-Nt}\,\frac{e^t-1-te^t}{(e^t-1)^2}\,dt\right]=N\int_0^{\infty}e^{-Nt}\,\frac{e^t-1-te^t}{(e^t-1)^2}\,dt\\
&= N\left[-\frac{e^{-Nt}}{N}\,\frac{e^t-1-te^t}{(e^t-1)^2}\Big|_0^{\infty}+\frac{1}{N}\int_0^{\infty}e^{-Nt}\,\underbrace{\frac{-(e^t-1)te^t-(e^t-1-te^t)2e^t}{(e^t-1)^3}}_{:=b(t)}\,dt\right]\\
&\overset{t\to 0}{\underset{\text{l'Hospital}}{=}} -\frac{1}{2}+\int_0^{\infty}e^{-Nt}b(t)dt=-\frac{1}{2}+\varepsilon_N.
\end{aligned}$$

Since b is bounded on $]0,\infty[$ and $\int_0^{\infty}e^{-Nt}dt=\frac{1}{N}$, we have that $\varepsilon_N\to 0$.

4965. *Proposed by Ángel Plaza.*

Prove that the following identities hold:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+4}-\cdots\right)=\frac{\ln^2 2}{8}-\frac{7\pi^2}{96}$,

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+4}-\cdots\right)=\frac{\ln^2 2}{8}+\frac{11\pi^2}{96}$,

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We observe that the sign in a) is false.

Step 1 *Reduction to the computation of an integral*

We use that for $a>0$,

$$\frac{1}{a}=\int_0^\infty e^{-ax}dx.$$

Since the moduli of the partial sums $\sum_{k=0}^{N} e^{-nx}(-1)^k e^{-2kx}$ are bounded by the $L^1([0,\infty[)$-function $2e^{-nx}$, we have $\sum\int=\int\sum$, and so

$$\begin{aligned} S_n:=\frac{1}{n}-\frac{1}{n+2}+\dots &= \sum_{k=0}^{\infty}\frac{(-1)^k}{n+2k}=\sum_{k=0}^{\infty}(-1)^k\int_0^\infty e^{-(n+2k)x}dx \\ &= \int_0^\infty e^{-nx}\sum_{k=0}^{\infty}(-1)^k e^{-2kx}dx=\int_0^\infty \frac{e^{-nx}}{1+e^{-2x}}dx. \end{aligned}$$

Now let

$$A:=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{n}-\frac{1}{n+2}+\dots\right)$$

and

$$B:=\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+2}+\dots\right).$$

Note that S_n is decreasing, so the series A converges (Leibniz rule). But actually,

$$0\le S_n\le\int_0^\infty e^{-nx}dx=\frac{1}{n}.$$

So the series A and B converge absolutely as the general term is $\mathcal{O}(1/n^2)$.

Due to the boundedness of the partial sums of the series $\sum_{n=1}^{\infty}\frac{1}{n}e^{-nx}$ by the $L^1(]0,\infty[)$-function $|\log(1-e^{-x})|$, we have $\sum\int=\int\sum$ in both cases. Hence

$$\begin{aligned} B &= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty\frac{e^{-nx}}{1+e^{-2x}}dx=\int_0^\infty\frac{1}{1+e^{-2x}}\sum_{n=1}^{\infty}\frac{1}{n}e^{-nx}dx \\ &= -\int_0^\infty\frac{\log(1-e^{-x})}{1+e^{-2x}}dx \underset{x\mapsto e^{-x}}{=} -\int_0^1\frac{\log(1-x)}{x(1+x^2)}\,dx. \end{aligned}$$

$$\begin{aligned} A &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^\infty\frac{e^{-nx}}{1+e^{-2x}}dx=\int_0^\infty\frac{1}{1+e^{-2x}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-nx}dx \\ &= \int_0^\infty\frac{\log(1+e^{-x})}{1+e^{-2x}}dx \underset{x\mapsto e^{-x}}{=} \int_0^1\frac{\log(1+x)}{x(1+x^2)}\,dx. \end{aligned}$$

Step 2 *Calculating the integrals*

To evaluate the integrals, we calculate $I := A + B$ and $J := A - B$ giving $A = (I+J)/2$ and $B = (I-J)/2$.

$$\begin{aligned}
I &= \int_0^1 \frac{\log\left(\frac{1+x}{1-x}\right)}{x(1+x^2)}dx \overset{\frac{1-x}{1+x}=t}{\underset{x=\frac{1-t}{1+t}}{=}} \int_0^1 \frac{-\log t}{\left(\frac{1-t}{1+t}\right)\left(1+\left(\frac{1-t}{1+t}\right)^2\right)} \frac{2dt}{(1+t)^2} \\
&= -2\int_0^1 \frac{(1+t)\log t}{(1-t)\big((1+t)^2+(1-t)^2\big)}dt = -2\int_0^1 \frac{(1+t)\log t}{(1-t)(2+2t^2)}dt \\
&= -\int_0^1 \frac{(1+t)\log t}{(1-t)(1+t^2)}dt = -\left(\int_0^1 \frac{\log t}{1-t}dt + \int_0^1 \frac{t\log t}{1+t^2}dt\right) \\
&\overset{s=t^2}{=} -\int_0^1 \frac{\log t}{1-t}dt - \frac{1}{4}\int_0^1 \frac{\log s}{1+s}ds.
\end{aligned}$$

$$\begin{aligned}
J &= \int_0^1 \frac{\log(1-x^2)}{x(1+x^2)}dx \overset{x^2=s}{=} \frac{1}{2}\int_0^1 \frac{\log(1-s)}{s(1+s)}ds \\
&= \frac{1}{2}\int_0^1 \frac{\log(1-s)}{s}ds - \frac{1}{2}\int_0^1 \frac{\log(1-s)}{1+s}ds \\
&= \frac{1}{2}\int_0^1 \frac{\log t}{1-t}dt - \frac{1}{2}\int_0^1 \frac{\log t}{2-t}dt.
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^1 \frac{\log t}{2-t}dt &\overset{t=2u}{=} \int_0^{1/2} \frac{\log(2u)}{2-2u}2du = \int_0^{1/2} \frac{\log 2 + \log u}{1-u}du \\
&= -(\log 2)\Big[\log(1-u)\Big]_0^{1/2} + \int_0^{1/2} \frac{\log u}{1-u}\,du \\
&= \log^2 2 + \int_0^{1/2} \frac{\log u}{1-u}\,du.
\end{aligned}$$

Hence, by using that for $0 < u \le 1$, $-\int_0^u \frac{\log(1-x)}{x}dx = \mathrm{Li}_2(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^2}$ (di-logarithm), we obtain with $0 < v < 1$,

$$\begin{aligned}
\int_0^v \frac{\log x}{1-x}dx &\underset{x=1-s}{=} \int_{1-v}^1 \frac{\log(1-s)}{s}ds = \int_0^1 \frac{\log(1-s)}{s}ds - \int_0^{1-v} \frac{\log(1-s)}{s}ds \\
&= -\mathrm{Li}_2(1) + \mathrm{Li}_2(1-v).
\end{aligned}$$

Consequently, by additionally using that

$$\int_0^1 \frac{\log x}{1+x}dx = \sum_{n=0}^{\infty}\int_0^1 (-1)^n(\log x)x^n dx = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(n+1)^2} = -\frac{\pi^2}{12},$$

$$\begin{aligned}
2A = I + J &= \left[-\int_0^1 \frac{\log x}{1-x}dx - \frac{1}{4}\int_0^1 \frac{\log x}{1+x}dx\right] + \left[\frac{1}{2}\int_0^1 \frac{\log x}{1-x}dx - \frac{1}{2}\left(\log^2 2 + \int_0^{1/2} \frac{\log x}{1-x}\right)\right] \\
&= -\frac{1}{2}\log^2 2 + \frac{1}{2}\,\mathrm{Li}_2(1) - \frac{1}{2}\,\big(-\mathrm{Li}_2(1) + \mathrm{Li}_2(\frac{1}{2})\big) - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2} \\
&= -\frac{1}{2}\log^2 2 + \frac{\pi^2}{6} - \frac{1}{2}\left(\frac{\pi^2}{12} - \frac{\log^2 2}{2}\right) + \frac{\pi^2}{48} \\
&= -\frac{1}{4}\log^2 2 + \frac{7}{48}\pi^2,
\end{aligned}$$

and

$$\begin{aligned}
2B = I - J &= \left[-\int_0^1 \frac{\log x}{1-x}dx - \frac{1}{4}\int_0^1 \frac{\log x}{1+x}dx\right] - \left[\frac{1}{2}\int_0^1 \frac{\log x}{1-x}dx - \frac{1}{2}\left(\log^2 2 + \int_0^{1/2} \frac{\log x}{1-x}\right)\right] \\
&= \frac{1}{2}\log^2 2 + \frac{3}{2}\,\mathrm{Li}_2(1) + \frac{1}{2}\,\big(-\mathrm{Li}_2(1) + \mathrm{Li}_2(\frac{1}{2})\big) - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2} \\
&= \frac{1}{2}\log^2 2 + \frac{\pi^2}{6} + \frac{1}{2}\left(\frac{\pi^2}{12} - \frac{\log^2 2}{2}\right) + \frac{\pi^2}{48} \\
&= \frac{1}{4}\log^2 2 + \frac{11}{48}\pi^2.
\end{aligned}$$

Consequently

$$A = -\frac{1}{8}\log^2 2 + \frac{7\pi^2}{96} \sim 0.659602\ldots$$

and

$$B = \frac{1}{8}\log^2 2 + \frac{11\pi^2}{96} \sim 1.19095\ldots$$

4959. *Proposed by Jeromin Rocklage.*

For $\alpha > 0$, evaluate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \left(\frac{k}{2n}\right)^\alpha .$$

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For $\alpha > 0$, let $S_n := \sum_{k=1}^{2n} (-1)^k \left(\frac{k}{2n}\right)^\alpha$. We show that $\boxed{\lim_{n\to\infty} S_n = \frac{1}{2}.}$

We will use the higher order Euler-MacLaurin formula applied to the function $f(x) = x^\alpha$. This formula tells us that for $n \geq m$,

$$\sum_{i=m}^{n} f(i) - \int_m^n f(x)dx \quad = \quad \frac{f(m)+f(n)}{2} + \frac{1}{6}\frac{f'(n)-f'(m)}{2!} - \int_m^n f''(x)\frac{P_2(x)}{2!}dx$$

where P_2 is the periodized Bernoulli function

$$P_2(x) = B_2(x - \lfloor x \rfloor),$$

and $B_2(x) = x^2 - x + \frac{1}{6}$, the second Bernoulli polynomial.

Let us first rewrite S_n, splitting the sum into odd and even indices:

$$\begin{aligned} S_n = \sum_{k=1}^{2n} (-1)^k \left(\frac{k}{2n}\right)^\alpha &= \sum_{j=1}^{n} \left(\frac{j}{n}\right)^\alpha - \sum_{j=1}^{n} \left(\frac{2j-1}{2n}\right)^\alpha \\ &= \frac{1}{n^\alpha}\sum_{j=1}^{n} j^\alpha - \frac{1}{(2n)^\alpha}\sum_{j=1}^{n}(2j-1)^\alpha. \end{aligned}$$

Case 1 $\alpha = 1$. Then

$$\begin{aligned} S_n &= \sum_{k=1}^{2n} (-1)^k \frac{k}{2n} = \frac{1}{n}\sum_{j=1}^{n} j - \frac{1}{2n}\sum_{j=1}^{n}(2j-1) \\ &= \frac{1}{n}\sum_{j=1}^{n} j - \frac{1}{n}\sum_{j=1}^{n} j + \frac{1}{2n}\cdot n = \frac{1}{2}. \end{aligned}$$

Case 2 $\alpha \neq 1$.

By choosing $m = 1$ in the Euler-MacLaurin formula, and estimating

$$\left|\int_1^n f''(x)\frac{P_2(x)}{2!}dx\right| \leq C\int_1^n x^{\alpha-2}dx = \begin{cases} \frac{C}{\alpha-1}\left(n^{\alpha-1}-1\right) & \text{if } \alpha > 1 \\ C\log n & \text{if } \alpha = 1 \\ \frac{C}{1-\alpha}\left(1-n^{\alpha-1}\right) & \text{if } 0 < \alpha < 1, \end{cases}$$

we obtain

$$\sum_{j=1}^{n} j^\alpha = \left(\frac{n^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1}\right) + \frac{n^\alpha+1}{2} + \frac{1}{12}\left(\alpha n^{\alpha-1} - \alpha\right) + \begin{cases} \mathcal{O}(n^{\alpha-1}) & \text{if } \alpha > 1 \\ \mathcal{O}(1) & \text{if } 0 < \alpha < 1. \end{cases}$$

Hence

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{j=1}^{n} j^\alpha &= \left(\frac{n}{\alpha+1}+\frac{1}{2}+\frac{\alpha}{12}\frac{1}{n}\right)+\mathcal{O}(n^{-\alpha})+\begin{cases}\mathcal{O}(n^{-1}) & \text{if } \alpha>1\\ \mathcal{O}(n^{-\alpha}) & \text{if } 0<\alpha<1\end{cases}\\ &= \frac{n}{\alpha+1}+\frac{1}{2}+o(1).\end{aligned}$$

Using, additionally, the formula

$$\begin{aligned}\frac{1}{(2n)^\alpha}\sum_{j=1}^{n}(2j-1)^\alpha &= \frac{1}{(2n)^\alpha}\left(\sum_{j=1}^{2n} j^\alpha - 2^\alpha\sum_{j=1}^{n} j^\alpha\right)\\ &= \frac{1}{(2n)^\alpha}\sum_{j=1}^{2n} j^\alpha - \frac{1}{n^\alpha}\sum_{j=1}^{n} j^\alpha\end{aligned}$$

we conclude that

$$\begin{aligned}S_n &= \left(\frac{n}{\alpha+1}+\frac{1}{2}\right)-\left(\frac{2n}{\alpha+1}+\frac{1}{2}-\left(\frac{n}{\alpha+1}+\frac{1}{2}\right)\right)+o(1)\\ &= \frac{1}{2}+o(1).\end{aligned}$$

4953. *Proposed by Michel Bataille.*

Let $a \in \mathbb{R} - \{0, 1\}$. Find all a and all functions $f : \mathbb{R} \to \mathbb{R}$ that are not identically zero such that

$$(x + y)f(x + y) - xf(x) - yf(y) = a(yf(x) + xf(y))$$

for all x, y.

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We show that under the assumption $a \in \mathbb{R} \setminus \{0, 1\}$, the number $\boxed{a = 3}$ is the only one for which the functional equation

(63) $$(x + y)f(x + y) - xf(x) - yf(y) = a(yf(x) + xf(y))$$

equivalently

(64) $$(x + y)f(x + y) = f(x)(x + ay) + f(y)(y + ax)$$

has non-trivial solutions. These are given by $\boxed{f(x) = cx^2}$ for $c \neq 0$.

It is obvious that the set of solutions of (63) is a vector space; may be the trivial one $\{0\}$. Moreover, if f is a solution then $f(0) = 0$: put $y = 0$, and so, $xf(x) = xf(x) + f(0)ax$. As $a \neq 0$, $f(0) = 0$.

Note that if $a = 3$, then $f(x) = x^2$ is a solution since

$$(x + y)^3 - x^3 - y^3 - 3yx^2 - 3xy^2 = 0.$$

Now suppose that f is a solution whenever $a = 3$. Then f is even. In fact, putting $y = -x$, and using that $f(0) = 0$, yields $0 = f(x)(-2x) + f(-x)(2x)$ and so $f(-x) = f(x)$.

Next assume that $f(1) = 1$ and let us introduce the auxiliary function $h(x) := f(x) - x^2$. Then h is a solution, too, satisfying $h(0) = h(1) = 0$. We show that $h \equiv 0$ (yielding the assertion that $f(x) = x^2$, and so, if $f(1) = c \neq 0$, $f(x) = cx^2$, and if $f(1) = 0$, $f \equiv 0$).

To see this, take $y = 1$ and $x \to -x$. Then

$$(-x + 1)h(-x + 1) + xh(-x) = 3h(-x),$$

hence, by using that h is even,

(65) $$h(1 - x) = \frac{3 - x}{1 - x}h(x), \quad x \neq 1.$$

On the other hand, if $x + y = 1$, then by (64),

$$0 = h(x)(x + 3(1 - x)) + h(1 - x)(1 - x + 3x).$$

That is

(66) $$h(1 - x) = \frac{2x - 3}{2x + 1}\, h(x), \quad x \neq -\frac{1}{2}.$$

Consequently,

$$\frac{2x - 3}{2x + 1}h(x) = \frac{3 - x}{1 - x}\, h(x).$$

Since $\frac{2x-3}{2x+1} = \frac{3-x}{1-x} \iff -3 = 3$ is impossible, $h(x) = 0$ for $x \notin \{1, -1/2\}$. By (65), $0 = h(1/2) = h(-1/2)$. As $h(1) = 0$ we are done.

It remains to see that a necessarily is 3 for the existence of nontrivial solutions. So let f be a solution to (63), $f \not\equiv 0$.

• Put $y = x$. Then $2xf(2x) - 2xf(x) = 2axf(x)$, implying that

$$f(2x) = (1 + a)f(x),$$

valid also for $x = 0$.

• Put $y = 2x$. Then $3xf(3x) - xf(x) - 2xf(2x) = a(2xf(x) + xf(2x))$. Hence

$$3f(3x) = (1 + 2a)f(x) + (2 + a)f(2x),$$

valid also for $x = 0$.

Pulling in $f(2x)$ gives

$$3f(3x) = (2a + 1)f(x) + (2 + a)(1 + a)f(x) = (3 + 5a + a^2)f(x).$$

• Put $y = 3x$. Then $4xf(4x) - xf(x) - 3xf(3x) = a(3xf(x) + xf(3x))$. Hence

$$4f(4x) = (1 + 3a)f(x) + (3 + a)f(3x),$$

valid also for $x = 0$. Since $f(4x) = f(2(2x)) = (1 + a)f(2x) = (1 + a)^2 f(x)$, we obtain with the formula for $f(3x)$ that

$$4(1 + a)^2 f(x) = (1 + 3a)f(x) + \frac{1}{3}\,(3 + a)(3 + 5a + a^2)f(x).$$

Consequently, as $f \not\equiv 0$,

$$3(3 + 5a + 4a^2) - (3 + a)(3 + 5a + a^2) = 0,$$

or equivalently

$$a(a - 1)(a - 3) = 0.$$

Thus $a = 3$.

Remark What happens for $a = 0$ or $a = 1$? If $a = 0$, then the solutions f are those functions for which xf is additive. If $a = 1$ then the solutions are exactly the additive functions: in fact, by (64), $(x+y)f(x+y) = f(x)(x+y)+f(y)(y+x)$. Hence, if $x+y \neq 0$, $f(x+y) = f(x)+f(y)$. Now let $x + y = 0$. Since $f(2x) = f(x + x) = f(x) + f(x) = 2f(x)$ we conclude that for $x \neq 0$,

$$\begin{aligned} f(x) + f(-x) &= f((x-1)+1) + f((-1-x)+1) = f(x-1) + f(1) + f(-1-x) + f(1) \\ &= f((x-1) + (-1-x)) + 2f(1) = f(-2) + 2f(1) \\ &= 2(f(-1) + f(1)). \end{aligned}$$

In particular, if $x = 1$, we deduce that $f(-1) + f(1) = 0$, and so

$$f(x) + f(-x) = 0 = f(0) = f(x + (-x)).$$

Remark It is quite astonishing for us that in the these two exceptional cases there are so many solutions, as the set of additive functions is in a one to one correspondance with the $\mathbb{Q}$-linear functions on the $\mathbb{Q}$-vector space V determined by the real numbers.

4951. *Proposed by Michel Bataille.*

Let n be a positive integer. Prove that the sums

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}^{-1} \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k-1}$$

are equal and find their common value.

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We show that the common value is $\boxed{\dfrac{1+(-1)^{n+1}}{n+1}}$.

Let

$$P_1(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} x^{k-1}.$$

To be calculated is $P_1(1)$. To this end, note that

$$\begin{aligned} (xP_1(x))' &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k-1} x^{k-1} = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} x^j \\ &= -(-x)^n + \sum_{j=0}^{n} \binom{n}{j} (-x)^j = (1-x)^n - (-x)^n. \end{aligned}$$

Hence

$$xP_1(x) = \int_0^x \left((1-t)^n - (-t)^n\right) dt,$$

from which we conclude that

$$P_1(x) = \frac{1}{x(n+1)} \left(-(1-x)^{n+1} + (-1)^{n+1} x^{n+1} + 1\right).$$

Consequently $P_1(1) = \dfrac{1+(-1)^{n+1}}{n+1}$.

To calculate $P_2 := \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}^{-1}$, we use that (see below)

$$\frac{1}{k}\binom{n}{k}^{-1} = \beta(k, n-k+1) = \int_0^1 (1-x)^{k-1} x^{n-k} dx =: I.$$

Hence

$$\begin{aligned} P_2 &= \int_0^1 \sum_{k=1}^{n} (x-1)^{k-1} x^{n-k} dx = \int_0^1 \sum_{j=0}^{n-1} (x-1)^j x^{n-j-1} dx \\ &= -\int_0^1 \left((x-1)^n - x^n\right) dx = \frac{1+(-1)^{n+1}}{n+1}. \end{aligned}$$

Addendum A classical formula tells us that for $\mu, \nu \in \mathbb{N}$ one has

$$\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma\big((\mu+1)+(\nu+1)\big)} = \beta(\mu+1, \nu+1) := \int_0^1 (1-x)^\mu x^\nu dx = \frac{\mu!\,\nu!}{(\mu+\nu+1)!}.$$

Hence, for $\mu = k-1$ and $\nu = n-k$,

$$I = \beta(k, n-k+1) = \frac{(k-1)!\,(n-k)!}{n!} = \frac{1}{k}\frac{k!\,(n-k)!}{n!} = \frac{1}{k}\binom{n}{k}^{-1}.$$

4937. *Proposed by Michel Bataille.*

Let a be a positive real number and let $f:[0,\infty)\to\mathbb{R}$ be a continuous, a-periodic function. Prove that if $b>a$, then

$$\int_0^b\int_0^a \frac{f(x+y)}{x+y}\,dxdy = a\int_a^{a+b}\frac{f(t)}{t}\,dt + b\int_b^{a+b}\frac{f(t)}{t}\,dt.$$

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Let $F(y)=\displaystyle\int_a^y \frac{f(t)}{t}dt$ be a primitive of $f(t)/t$ on $]0,\infty[$. Note that $F(y)$ may be unbounded near 0 (for $f\equiv 1$ for instance). But one can control its behaviour near 0:

$$R(y):=y(F(a+y)-F(y))\to 0 \text{ as } y\to 0.$$

In fact, as f is continuous, $|f|\le M$ on $[0,a]$. Hence, for $0<y\le a$,

$$y|F(y)|\le My\int_y^a \frac{1}{t}dt = My\log a - My\log y\to 0,$$

and so $R(y)\to 0$ as F is continuous at a.

Using the transformation $x+y=t$ with $y\le t\le a+y$, we obtain for $\delta>0$ close to 0,

$$\begin{aligned}
\int_\delta^b\left(\int_0^a \frac{f(x+y)}{x+y}dx\right)dy &= \int_\delta^b\left(\int_y^{a+y}\frac{f(t)}{t}dt\right)dy = \int_\delta^b (F(a+y)-F(y))\,dy\\
&= \Big[y(F(a+y)-F(y))\Big]_\delta^b - \int_\delta^b y(F'(y+a)-F'(y))dy\\
&= b(F(a+b)-F(b)) - \underbrace{\delta(F(a+\delta)-F(\delta))}_{=R(\delta)} - \int_\delta^b y\left(\frac{f(a+y)}{a+y}-\frac{f(y)}{y}\right)dy\\
&\underset{\delta\to 0}{\longrightarrow} b(F(a+b)-F(b)) - \int_0^b \frac{a+y-a}{a+y}f(a+y)dy + \int_0^b f(y)dy\\
&= b(F(a+b)-F(b)) - \int_0^b f(a+y)dy + a\int_0^b\frac{f(a+y)}{a+y}dy + \int_0^b f(y)dy\\
&\overset{f\ a\text{-periodic}}{\underset{s=a+y}{=}} b(F(a+b)-F(b)) + a\int_a^{a+b}\frac{f(s)}{s}ds\\
&= b\left(\int_a^{a+b}\frac{f(t)}{t}dt - \int_a^b\frac{f(t)}{t}dt\right) + a\int_a^{a+b}\frac{f(s)}{s}ds\\
&= b\int_b^{a+b}\frac{f(t)}{t}dt + a\int_a^{a+b}\frac{f(s)}{s}ds.
\end{aligned}$$

4924. *Proposed by Yagub N. Aliyev.*

Let n be a positive integer. Find all possible values of $x \geq 0$ for which the inequality

$$1+\frac{n}{B} \leq \left(1+\frac{1}{B}\right)\left(1+\frac{1}{B+x}\right)\left(1+\frac{1}{B+2x}\right)\dots\left(1+\frac{1}{B+(n-1)x}\right),$$

holds true for all $B>0$. For which $x \geq 0$ is the reverse inequality true?

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We claim that for all $B>0$ and $n \geq 2$

$$\boxed{1+\frac{n}{B} \leq \prod_{j=0}^{n-1}\left(1+\frac{1}{B+jx}\right) \text{ if } 0 \leq x \leq 1}$$

and

$$\boxed{1+\frac{n}{B} \geq \prod_{j=0}^{n-1}\left(1+\frac{1}{B+jx}\right) \text{ if } x \geq 1.}$$

In particular, we have equality for $x=1$. Moreover, equality holds for all $B>0$ and all $x \geq 0$ if $n=1$.

Let L_n be the left hand side and R_n the right hand side.

• $n=1$. Then

$$L_1-R_1=1+\frac{1}{B}-(1+\frac{1}{B})=0.$$

• $n \geq 2$. We show the assertion above via induction on n. So let $n=2$.

$$\begin{aligned} L_2-R_2 &= 1+\frac{2}{B}-\left(1+\frac{1}{B}\right)\left(1+\frac{1}{B+x}\right) \\ &= \frac{x-1}{B(B+x)}. \end{aligned}$$

Hence the assertion is true for $n=2$.

• $n \to n+1$. Assume that the assertion is correct for some $n \in \mathbb{N}$. Then, for $x \geq 1$, $L_n \geq R_n$, and so

$$\begin{aligned} L_{n+1}-R_{n+1} &= 1+\frac{n+1}{B}-\prod_{j=0}^{n}\left(1+\frac{1}{B+jx}\right) \\ &\geq 1+\frac{n+1}{B}-\left(1+\frac{n}{B}\right)\left(1+\frac{1}{B+nx}\right) \\ &= \frac{1}{B}\left(1-\frac{B+n}{B+nx}\right)=\frac{1}{B}\,\frac{n(x-1)}{B+nx} \geq 0. \end{aligned}$$

The same estimates replacing $\geq$ by $\leq$ show that we also get the assertion for $0 \leq x \leq 1$.

4930. *Proposed by Toyesh Prakash Sharma.*

For positive integers a, b, c, show that

$$a^b b^c c^a \leq \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}.$$

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First we note that for $x, y \geq 0$, $xy \leq \frac{1}{2}(x^2+y^2)$. Hence

$$ab+bc+ca \leq \frac{1}{2}(a^2+b^2)+\frac{1}{2}(b^2+c^2)+\frac{1}{2}(c^2+a^2) = a^2+b^2+c^2.$$

As $\log x$ is concave, we know that $\log(\sum_{j=1}^n \varepsilon_j x_j) \geq \sum_{j=1}^n \varepsilon_j \log x_j$ whenever $\sum_{j=1}^n \varepsilon_j = 1$, $\varepsilon_j \geq 0$. Hence

$$\begin{aligned}\frac{b}{a+b+c}\log a + \frac{c}{a+b+c}\log b + \frac{a}{a+b+c}\log c &\leq \log\left(\frac{b}{a+b+c}a + \frac{c}{a+b+c}b + \frac{a}{a+b+c}c\right)\\ &\leq \log\left(\frac{a^2+b^2+c^2}{a+b+c}\right).\end{aligned}$$

Hence

$$a^b b^c c^a \leq \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}.$$

4925. *Proposed by Ivan Hadinata.*

Determine all possible real numbers a for which there exists a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y) + xf(f(y)) = f(f(x)) + f(y) + axy$$

for all $x, y \in \mathbb{R}$.

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We show that the functional equation

$$f(x+y) + xf(f(y)) = f(f(x)) + f(y) + axy \tag{67}$$

admits a solution if and only if $a \in \{0, 1\}$. If $a = 1$, then the identity is the only solution and if $a = 0$, then the zero function is the only solution.

Put $x = y = 0$. Then $f(0) + 0 = f(f(0)) + f(0)$ implies that $f(f(0)) = 0$. Now put $y = 0$ in (67). Then

$$f(x) = f(f(x)) + f(0). \tag{68}$$

This yields the new equation

$$f(x+y) + x\big(f(y) - f(0)\big) = f(x) - f(0) + f(y) + axy. \tag{69}$$

Now put $x = 1$. Then

$$f(1+y) + f(y) - f(0) = f(1) - f(0) + f(y) + ay, \tag{70}$$

from which we conclude that $f(y+1) = f(1) + ay$, or in other words,

$$f(u) = f(1) + a(u-1) =: au + b, \tag{71}$$

that is, f is linear-affine.

Since $f(f(0)) = 0$, we have $a(a \cdot 0 + b) + b = 0$, and so $ab + b = 0$. Thus $b = 0$ or $a = -1$. Due to (68),

$$ax + b = a(ax+b) + b + b.$$

That is

$$ax = a^2x + (ab + b) = a^2x,$$

from which we deduce $a \in \{0, 1\}$. Hence, as $a \neq -1$, $b = 0$. Consequently, $f(x) = x$ (if $a = 1$) or $f \equiv 0$ if $a = 0$. It is straightforward to check that these are actually solutions.

4929. *Proposed by Seán M. Stewart.*

Evaluate

$$\int_0^1 \frac{\log(1+\sqrt{1-u^2})}{1+u}\, du.$$

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We show that

$$\boxed{I := \int_0^1 \frac{\log(1+\sqrt{1-u^2})}{1+u}\, du = \frac{\pi^2}{24} \sim 0.4112335167\cdots}.$$

In fact,

$$I \quad = \quad \int_0^1 \log\left(1+\sqrt{1-u^2}\right) \sum_{n=0}^\infty (-1)^n u^n \, du = \sum_{n=0}^\infty (-1)^n \int_0^1 \log\left(1+\sqrt{1-u^2}\right) u^n \, du,$$

where the interchanging $\lim_n \int S_n = \int \lim S_n$ is allowed as the partial sums

$$S_n = \sum_{j=0}^n \log\left(1+\sqrt{1-u^2}\right)(-1)^j u^j$$

are bounded by the $L^1[0,1]$-function $\frac{\log\left(1+\sqrt{1-u^2}\right)}{1+u}$. Now put

$$I_n := \int_0^1 \log\left(1+\sqrt{1-u^2}\right) u^n \, du,$$

and use partial integration with $f(u) = \log\left(1+\sqrt{1-u^2}\right)$ and $g'(u) = u^n$. Note that $g(u) = \frac{u^{n+1}}{n+1}$ and

$$f'(u) = \frac{\frac{-u}{\sqrt{1-u^2}}}{1+\sqrt{1-u^2}} = -\frac{u(1-\sqrt{1-u^2})}{\sqrt{1-u^2}\, u^2} = -\frac{1}{u\sqrt{1-u^2}} + \frac{1}{u}.$$

Hence

$$\begin{aligned} I_n &= -\frac{1}{n+1}\int_0^1 \left(u^n - \frac{u^n}{\sqrt{1-u^2}}\right) du \\ &\underset{u=\sin t}{=} -\frac{1}{(n+1)^2} + \frac{1}{n+1}\int_0^{\pi/2} \sin^n t \, dt := -\frac{1}{(n+1)^2} + J_n. \end{aligned}$$

Method 1.
By Lemma 14 below,

$$\begin{aligned} I = -\sum_{n=0}^\infty (-1)^n \frac{1}{(n+1)^2} + \sum_{n=0}^\infty (-1)^n J_n &= -\frac{\pi^2}{12} + \sum_{m=0}^\infty J_{2m} - \sum_{m=0}^\infty J_{2m+1} \\ &= -\frac{\pi^2}{12} + \sum_{m=0}^\infty \frac{\binom{2m}{m}}{(2m+1)4^m}\cdot\frac{\pi}{2} - \sum_{m=0}^\infty \frac{4^m}{(2m+1)(2m+2)\binom{2m}{m}} \\ &= -\frac{\pi^2}{12} + \frac{\pi^2}{4} - \frac{\pi^2}{8} = \frac{\pi^2}{24}. \end{aligned}$$

Here we have used the Taylor series (see e.g. [60] and [61]):

$$\arcsin x = \sum_{n=0}^\infty \frac{\binom{2n}{n}}{4^n}\frac{x^{2n+1}}{2n+1}$$

$$
\begin{aligned}
\arcsin^2 x &= 2\sum_{n=0}^{\infty} \frac{4^n x^{2n+2}}{(2n+1)(2n+2)\binom{2n}{n}} \\
&= \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!} x^{2n+2} = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2\binom{2n}{n}} x^{2n}.
\end{aligned}
$$

evaluated at $x = 1$.

Lemma 14. *Let* $I_n := \int_0^{\pi/2} (\sin x)^n dx$. *Then* $I_0 = \pi/2$, $I_1 = 1$ *and for* $n \in \mathbb{N}^*$,

(1) $I_{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \frac{\pi}{2} = \frac{(2n)!}{4^n (n!)^2} \cdot \frac{\pi}{2} = \frac{\binom{2n}{n}}{4^n} \cdot \frac{\pi}{2}$.

(2) $I_{2n+1} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1} = \frac{4^n (n!)^2}{(2n+1)!} = \frac{4^n}{(2n+1)\binom{2n}{n}}$.

Proof. (1) $I_{2n} = \frac{2n-1}{2n} I_{2n-2}$ for $n \in \mathbb{N}^*$ and $I_0 = \frac{\pi}{2}$, because

$$
\begin{aligned}
2nI_{2n} - (2n-1)I_{2n-2} &= \int_0^{\pi/2} (\sin x)^{2n-2} \left(2n \sin^2 x - (2n-1)\right) dx \\
&= -\int_0^{\pi/2} (\sin x)^{2n-2} \left((2n-1)\cos^2 x - \sin^2 x\right) dx \\
&= -\left[(\sin x)^{2n-1} \cos x\right]_0^{\pi/2} = 0.
\end{aligned}
$$

(2) $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$ for $n \in \mathbb{N}^*$ and $I_1 = 1$, because

$$
\begin{aligned}
(2n+1)I_{2n+1} - 2nI_{2n-1} &= \int_0^{\pi/2} (\sin x)^{2n-1} \left((2n+1)\sin^2 x - 2n\right) dx \\
&= -\int_0^{\pi/2} (\sin x)^{2n-1} \left(2n\cos^2 x - \sin^2 x\right) dx \\
&= -\left[(\sin x)^{2n} \cos x\right]_0^{\pi/2} = 0.
\end{aligned}
$$

□

Next we note that for $0 \le x < 1$, $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$. In fact

Lemma 15. *For* $0 \le x < 1$ *we have*

(1) $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

(2) $\dfrac{\arcsin x}{\sqrt{1-x^2}} = \displaystyle\sum_{n=0}^{\infty} x^{2n+1} \int_0^{\pi/2} (\sin t)^{2n+1}\, dt$.

(3) $(\arcsin x)^2 = \displaystyle\sum_{n=0}^{\infty} \left(\frac{\int_0^{\pi/2} (\sin t)^{2n+1} dt}{n+1}\right) x^{2n+2}$.

Proof. (1) Obvious, as $\tan(\arcsin x) = \sin(\arcsin x)/(\cos \arcsin x)$.

(2) As in [60]): we shall use that for $0 \le y < 1$

$$
\frac{y}{1-y^2} = \frac{1}{2}\left(\frac{1}{1-y} - \frac{1}{1+y}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(y^n - (-1)^n y^n) = \sum_{n=0}^{\infty} y^{2n+1}.
$$

Now let

$$
I(t) := \frac{1}{\sqrt{1-x^2}} \ \arctan \frac{x \sin t}{\sqrt{1-x^2}}.
$$

Then, with $0 < x < 1$,

$$
\begin{aligned}
\frac{\arcsin x}{\sqrt{1-x^2}} &= I\left(\frac{\pi}{2}\right) - I(0) = \int_0^{\frac{\pi}{2}} \frac{\partial I}{\partial t} dt = \int_0^{\frac{\pi}{2}} \frac{x \cos t}{1 - x^2 \cos^2 t}\, dt \\
&= \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} (x \cos t)^{2n+1}\, dt = \sum_{n=0}^{\infty} x^{2n+1} \int_0^{\frac{\pi}{2}} (\cos t)^{2n+1}\, dt.
\end{aligned}
$$

Here $\sum\int = \int\sum$, as all the terms are positive.

(3) Just use that $\dfrac{d}{dx}(\arcsin x)^2 = 2\dfrac{\arcsin x}{\sqrt{1-x^2}}$. □

Method 2

Note that

$$I = -\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n+1)^2} + \sum_{n=0}^{\infty}(-1)^n J_n.$$

Consider the auxiliary function

$$h(x) = \sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}} \frac{(\sin t)^n}{n+1}\, dt\right)(-1)^n x^{n+1}.$$

Then

$$\begin{aligned} h'(x) &= \sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}} (\sin t)^n\, dt\right)(-1)^n x^n \\ &= \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty}(-x\sin t)^n\, dt = \int_0^{\frac{\pi}{2}} \frac{1}{1+x\sin t}\, dt \overset{(*)}{=} \frac{\arccos x}{\sqrt{1-x^2}} \end{aligned}$$

Hence,

$$h(1) = h(1) - h(0) = \int_0^1 \frac{\arccos x}{\sqrt{1-x^2}} dx = \left[-\frac{1}{2}\arccos^2 x\right]_0^1 = \frac{\pi^2}{8}.$$

We conclude that $I = -\dfrac{\pi^2}{12} + \dfrac{\pi^2}{8} = \dfrac{\pi^2}{24}$.

To prove (*), just use the transformation $\tan(t/2) = y$ to get

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{1}{1+x\sin t}\, dt &= 2\int_0^1 \frac{1}{1+y^2+2xy} dy = \frac{2}{1-x^2}\int_0^1 \frac{1}{1+\frac{(y+x)^2}{1-x^2}} dy \\ &= 2\left[\frac{\arctan\frac{x+y}{\sqrt{1-x^2}}}{\sqrt{1-x^2}}\right]_0^1 = 2\frac{\arctan\frac{x+1}{\sqrt{1-x^2}} - \arctan\frac{x}{\sqrt{1-x^2}}}{\sqrt{1-x^2}} \\ &= \frac{\arccos x}{\sqrt{1-x^2}}. \end{aligned}$$

The latter is verified by calculating the derivatives of the numerators and by using that for $x=0$, $2\arctan 1 = \pi/2 = \arccos 0$.

Method 3

$$\begin{aligned} f(1) &= \int_0^1\int_0^{\frac{\pi}{2}} \frac{1}{1+\xi\sin t}\, dt d\xi = \int_0^{\frac{\pi}{2}}\int_0^1 \frac{1}{1+\xi\sin t} d\xi dt \\ &= \int_0^{\frac{\pi}{2}} \frac{\log(1+\sin t)}{\sin t} dt \\ &= \int_0^1 \frac{\log(1+y)}{y\sqrt{1-y^2}} dy \\ &= \frac{\pi^2}{8}. \end{aligned}$$

From [60]

For simplicity we will use $\alpha = \sin a$ so let's consider:

$$I(a) := \int_0^{\frac{\pi}{2}} \frac{\ln(1+\sin a \sin x)}{\sin x} dx$$

Note that $\sin a$ is always inside $[-1, 1]$ so it's equivalent to $|\alpha| \le 1$. Also put $x \to \pi - x$, then average the two integrals to see that:

$$2\int_0^{\frac{\pi}{2}} \frac{\ln(1+\sin a \sin x)}{\sin x} dx = \int_0^{\pi} \frac{\ln(1+\sin a \sin x)}{\sin x} dx$$

$$\Rightarrow I(a) = \frac{1}{2}\int_0^{\pi} \frac{\ln(1+\sin a \sin x)}{\sin x} dx \Rightarrow I'(a) = \frac{1}{2}\int_0^{\pi} \frac{\cos a}{1+\sin a \sin x} dx$$

$$\overset{\tan\frac{x}{2}=t}{=} \int_0^{\infty} \frac{\cos a}{1+\sin a \frac{2t}{1+t^2}} \frac{1}{1+t^2} dt = \int_0^{\infty} \frac{\cos a}{(t+\sin a)^2 + \cos^2 a} dt$$

$$= \arctan\left(\frac{t+\sin a}{\cos a}\right)\Big|_0^{\infty} = \frac{\pi}{2} - a$$

Now we integrate to get back:

$$I(a) = \int \left(\frac{\pi}{2} - a\right) da = \frac{\pi a}{2} - \frac{a^2}{2} + C$$

$$I(0) = 0 \Rightarrow C = 0 \Rightarrow I(a) = \frac{a}{2}(\pi - a)$$

4920. *Proposed by Ángel Plaza.*

If $k > 1$ and $n \in \mathbb{N}$, evaluate $\displaystyle\int_0^1 \frac{\log(1+x^k+x^{2k}+\cdots+x^{nk})}{x} dx.$

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Let

$$I := \int_0^1 \frac{\log(1+x^k+x^{2k}+\cdots+x^{nk})}{x}\, dx.$$

We show that

$$\boxed{I = \frac{\pi^2}{6}\frac{n}{k(n+1)}}.$$

For the proof, we use the power series representation

$$\log(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}, \quad |x| < 1.$$

So, if $0 < x < 1$ we have

$$f(x) := \frac{\log(1 + x^k + x^{2k} + \cdots + x^{nk})}{x} = \frac{1}{x} \log\left(\frac{1 - x^{k(n+1)}}{1 - x^k}\right) = -\sum_{j=1}^{\infty} \frac{x^{k(n+1)j-1}}{j} + \sum_{j=1}^{\infty} \frac{x^{kj-1}}{j}.$$

Hence, a primitive is given by

$$-\sum_{j=1}^{\infty} \frac{x^{k(n+1)j}}{j^2 k(n+1)} + \sum_{j=1}^{\infty} \frac{x^{kj}}{j^2 k}.$$

We conclude that

$$I = \int_0^1 f(x)dx = \sum_{j=1}^{\infty} \left(-\frac{1}{k(n+1)} \frac{1}{j^2} + \frac{1}{k} \frac{1}{j^2}\right) = \frac{\pi^2}{6} \left(\frac{1}{k} - \frac{1}{k(n+1)}\right) = \frac{\pi^2}{6} \frac{n}{k(n+1)}.$$

4918. *Proposed by Yagub Aliyev.*

Let $L = \lim_{\lambda\to+\infty} \dfrac{\lambda^{x^2}}{\int_a^b \lambda^{t^2}\,dt}$.

a) Show that if $0 \le a \le x < b$, then $L = 0$.

b) Show that if $0 \le a < x = b$, then $L = +\infty$.

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Let $0 \le a < b$.

a) Suppose that $a \le x < b$. Then for $a \le t \le b$ we have $1 \ge t/b$, and so, for $\lambda > 1$,

$$0 \le I(\lambda) := \frac{\lambda^{x^2}}{\int_a^b \lambda^{t^2}\,dt} \quad \le \quad \frac{b\lambda^{x^2}}{\int_a^b t\lambda^{t^2}\,dt} = \frac{2b(\log\lambda)\,\lambda^{x^2}}{\lambda^{b^2}-\lambda^{a^2}} = \frac{2b(\log\lambda)\,\lambda^{x^2-b^2}}{1-\lambda^{a^2-b^2}}.$$

Since

$$(\log\lambda)e^{(\log\lambda)(x^2-b^2)} = \frac{\log\lambda}{e^{(\log\lambda)(b^2-x^2)}} \to 0 \text{ as } \lambda\to\infty,$$

we have that $\lim_{\lambda\to\infty} I(\lambda) = 0$.

b) Suppose that $x = b$. Then for $a \le t \le b$ we have $1 \le t/a$ and so

$$\begin{aligned} I(\lambda) = \frac{\lambda^{b^2}}{\int_a^b \lambda^{t^2}\,dt} \quad &\ge \quad \frac{a\lambda^{b^2}}{\int_a^b t\lambda^{t^2}\,dt} = \frac{2a(\log\lambda)\,\lambda^{b^2}}{\lambda^{b^2}-\lambda^{a^2}} \\ &\ge \quad \frac{2a(\log\lambda)\,\lambda^{b^2}}{\lambda^{b^2}} = 2a\log\lambda \\ &\to \quad \infty \text{ as } \lambda\to\infty. \end{aligned}$$

4915. *Proposed by Michel Bataille.*

Let $S_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+n+1)}$, where n is a nonnegative integer. Find real numbers a, b, c such that $\lim_{n\to\infty} \left(n^3 S_n - (an^2+bn+c)\right) = 0$.

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We claim that $a = \log 2$, $b = -\frac{1}{2} - \log 2$ and $c = \log 2 + \frac{5}{4}$; that is

$$\lim_{n\to\infty} \left(n^3 S_n - \left((\log 2)n^2 + \left(-\frac{1}{2} - \log 2\right)n + \log 2 + \frac{5}{4}\right)\right) = 0.$$

$$\begin{aligned}
S_n := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+n+1)} &= \frac{1}{n+1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{n+1+k-k}{k(k+n+1)} \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k} - \frac{1}{n+k+1}\right) \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} \left((-1)^{k+1} \int_0^1 x^{k-1} dx - (-1)^{k+1} \int_0^1 x^{n+k} dx\right) \\
&\overset{(*)}{=} \frac{1}{n+1} \int_0^1 \left(\sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1} - \sum_{k=1}^{\infty} (-1)^{k+1} x^{n+k}\right) dx \\
&= \frac{1}{n+1} \int_0^1 \left(\frac{1}{1+x} - x^{n+1} \sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1}\right) dx = \frac{1}{n+1} \int_0^1 \frac{1-x^{n+1}}{1+x} dx \\
&= \frac{\log(2)}{n+1} - \frac{1}{n+1} \int_0^1 \frac{x^{n+1}}{1+x} dx \\
&= \frac{\log(2)}{n+1} - \frac{1}{n+1} \left[\frac{x^{n+2}}{n+2} \frac{1}{1+x}\Big|_0^1 + \frac{1}{n+2} \int_0^1 \frac{x^{n+2}}{(1+x)^2} dx\right] \\
&= \frac{\log(2)}{n+1} - \frac{1}{2(n+1)(n+2)} - \frac{1}{(n+2)(n+1)} \int_0^1 \frac{x^{n+2}}{(1+x)^2} dx \\
&= \frac{\log(2)}{n+1} - \frac{1}{2(n+1)(n+2)} - \frac{1}{(n+2)(n+1)} \left[\frac{x^{n+3}}{n+3} \frac{1}{(1+x)^2}\Big|_0^1 + \frac{2}{n+3} \int_0^1 \frac{x^{n+2}}{(1+x)^3} dx\right] \\
&= \frac{\log(2)}{n+1} - \frac{2n+7}{4(n+1)(n+2)(n+3)} - \frac{2}{(n+1)(n+2)(n+3)} \int_0^1 \frac{x^{n+2}}{(1+x)^3} dx.
\end{aligned}$$

Here the interchanging of $\int\sum = \sum\int$ in $(*)$ is possible since the partial sums

$$\sum_{k=1}^{N}(-1)^{k+1}(x^{k-1} - x^{n+k})$$

are bounded. Hence [14]

$$\begin{aligned}
n^3 S_n &= \frac{n^3}{n+1}\log(2) - \frac{n^3(2n+7)}{4(n+1)(n+2)(n+3)} - \underbrace{\frac{2n^3}{(n+1)(n+2)(n+3)}}_{\leq 2}\underbrace{\int_0^1 \frac{x^{n+2}}{(1+x)^3}dx}_{\leq 1/(n+3)} \\
&= (n^2 - n + 1 + \mathcal{O}(1/n))\log(2) - \frac{n}{2} + \frac{5}{4} + \mathcal{O}(1/n) + \mathcal{O}(1/n) \\
&= n^2\log(2) + n\left(-\frac{1}{2} - \log(2)\right) + \log(2) + \frac{5}{4} + \mathcal{O}(1/n) \\
&= an^2 + bn + c + o(1),
\end{aligned}$$

where the asymptotics are obtained by calculating the partial fraction decomposition of the rational functions in n.

[14] Here $\mathcal{O}$ and o denote the Landau symbols: $\mathcal{O}(1/n)$ is a function $n \mapsto h(n)$ satisfying $\frac{|h(n)|}{1/n} \leq C$ and $o(1)$ is a function $n \mapsto g(n)$ with $\lim_{n\to\infty} g(n) = 0$. In particular, $\mathcal{O}(1/n)$ implies $o(1)$.

4914. *Proposed by Ivan Hadinata.*

Let $\mathbb{R}_{\geq 0}$ be the set of all non-negative real numbers. Find all possible monotonically increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$f(x^2 + y + 1) = xf(x) + f(y) + 1, \quad \forall x, y \in \mathbb{R}_{\geq 0}.$$

Solution to problem 4914 Crux Math. 50 (2) 2024, 82

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We show that the *identity* is the only monotonically increasing solution on $[0, \infty[$.

First we note that $f(x) = x$ trivially satisfies the functional equation on $[0, \infty[$. Now suppose that f is a solution and that f is increasing on $[0, \infty[$.

• Put $y = 0$, respectively $x = 0$. Then

$$f(x^2 + 1) = xf(x) + f(0) + 1 \text{ and } f(y + 1) = f(y) + 1.$$

Hence, with $y = x^2$, respectively $y = 0$,

$$f(x^2 + 1) = f(x^2) + 1 \text{ and } f(1) = f(0) + 1$$

and so

(72) $$f(x^2) + 1 = f(x^2 + 1) = xf(x) + f(1).$$

This implies that, with $x = 1$,

$$f(1) + 1 = 1 \cdot f(1) + f(1) = 2f(1).$$

Hence $f(1) = 1$ and so $f(0) = 0$. Consequently

(73) $$f(x^2) = f(x^2 + 1) - 1 = xf(x) + f(1) - 1 = xf(x).$$

• Next we note that f is right-continuous at $x_1 = 1$. In fact, since f is increasing, f is bounded in a right-neighborhood of $x_0 = 0$ [15]. Hence, if $x \to 0$, we have that $xf(x) \to 0$. Consequently, by the second identity in (72)

$$\lim_{x \to 0} f(x^2 + 1) = f(1) = 1.$$

• Via induction we obtain from $f(x^2) = xf(x)$, or equivalently $f(x) = \sqrt{x}\, f(\sqrt{x})$ (just replace x^2 by x), that

$$\frac{f(\sqrt[2^n]{x})}{\sqrt[2^n]{x}} = \frac{f(x)}{x}.$$

Now $\sqrt[2^n]{x} \to 1$ for $x > 0$ and $\sqrt[2^n]{x} \geq 1$ for $x \geq 1$. Thus, the right-continuity of f at $x_1 = 1$ yields that

$$\frac{f(1)}{1} = \frac{f(x)}{x},$$

from which we conclude that $f(x) = x$ for $x \geq 1$.

Now let $0 \leq x \leq 1$. Then $x + 1 \geq 1$ and so, due to $f(y + 1) = f(y) + 1$,

$$x + 1 = f(x + 1) = f(x) + 1.$$

Consequently $f(x) = x$, too.

[15] For this, it is important that f is defined at 0.

Remark 1 Our proof shows that instead of f being increasing, we may have assumed merely that f is bounded in a right neighborhood of the origin.

Remark 2 A small modification of the proof (see below) shows that any solution to the functional equation

$$(E) \qquad f(x^2+y+1) = xf(x) + f(y) + 1, \quad (x, y \geq 0)$$

actually is additive on $[0, \infty[$; that is satisfies $f(u+v) = f(u)+f(v)$ with $f(0) = 0$ and $f(1) = 1$. Thus we obtain from the well known fact on the Cauchy functional equation (restricted to the non-negative reals) that actually every measurable solution of (E) coincides with the identity.

In fact, $f(x^2) \overset{(73)}{=} xf(x)$ and $f(y+1) = f(y) + 1$ imply that (E) becomes

$$(74) \qquad f(x^2+y+1) = f(x^2) + f(y) + 1 = f(x^2) + f(y+1), \quad (x, y \geq 0),$$

and so

$$f(u+v) = f(u) + f(v) \text{ for } u, v \geq 0,$$

due to the following reason: since for $y \geq 1$, $f(y-1) = f(y) - 1$,

$$\begin{aligned} f(u) + f(v) &= f(u) + f\big((v+1) - 1\big) = f(u) + f(v+1) - 1 \\ &\underset{(74)}{=} f(u + (v+1)) - 1 = f(u+v) + 1 - 1 \\ &= f(u+v). \end{aligned}$$

4913. *Proposed by Albert Natian.*

Suppose the continuous function f satisfies the integral equation

$$\int_0^{xf(7)} f\left(\frac{tx^2}{f(7)}\right) dt = 3f(7)\,x^4.$$

Find $f(7)$.

Solution to problem 4913 Crux Math. 50 (2) 2024, 82

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Let $a \in \mathbb{R}$ and, for $f \in C(\mathbb{R})$, let

$$I_f(x,a) := \int_0^{xa} f(tx^2a^{-1})dt.$$

We show that $\boxed{f(7) = 42 \text{ whenever } I_f(x,a) = 3ax^4.}$

Proof. Using for $x \neq 0$ the linear substitution $t \to u$ with $u := tx^2a^{-1}$ and $dt = ax^{-2}du$, we obtain

$$I_f(x,a) \quad = \quad ax^{-2} \int_0^{x^3} f(u)du.$$

Now $I_f(x,a) = 3ax^4$ if and only if

$$\int_0^{x^3} f(u)du = 3x^6.$$

Differentiating yields

$$3x^2 f(x^3) = 18x^5,$$

equivalently

$$f(x^3) = 6x^3.$$

As $x \mapsto x^3$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}$, we obtain $f(u) = 6u$. Conversely, it is straightforward to check that this f satisfies for every a the given integral equation. So $f(7) = 42$ independently of a.

4910. *Proposed by Paul Bracken.* Let m and n be non-negative integers and let

$$J_{m,n} = \int_0^\infty \left(\left(\frac{\sin t}{t} \right)^m - \left(\frac{\sin t}{t} \right)^n \right) \frac{dt}{t^2}.$$

Prove that the $J_{m,n}$ are rational multiples of π.

Solution to problem 4910 Crux Math. 50 (1) 2024, 38

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It is sufficient to consider the case $n = 0$, otherwise write

$$\int \frac{S^m - S^n}{x^2} dx = \int \frac{S^m - 1}{x^2} dx + \int \frac{1 - S^n}{x^2} dx,$$

where $S = \dfrac{\sin x}{x}$. Since $|S| \leq 1$, we see that $\int_1^\infty \frac{S^m - 1}{x^2} dx$ converges. Now use that

$$1 - S = \frac{x^2}{3!} - \frac{x^4}{5!} \pm \cdots = x^2 \left(\frac{1}{6} + o(x) \right) \quad \text{as} \quad x \to 0,$$

and

$$|S^m - 1| = |S - 1| \left| \sum_{j=0}^{m-1} S^j \right| \leq m|S - 1|,$$

to conclude that $\displaystyle\int_0^1 \frac{S^m - 1}{x^2} dx$ converges, too. Hence

$$I(m) := \int_0^\infty \frac{S^m - 1}{x^2} dx$$

converges. Next we write

$$J := \frac{S^m - 1}{x^2} = \frac{(\sin x)^m - x^m}{x^{m+2}} =: \frac{f(x)}{x^{m+2}}.$$

Now we apply Apostol's method (see [63]). Integration by parts $\int uv' = uv - \int u'v$ with $u = f$ and $v' = x^{-m-2}$ yields:

$$I(m) = \frac{1}{m+1} \int_0^\infty \frac{f'(x)}{x^{m+1}} dx,$$

since $\displaystyle\lim_{x \to 0} \frac{f(x)}{x^{m+1}} = 0$ (and $\displaystyle\lim_{x \to \infty} \frac{f(x)}{x^{m+1}} = 0$) because

$$\left| \frac{f(x)}{x^{m+1}} \right| \quad \leq \quad m \frac{|S - 1|}{x} = mx \left(\frac{1}{6} + o(x) \right) \quad \text{as} \quad x \to 0.$$

Similarly, since 0 is a zero of order 1 of the analytic [16] function $J(z) := \dfrac{(\sin z)^m - z^m}{z^{m+1}}$, we have that for all $j = 0, 1, \ldots, m$

$$\lim_{x \to 0} \frac{f^{(j)}(x)}{x^{m+1-j}} = 0.$$

Hence, by repeating this procedure another m-times, we obtain

$$I(m) = \frac{1}{(m+1)!} \int_0^\infty \frac{f^{(m+1)}(x)}{x} dx.$$

Now $f^{(m+1)}(x) = \frac{d^{m+1}}{dx}(\sin x)^m - 0$. Next we "linearize" the sinus-power:

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[16] Note that $J(z) = z \dfrac{\left(\frac{\sin z}{z}\right)^m - 1}{z^2} = z \left(-\frac{1}{3!} + \frac{z^2}{5!} + \cdots \right) R(z)$, where $\displaystyle\lim_{z \to 0} R(z) = \lim_{z \to 0} \sum_{j=0}^{m-1} \left(\frac{\sin z}{z} \right)^j = m$.

$$\begin{aligned}(\sin x)^m &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^m = \left(\frac{1}{2i}\right)^m \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}e^{ijx}e^{-i(m-j)x}\\ &= (-1)^m\frac{1}{(2i)^m}\sum_{j=0}^{m}(-1)^{-j}\binom{m}{j}e^{ix(2j-m)}.\end{aligned}$$

Since the "constant" term (appearing for $j = m/2$ when m is even) is annihilated by the derivative, we find

$$\begin{aligned}\frac{d^{m+1}}{dx}(\sin x)^m &= (-1)^m(-i)^{m+1}\frac{1}{(2i)^m}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j}e^{-ix(m-2j)}\\ &\quad +(-1)^m i^{m+1}\frac{1}{(2i)^m}\sum_{\frac{m}{2}<j\le m}(2j-m)^{m+1}(-1)^j\binom{m}{j}e^{ix(2j-m)}.\end{aligned}$$

As the left hand side is real, we may take the real part on the right hand side and get (by observing $\operatorname{Re} iz = -\operatorname{Im} z$)

$$\begin{aligned}\frac{d^{m+1}}{dx}(\sin x)^m &= -\frac{1}{2^m}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j}\sin((m-2j)x)\\ &\quad +(-1)^{m+1}\frac{1}{2^m}\sum_{\frac{m}{2}<j\le m}(2j-m)^{m+1}(-1)^j\binom{m}{j}\sin((2j-m)x).\end{aligned}$$

Finally, as

$$\int_0^\infty \frac{\sin(px)}{x}dx = \int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

whenever $p > 0$, we deduce that

$$\begin{aligned}I(m) = \frac{1}{(m+1)!}\int_0^\infty \frac{f^{(m+1)}(x)}{x}dx &= \frac{\pi}{2}\frac{1}{(m+1)!}\Bigg[-\frac{1}{2^m}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j}\\ &\quad +(-1)^{m+1}\frac{1}{2^m}\sum_{\frac{m}{2}<j\le m}(2j-m)^{m+1}(-1)^j\binom{m}{j}\Bigg]\end{aligned}$$

which surely is a rational multiple of π. Making in the second summand the substitution $k = m - j$, then we obtain

$$I(m) = -\frac{\pi}{2^m(m+1)!}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j}.$$

For instance $I(1) = -\frac{\pi}{4}, I(2) = -\frac{\pi}{3}, I(3) = -\frac{13\pi}{32}$.

4909. *Proposed by Michel Bataille.*

For each positive integer n, let $P_n(x) = (x-1)^{2n+1}(x^2-(2n+1)x-1)$. Show that the equation $P_n(x) = 1$ has a unique solution x_n in the interval $(0,\infty)$. Prove that $\lim_{n\to\infty}(x_n - 2n) = 1$ and find $\lim_{n\to\infty} n(x_n - 2n - 1)$.

Solution to problem 4909 Crux Math. 50 (1) 2024, 38

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First we note that P_n is continuous on $[0,\infty[$, $P_n(0) = 1$ and that $\lim_{n\to\infty} P_n(x) = \infty$. Now

$$\begin{aligned} P_n'(x) &= x(x-1)^{2n}\Big((2n+3)x - (4n^2+6n+4)\Big) \\ &= x(x-1)^{2n}(2n+3)\Big(x - \big(2n + \frac{4}{2n+3}\big)\Big). \end{aligned}$$

Hence P_n is strictly decreasing on $\left[0, 2n+\dfrac{4}{2n+3}\right]$ and strictly increasing on $\left[2n+\dfrac{4}{2n+3}, \infty\right]$. Combining all this, and thanks to the intermediate value theorem, we deduce there exists a unique $x_n \in]0,\infty[$ with $P_n(x_n) = 1$. Next we discuss the assymptotics of the sequence (x_n). Since $x_n > 2n + \frac{4}{2n+3}$, we see that $x_n \to \infty$.

- As

$$\begin{aligned} 1 = P_n(x_n) &= (x_n-1)^{2n+1}(x_n^2 - (2n+1)x_n - 1) \\ &= x_n(x_n-1)^{2n+1}(x_n - 2n - 1 - x_n^{-1}) \end{aligned}$$

we get

$$\frac{1}{x_n(x_n-1)^{2n+1}} + \frac{1}{x_n} = x_n - 2n - 1. \tag{75}$$

But $x_n \to \infty$. Thus $x_n - 2n - 1 \to 0$, from which we conclude that $\boxed{x_n - 2n \to 1}$. In particular,

$$\frac{x_n}{n} - 2 = \frac{x_n - 2n}{n} \to 0.$$

- By (75),

$$\begin{aligned} n\,(x_n - 2n - 1) &= \frac{n}{x_n} + \frac{n}{x_n(x_n-1)^{2n+1}} \\ &= \frac{n}{x_n}\left(1 + \frac{1}{(x_n-1)^{2n+1}}\right) \to \frac{1}{2}(1+0) = \frac{1}{2}. \end{aligned}$$

4905. *Proposed by Aravind Mahadevan.*

In a right-angled triangle, the acute angles x and y satisfy the following equation:

$$\tan x + \tan y + \tan^2 x + \tan^2 y + \tan^3 x + \tan^3 y = 70.$$

Find x and y.

Solution to problem 4905 Crux Math. 50 (1) 2024, 37

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We show that $\boxed{(x,y) = \left(\frac{\pi}{12}, \frac{5\pi}{12}\right)}$, or in terms of degrees $\boxed{15^\circ \text{ and } 75^\circ}$.

We may assume that $0 \le x \le y \le \pi/2$ and $x + y = \pi/2$. Now

$$\tan y = \tan\left(\frac{\pi}{2} - x\right) = \cot(x) = \frac{1}{\tan x}.$$

So we have to solve for $t = \tan x$ the equation

$$t + t^2 + t^3 + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} = 70 \tag{76}$$

(or equivalently $1 + t + t^2 - 70t^3 + t^4 + t^5 + t^6 = 0$). Such symmetric equations are solved using the substitution $s := t + 1/t$. Now $s^2 = (t + \frac{1}{t})^2 = t^2 + \frac{1}{t^2} + 2$; hence $t^2 + \frac{1}{t^2} = s^2 - 2$. Moreover,

$$s^3 = \left(t + \frac{1}{t}\right)^3 = t^3 + 3t + \frac{3}{t} + \frac{1}{t^3}$$

and so $t^3 + \frac{1}{t^3} = s^3 - 3s$. This yields the equation $s + s^2 - 2 + s^3 - 3s = 70$, or equivalently $s^3 + s^2 - 2s - 72 = 0$. As $s = 4$ is a solution, we obtain the factorization

$$0 = (s-4)(s^2 + 5s + 18) = (s-4)\left(\left(s + \frac{5}{2}\right)^2 + \frac{47}{4}\right).$$

So $s = 4$ is the only real solution. The equation $4 = t + \frac{1}{t}$ now is equivalent to $t^2 - 4t + 1 = 0$, which has $2 \pm \sqrt{3}$ as solutions. Now we have to calculate the values x for which $\tan x = 2 \pm \sqrt{3}$. As is well known, $\arctan(2 - \sqrt{3}) = \pi/12$ and $\arctan(2 + \sqrt{3}) = 5\pi/12$. This can be verified by using the formulas

$$\sin x = \sqrt{\frac{1 - \cos 2x}{2}} \text{ and } \cos x = \sqrt{\frac{1 + \cos 2x}{2}}.$$

4904. *Proposed by Ivan Hadinata.*

Find all pairs (x, y) of prime numbers x and y such that $x \geq y$, $x+y$ is prime and $x^x + y^y$ is divisible by $x+y$.

Solution to problem 4904 Crux Math. 50 (1) 2024, 37

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This problem has no solution [17]. In fact, suppose that $x+y$ is prime whenever x and y are prime, $y \leq x$. Since the sum of two odd prime numbers is even, it cannot be prime. Hence $y = 2$. Let $n := x$ [18]. Then we have to discuss the property $4 + n^n$ is divisible by $2 + n$.

$$\begin{aligned}
\frac{4+n^n}{2+n} &= \frac{4+((n+2)-2)^n}{2+n} = \frac{4+\sum_{j=0}^{n}(n+2)^j\binom{n}{j}(-1)^{n-j}2^{n-j}}{n+2} \\
&= \frac{4+(-1)^n 2^n + \sum_{j=1}^{n}(n+2)^j\binom{n}{j}(-1)^{n-j}2^{n-j}}{n+2} \\
&= \frac{4+(-1)^n 2^n}{n+2} + \sum_{j=1}^{n}(n+2)^{j-1}\binom{n}{j}(-1)^{n-j}2^{n-j} \\
&=: \frac{4+(-1)^n 2^n}{n+2} + m,
\end{aligned}$$

where $m \in \mathbb{Z}$ (note that the binomial coefficients belong to $\mathbb{N}$). If $n \geq 3$ is odd, then $2+n$ is odd and therefore $2+n$ cannot divide (in $\mathbb{Z}$) the even number $4 + (-1)^n 2^n$. Hence the primeness of n implies that $n = 2$. Since $x + y = 2 + 2 = 4$ is not prime, the pair $(2, 2)$ is not a solution either.

[17] Under the usual assumption that the number 1 is not considered as a prime number.

[18] The symbol x for a natural number hurts my eyes ☺.

4903. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate
$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1}{2n+3}-\cdots\right)-\frac{1}{4n}\right].$$

Solution to problem 4903 Crux Math. 50 (1) 2024, 37

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Let
$$S := \sum_{n=1}^{\infty}\left(-\frac{1}{4n}+\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}\right)$$
which is the concise form of the sum in the problem. We claim that
$$\boxed{S=\frac{\log 2}{2}+\frac{\pi}{8}}.$$

Solution

For $n\geq 1$, let
$$F_n(x) := \sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}x^{2n+2k-1}$$
be the generating function. Then $F_n(x)$ converges for $0\leq x\leq 1$ (Leibniz rule for the alternating series at $x=1$), and by Abel's rule, F_n is continuous on $[0,1]$. Now
$$F_n'(x)=\sum_{k=0}^{\infty}(-1)^k x^{2n+2k-2}=\frac{x^{2n-2}}{1+x^2}.$$
Since $F_n(0)=0$, we obtain
$$F_n(1)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}=\int_0^1\frac{x^{2n-2}}{1+x^2}\,dx.$$
Note that $S=\sum_{n=1}^{\infty}\left(F_n(1)-\frac{1}{4n}\right)$. Partial integration $\int u'v=uv-\int uv'$with
$$u'=x^{2n-2}\text{ and }v=(1+x^2)^{-1}$$
yields
$$\begin{aligned}\int_0^1\frac{x^{2n-2}}{1+x^2}\,dx &= \frac{x^{2n-1}}{2n-1}\,\frac{1}{1+x^2}\Big|_0^1+\frac{1}{2n-1}\int_0^1 x^{2n-1}\frac{2x}{(1+x^2)^2}\,dx\\ &= \frac{1}{2(2n-1)}+\frac{2}{2n-1}\int_0^1\frac{x^{2n}}{(1+x^2)^2}\,dx.\end{aligned}$$
Hence, by using that $\sum\int=\int\sum$ as all factors are positive, and the fact that
$$\sum_{n=1}^{\infty}\left(\frac{1}{2(2n-1)}-\frac{1}{4n}\right)=\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{2(2n-1)}-\frac{1}{4n}\right)=\frac{1}{2}\lim_{N\to\infty}\sum_{j=1}^{2N}\frac{(-1)^{j-1}}{j}=\frac{\log 2}{2},$$
we obtain
$$\begin{aligned}S &= \sum_{n=1}^{\infty}\left(\frac{1}{2(2n-1)}-\frac{1}{4n}\right)+2\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}\right)\frac{x}{(1+x^2)^2}\,dx\\ &= \frac{\log 2}{2}+\int_0^1\left(\log\left(\frac{1+x}{1-x}\right)\,\frac{x}{(1+x^2)^2}\right)dx\\ &=: \frac{\log 2}{2}+\int_0^1 I(x)dx.\end{aligned}$$

To calculate a primitive of $I(x)$, we use partial integration with $u=\log\left(\frac{1+x}{1-x}\right)$ and $v'=\frac{x}{(1+x^2)^2}$. Hence

$$\begin{aligned}
\int I(x)dx &= -\frac{1}{2}\frac{1}{1+x^2}\log\left(\frac{1+x}{1-x}\right)+\int\frac{1}{1-x^4}dx \\
&= -\frac{1}{2}\frac{1}{1+x^2}\log\left(\frac{1+x}{1-x}\right)+\frac{1}{4}\log\left(\frac{1+x}{1-x}\right)+\frac{1}{2}\arctan x \\
&= \frac{1}{4}\,\frac{x^2-1}{x^2+1}\,\log\left(\frac{1+x}{1-x}\right)+\frac{1}{2}\arctan x \\
&=: R(x).
\end{aligned}$$

Hence

$$\int_0^1 I(x)dx = \lim_{x\to 1} R(x) - R(0) = \frac{\pi}{8}.$$

4900. *Proposed by Daniel Sitaru.*

For a positive integer m, let H_m denote the m-th harmonic number, that is, $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. For m, n, p, q positive integers, prove that

$$H_m + H_n + H_p + H_q \leq 3 + H_{mnpq}.$$

Solution to problem 4900 Crux Math. 49 (10) 2023, 541

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We first show that

(77) $$H_p + H_q \leq 1 + H_{pq} \text{ for } 1 \leq p \leq q.$$

In fact, if $1 = p = q$, then nothing has to be shown. So let us assume that $q \geq 2$. Then

$$\begin{aligned} 1 + H_{pq} &= 1 + H_p + \left(\frac{1}{p+1} + \cdots + \frac{1}{2p}\right) + \left(\frac{1}{2p+1} + \cdots + \frac{1}{3p}\right) + \cdots + \left(\frac{1}{(q-1)p+1} + \cdots + \frac{1}{qp}\right) \\ &\geq 1 + H_p + p \cdot \frac{1}{2p} + p \cdot \frac{1}{3p} + \cdots + p \cdot \frac{1}{qp} \\ &= H_p + H_q. \end{aligned}$$

Now let $1 \leq m \leq n \leq p \leq q$ (of course this is without loss of generality). Then by (77),

$$\begin{aligned} (H_m + H_n) + (H_p + H_q) &\leq (1 + H_{mn}) + (1 + H_{pq}) = 2 + H_{mn} + H_{pq} \\ &\leq 2 + 1 + H_{(mn)(pq)} = 3 + H_{mnpq}. \end{aligned}$$

Remark 1 More generally, one can show that

$$\sum_{j=1}^{n} H_{n_j} \leq (n-1) + H_{\prod_{j=1}^{n} n_j}.$$

Remark 2 Solutions to the special case (77) above also appeared in Amer. Math. Monthly 56 (2) 1949, 109-110, Problem E819 Euler's constant.

Remark 3 A different proof of $H_p + H_q \leq 1 + H_{pq}$, $p, q \in \mathbb{N} = \{1, 2, \dots\}$, can be given via induction on q: for $q = 1$, $H_p + H_1 = H_p + 1 \leq 1 + H_{p \cdot 1}$. Now for $q \to q+1$, we use that

$$H_{p(q+1)} = H_{pq} + \frac{1}{pq+1} + \cdots + \frac{1}{pq+p} \geq H_{pq} + \frac{p}{pq+p} = H_{pq} + \frac{1}{1+q}.$$

Hence

$$H_p + H_{q+1} = H_p + H_q + \frac{1}{q+1} \leq 1 + H_{pq} + \frac{1}{q+1} \leq 1 + H_{p(q+1)}.$$

4896. *Proposed by Ivan Hadinata.*

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$ the following equation holds:

$$f(f(x) + yf(z) - 1) + f(z + 1) = zf(y) + f(x + z).$$

Solution to problem 4896 Crux Math. 49 (10) 2023, 540

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We show that all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f\Big(f(x) + yf(z) - 1\Big) + f(z + 1) = zf(y) + f(x + z) \quad (x, y, z \in \mathbb{R}) \tag{78}$$

are given by

$$\boxed{f \equiv 0 \text{ or } f(x) = x \text{ for } x \in \mathbb{R}.}$$

Obviously $f \equiv 0$ and the identity are solutions. Now let f be a solution to (78) with $f(y_0) \neq 0$ for some $y_0 \in \mathbb{R}$. We claim that f is surjective.

In fact, put $x = 1$ and $y = y_0$ in (78). Then, for all $z \in \mathbb{R}$.

$$f\Big(f(1) + y_0 f(z) - 1\Big) + f(z + 1) = zf(y_0) + f(1 + z) \iff f\Big(f(1) + y_0 f(z) - 1\Big) = f(y_0)\, z. \tag{79}$$

As the function $z \mapsto f(y_0)\, z$ is surjective, the function $z \mapsto f\Big(f(1) + y_0 f(z) - 1\Big)$ is surjective, too. Hence $x \mapsto f(x)$ is surjective.

Next put $y = 0$ and $z = 0$ in (78). Then, for all $x \in \mathbb{R}$

$$f\big(f(x) - 1\big) + f(1) = f(x). \tag{80}$$

Now put $u := f(x)$. Note that if x runs through $\mathbb{R}$, the surjectivity of f implies that u runs through $\mathbb{R}$, too. In particular, $f(u - 1) = u - f(1)$ for every $u \in \mathbb{R}$ and so, with $v := u - 1$, $f(v) = v + 1 - f(1)$. Now $v = 1$ yields that $f(1) = 2 - f(1)$ and so $f(1) = 1$. Hence $f(v) = v$ for every $v \in \mathbb{R}$.

4894. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty} \frac{H_{n-1}H_{n+1}}{n(n+1)},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number and $H_0 = 0$.

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We show that

$$\boxed{\sum_{n=1}^{\infty} \frac{H_{n-1}H_{n+1}}{n(n+1)} = 3}.$$

For the proof we shall decompose the series into several telescoping series.

$$\begin{aligned}
\frac{H_{n-1}H_{n+1}}{n(n+1)} &= \frac{H_{n-1}H_{n+1}}{n} - \frac{H_{n-1}H_{n+1}}{n+1} = \frac{H_{n-1}H_{n+1}}{n} - \frac{(H_n - \frac{1}{n})(H_{n+2} - \frac{1}{n+2})}{n+1} \\
&= \frac{H_{n-1}H_{n+1}}{n} - \frac{H_nH_{n+2} - \frac{1}{n}H_{n+2} - \frac{1}{n+2}H_n + \frac{1}{n(n+2)}}{n+1} \\
&= \left(\frac{H_{n-1}H_{n+1}}{n} - \frac{H_nH_{n+2}}{n+1}\right) + \frac{H_{n+2}}{n(n+1)} + \frac{H_n}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}.
\end{aligned}$$

Now

$$\frac{H_{n+2}}{n(n+1)} = \left(\frac{H_{n+1}}{n} + \frac{\frac{1}{n+2}}{n}\right) - \frac{H_{n+2}}{n+1} = \left(\frac{H_{n+1}}{n} - \frac{H_{n+2}}{n+1}\right) + \frac{1}{n(n+2)}$$

and

$$\frac{H_n}{(n+1)(n+2)} = \left(\frac{H_{n-1}}{n+1} + \frac{\frac{1}{n}}{n+1}\right) - \frac{H_n}{n+2} = \left(\frac{H_{n-1}}{n+1} - \frac{H_n}{n+2}\right) + \frac{1}{n(n+1)}.$$

Moreover,

$$\begin{aligned}
\frac{2}{n(n+2)} &= \frac{1}{n} - \frac{1}{n+2} = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\
\frac{2}{n(n+1)(n+2)} &= \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{H_{n-1}H_{n+1}}{n(n+1)} &= \left(\frac{H_{n-1}H_{n+1}}{n} - \frac{H_nH_{n+2}}{n+1}\right) + \left(\frac{H_{n+1}}{n} - \frac{H_{n+2}}{n+1}\right) + \left(\frac{H_{n-1}}{n+1} - \frac{H_n}{n+2}\right) \\
&\quad + \frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).
\end{aligned}$$

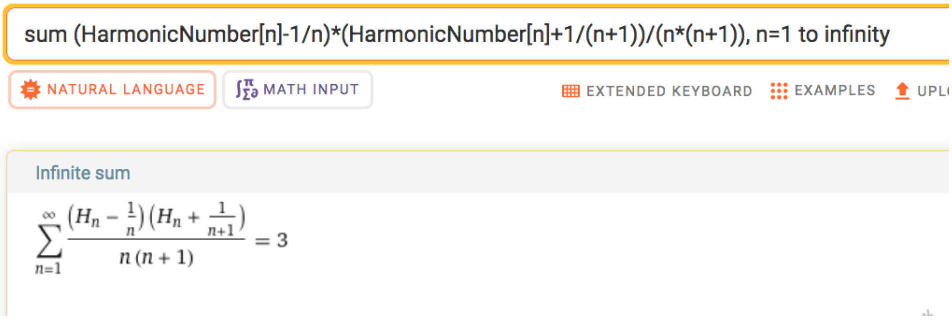
Consequently

$$\begin{aligned}
\lim_{N\to\infty} \sum_{n=1}^{N} \frac{H_{n-1}H_{n+1}}{n(n+1)} &= \left(H_0H_2 - \lim_{N\to\infty} \frac{H_NH_{N+2}}{N+1}\right) + \left(H_2 - \lim_{N\to\infty} \frac{H_{N+2}}{N+1}\right) + \left(\frac{H_0}{2} - \lim_{N\to\infty} \frac{H_N}{N+2}\right) \\
&\quad + \frac{3}{2} + \frac{1}{4} - \frac{1}{4}.
\end{aligned}$$

Since $\gamma = \lim_{n\to\infty}(H_n - \log n)$, we have that $\lim_{N\to\infty} \frac{H_NH_{N+2}}{N+1} = 0$ as well as $\lim_{N\to\infty} \frac{H_{N+2}}{N+1} = 0$ and $\lim_{N\to\infty} \frac{H_N}{N+2} = 0$. Consequently, by noticing that $H_0 = 0$,

$$\sum_{n=1}^{\infty} \frac{H_{n-1}H_{n+1}}{n(n+1)} = H_2 + \frac{3}{2} = 3.$$

We thank Roberto Tauraso for confirming the result via Maple and wolframalpha.com, the latter though using a different representation:



Calculating the sum beginning with index $n = 2$, this software obtains the wrong result (the actual sum is of course 3, too as the first summand $\frac{(H_1-1)(H_1+0.5)}{1*2} = 0$):

$$\sum_{n=2}^{\infty} \frac{\left(H_n - \frac{1}{n}\right)\left(H_n + \frac{1}{n+1}\right)}{n\,(n+1)} = \frac{7}{4} = 1.75$$

Very strange, too, is that the software does not give the correct value of the original sum but only very rough approximations:

Approximated sum

$$\sum_{n=1}^{\infty} \frac{H_{n-1}\, H_{n+1}}{n\,(n+1)} \approx 2.75949$$

Approximated sum

$$\sum_{n=1}^{\infty} \frac{H_{n-1}\, H_{n+1}}{n\,(n+1)} \approx$$

2.9135100717114100051500066686563672253196126664946473290321059642\
270541454027040860966879348133249579785186812243965

4893. *Proposed by Albert Natian.*

Find all continuous real functions f on $[-1,1]$ that satisfy the integral equation

$$x^2+\int_1^{\frac{1}{x}} f\left(x^2 t\right) dt=1.$$

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The statement of the problem is a bit ambiguous, as problems arise for $x=0$. Note that for $0<x\leq 1$ and $1\leq t\leq 1/x$ one has

$$0\leq x^2t\leq x^2\frac{1}{x}=x\leq 1,$$

so that the integral $\int_1^{1/x} f(x^2t)\,dt$ is well defined for $0<x\leq 1$. Moreover, for $-1\leq x<0$ and $1/x\leq t\leq 0$, one has

$$-1\leq x=x^2\frac{1}{x}\leq x^2t\leq 0,$$

and so the integral $\int_1^{1/x} f(x^2t)\,dt=-\int_0^1 f(x^2t)\,dt-\int_{1/x}^0 f(x^2t)\,dt$ is well defined for $-1\leq x<0$, too. If $x=0$, though, then the symbol $\int_1^{1/x}$ is not well defined as $1/0^+=\infty$ and $1/0^-=-\infty$. Actually no function can be a solution to $x^2+\int_1^{1/x} f(x^2t)dt=1$ also at this point, as $\int_1^{\pm\infty} f(0)dt$ is divergent if $f(0)\neq 0$, and if $f(0)=0$, then $\int_1^{\pm\infty} 0\,dt=0$ but $0+0\neq 1$.
Thus we need to interprete at $x=0$ this functional equation as

$$\lim_{x\to 0}\left(x^2+\int_1^{1/x} f(x^2t)dt\right)=1.$$

We show that

$$\boxed{f(x)=2x}$$

is the only continuous function $f:[-1,1]\to\mathbb{R}$ satisfying for $x\in[-1,1]\setminus\{0\}=:X$ the integral equation

(81) $$x^2+\int_1^{1/x} f(x^2t)\,dt=1,$$

and

(82) $$\lim_{x\to 0}\left(x^2+\int_1^{1/x} f(x^2t)dt\right)=1.$$

Proof. For $x\in X$ and $f\in C[-1,1]$, let $F(x):=x^2+\int_1^{1/x} f(x^2t)dt$. By the change of variable $u:=x^2t$ we obtain

$$x^2F(x)-x^4=\int_{x^2}^x f(u)du.$$

So $F\equiv 1$ on X if and only if $\int_{x^2}^x f(u)du=x^2-x^4$ on X, hence also on $[-1,1]$. Hence, if $f\in C[-1,1]$ is a solution on X to (81) then, by taking derivatives, $f(x)-f(x^2)=2x-4x^3$ on $[-1,1]$. From this, we guess that $f(x)=2x$. To this end, let $g(x):=f(x)-2x$, $x\in[-1,1]$. Then $g\in C[-1,1]$ and $g(x)=2xg(x^2)$ for $x\in[-1,1]$. Since $g(-x)=-2xg(x^2)$, it suffices to determine g for $x\in[0,1]$.

By induction, for each $x\in[0,1]$,

$$g(x)=2^nx^{2^n-1}g(x^{2^n}).$$

Now, for $0\leq x<1$ we may let $n\to\infty$ and conclude (due to the continuity of g at 0 and $my^m\to 0$ for $0\leq y<1$) that $g(x)=\lim_{n\to\infty} 2^nx^{2^n-1}\,g(0)=0$. As g is continuous at 1, we deduce that $g\equiv 0$ on $[0,1]$, hence on $[-1,1]$, and so $f(x)=2x$ for $x\in[-1,1]$ whenever f satisfies (81) on X. Now it is straightforward to show that $2x$ also satisfies (82), that is

$$\lim_{x\to 0}\left(x^2+\int_1^{1/x} f(x^2t)dt\right)=1.$$

4889. ***Proposed by Ovidiu Furdui and Alina Sîntămărian.***

Find all non-constant continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{y-x}\int_x^y f(g(t))\mathrm{d}t = f\left(\frac{x+y}{2}\right), \quad \forall x, y \in \mathbb{R},\ x \neq y. \qquad (1)$$

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Consider the functional equation

(83) $$\frac{1}{y-x}\int_x^y f(g(t))dt = f\left(\frac{x+y}{2}\right), x \neq y.$$

We show that all nonconstant continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying (1) are given by

$$\boxed{f(x) = ax + b \text{ for } a \neq 0,\ b \in \mathbb{R} \text{ and } g(x) = x.}$$

In fact, for fixed x, let $y \to x$. Then (1) implies (for instance via l'Hospital's rule) that

(84) $$f(g(x)) = f(x)$$

for every $x \in R$. Hence

(85) $$\frac{1}{y-x}\int_x^y f(t)dt = f\left(\frac{x+y}{2}\right), x \neq y.$$

Next we observe that any continuous f satisfying (85), necessarily is C^∞. In fact, for all x,

$$\int_{x-1}^{x+1} f(t)dt = 2f(x).$$

As the function on the left obviously is differentiable by the fundamental theorem of calculus, we do have the same for the function on the right. A calculation gives $f(x+1) - f(x-1) = 2f'(x)$. Hence, the continuity of f implies that f is continuously differentiable. Inductively, we now conclude that f is C^∞.

Now let

$$H(x,y) := \int_{y-x}^{y+x} f(t)dt.$$

Then, by assumption,

$$H(x,y) = 2xf\left(\frac{y+x+(y-x)}{2}\right) = 2x\ f(y).$$

Therefore,

$$H_y = f(y+x) - f(y-x) = 2xf'(y),$$

and

$$H_x = f(y+x) + f(y-x) = 2f(y).$$

Addition yields

$$f(y+x) = xf'(y) + f(y).$$

Now

$$\begin{aligned} f'(y+x) &= \frac{\partial}{\partial y} f(y+x) = xf''(y) + f'(y) \\ f'(y+x) &= \frac{\partial}{\partial x} f(y+x) = f'(y). \end{aligned}$$

Hence, for all x, we must have $xf''(y) = 0$. As f'' is continuous, $f'' \equiv 0$, and so $f(x) = ax + b$ with $a \neq 0$ (since f is not constant) and $b \in \mathbb{R}$. Moreover, as we know from (84) that $f(g(x)) = f(x)$, the injectivity of the linear function f implies that g is the identity.

We note that an equivalent for (85), the mid-point mean value theorem for derivatives, was dealt with in [64].

4862. *Proposed by Michel Bataille.*

Let m be a nonnegative integer. Find

$$\lim_{n\to\infty} \frac{1}{2^n n^m} \sum_{k=0}^{n} \binom{m+k}{k}\binom{m+n+1}{n-k}.$$

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Let

$$L_m(n) := \frac{1}{2^n}\,\frac{1}{n^m} \sum_{k=0}^{n} \binom{m+k}{k}\binom{m+n+1}{n-k}.$$

We prove that

$$\boxed{\lim_{n\to\infty} L_m(n) = \frac{2}{m!}}.$$

To this end, note that

$$\begin{aligned}\binom{m+k}{k}\binom{m+n+1}{n-k} &= \frac{(m+k)!}{m!k!}\,\frac{(m+n+1)!}{(m+k+1)!(n-k)!}\\ &= \frac{(m+n+1)!}{m!}\,\frac{1}{k!(m+k+1)(n-k)!} = \frac{(m+n+1)!}{m!n!}\binom{n}{k}\frac{1}{m+k+1}.\end{aligned}$$

Hence

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m+n+1}{n-k} = \frac{(m+n+1)!}{m!n!}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{m+k+1}.$$

Put

$$f(x) := \sum_{k=0}^{n}\binom{n}{k}\frac{1}{m+k+1}\,x^{m+k+1}.$$

Then

$$f'(x) = \sum_{k=0}^{n}\binom{n}{k}x^{m+k} = x^m(1+x)^n.$$

Consequently, as $\int_0^1 f'(x)dx = f(1)-f(0)$ and $f(0)=0$,

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m+n+1}{n-k} = \frac{(m+n+1)!}{m!n!}\int_0^1 x^m(1+x)^n\,dx,$$

and so

(86)
$$L_m(n) = \frac{1}{n^m}\,\frac{(m+n+1)!}{m!n!}\int_0^1 x^m\left(\frac{1+x}{2}\right)^n\,dx.$$

Case 1 If $m=0$, then

$$L_0(n) = \frac{(n+1)!}{n!}\int_0^1 \left(\frac{1+x}{2}\right)^n dx = (n+1)\left[\frac{2}{n+1}\left(\frac{1+x}{2}\right)^{n+1}\right]_0^1 = 2\left(1-\frac{1}{2^{n+1}}\right)\to 2.$$

Case 2 $m\geq 1$. We claim that

$$R_n := (n+1)\int_0^1 x^m\left(\frac{1+x}{2}\right)^n dx \to 2 \quad \text{as } n\to\infty.$$

In fact, partial integration yields

$$\begin{aligned} R_n &= \left[x^m 2\left(\frac{1+x}{2}\right)^{n+1}\right]_0^1 - \int_0^1 m x^{m-1} 2\left(\frac{1+x}{2}\right)^{n+1}\,dx \\ &= 2 - m\underbrace{\int_0^1 x^{m-1} 2\left(\frac{1+x}{2}\right)^{n+1}\,dx}_{:=I_n}. \end{aligned}$$

Since

$$0 \le I_n \le 2\int_0^1 \left(\frac{1+x}{2}\right)^{n+1}\,dx = \frac{1}{n+2}\left[\left(\frac{1+x}{2}\right)^{n+2}\right]_0^1 \le \frac{1}{n+2},$$

we conclude that $I_n \to$ and so $R_n \to 2$.

Together with (86), this finally yields that

$$\begin{aligned} L_m(n) &= \frac{1}{n^m}\frac{(m+n+1)!}{m!n!}\int_0^1 x^m\left(\frac{1+x}{2}\right)^n\,dx \\ &= \frac{1}{m!}\,\frac{(m+n+1)(m+n)\dots(n+2)}{n^m}\,(n+1)\int_0^1 x^m\left(\frac{1+x}{2}\right)^n\,dx \\ &= \frac{1}{m!}\,\frac{m+n+1}{n}\,\frac{m+n}{n}\dots\frac{n+2}{n}\,R_n \\ &\to 1^m\cdot\frac{2}{m!} = \frac{2}{m!}. \end{aligned}$$

4870*. *Proposed by Borui Wang.*

Define the series $\{a_n\}$ by the following recursion: $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ for $n > 0, q > 0$. Find the constant number $c(q)$ such that

$$\lim_{n\to\infty} (a_n - \sqrt{c(q) \cdot n}) = 0.$$

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We will show the following:

Let $q, c > 0$ and define the sequence (not series like it is mentioned in the problem statement) (a_n) by $a_1 = c$ and $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ for $n \geq 1$. Then, with $c(q) = 2/q$

$$\lim_{n\to\infty} \left|a_n - \sqrt{c(q)n}\right| = 0.$$

Remark If one starts with $c < 0$, then all the a_n are negative too, and one obtains $\left|a_n + \sqrt{c(q)n}\right| \to 0$. (This is done by considering $b_n := -a_n$).

Solution Let $h(x) := x + \dfrac{1}{qx}$. Then $h > 0$ on $]0, \infty[$ and so $a_{n+1} = h(a_n)$ is well defined. Taking squares

$$a_{n+1}^2 = a_n^2 + \frac{2}{q} + \frac{1}{q^2 a_n^2},$$

or equivalently

$$a_{n+1}^2 - a_n^2 = \frac{2}{q} + \frac{1}{q^2 a_n^2},$$

we obtain the finite telescoping series:

$$a_{n+1}^2 - a_1^2 = \sum_{k=1}^{n} (a_{k+1}^2 - a_k^2) = \frac{2}{q}\, n + \frac{1}{q^2} \sum_{k=1}^{n} \frac{1}{a_k^2}.$$

Hence

$$a_{n+1}^2 = c^2 + \frac{2}{q}\, n + \frac{1}{q^2} \sum_{k=1}^{n} \frac{1}{a_k^2}. \tag{87}$$

This allows us to estimate a_{n+1}:

$$a_{n+1}^2 \geq c^2 + \frac{2}{q}\, n,$$

and so by using this,

$$a_{n+1}^2 \leq c^2 + \frac{2}{q}\, n + \frac{1}{q^2} \sum_{k=1}^{n} \frac{1}{c^2 + \frac{2}{q}(k-1)}. \tag{88}$$

Next consider the decreasing function $f(x) = \dfrac{1}{c^2 + \frac{2}{q}x}$, $x \geq 0$. Then

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{c^2 + \frac{2}{q}(k-1)} &= \frac{1}{c^2} + \sum_{k=1}^{n-1} f(k) \leq \frac{1}{c^2} + \int_0^{n-1} f(x)dx \\ &= \frac{1}{c^2} + \left[\frac{q}{2} \log\left(c^2 + \frac{2}{q}x\right)\right]_0^{n-1} \\ &= \frac{1}{c^2} + \frac{q}{2} \log\left(1 + \frac{2}{qc^2}(n-1)\right). \end{aligned}$$

Thus we have arrived at the following estimates:

$$c^2 + \frac{2}{q}\, n \le a_{n+1}^2 \le c^2 + \frac{2}{q}\, n + \frac{1}{q^2}\ \frac{1}{c^2} + \frac{1}{2q} \log\left(1 + \frac{2}{qc^2}(n-1)\right). \tag{89}$$

Hence

$$\begin{aligned}
\Delta_n := \left|a_{n+1} - \sqrt{\frac{2}{q} n}\right| &= \frac{\left|a_{n+1}^2 - \frac{2}{q} n\right|}{a_{n+1} + \sqrt{\frac{2}{q} n}} \\
&\le \frac{c^2 + \frac{1}{q^2}\ \frac{1}{c^2} + \frac{1}{2q}\log\left(1 + \frac{2}{qc^2}(n-1)\right)}{\sqrt{c^2 + \frac{2}{q}\, n} + \sqrt{\frac{2}{q} n}} \\
&\underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\left|a_n - \sqrt{\frac{2}{q} n}\right| &\le \left|a_n - \sqrt{\frac{2}{q}(n-1)}\right| + \left|\sqrt{\frac{2}{q}(n-1)} - \sqrt{\frac{2}{q} n}\right| \\
&= \Delta_{n-1} + \sqrt{\frac{2}{q}} \cdot \frac{1}{\sqrt{n-1} + \sqrt{n}} \\
&\underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

4866. *Proposed by Ivan Hadinata.*

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that the equation

$$f(xy+f(f(y)))=xf(y)+y$$

holds for all real numbers x and y.

Solution to problem 4866 Crux Math. 49 (7) 2023, 376

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This is entirely trivial. We claim that the identity is the only solution $f:\mathbb{R}\to\mathbb{R}$ to

$$f(xy+f(f(y)))=xf(y)+y. \tag{90}$$

We first show that $f(0)=0$. In fact, if $x=0$, then $f(f(f(y)))=y$. Now take $y=0$ in (90). Then $0=f(f(f(0)))=xf(0)$. Hence $f(0)=0$.

Next, we take $y=1$ in (90). Then

$$f(x+f(f(1)))=xf(1)+1. \tag{91}$$

Put $u:=x+f(f(1))$. This yields

$$f(u)=(u-f(f(1)))\,f(1)+1=:au+b \tag{92}$$

In other words, f necessarily is an affine function. As we already know, $f(0)=0$. Thus $b=0$. Now (90) yields

$$a(xy+a^2y)=xay+y.$$

Hence $a=1$.

4857. *Proposed by Toyesh Prakash Sharma.*

Let a, b, c be positive real numbers such that $a+b+c=\frac{3}{2}$. Show that

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

Solution to problem 4857 Crux Math. 49 (5) 2023, 323

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Since the function $\log x$ is concave on $]0,\infty[$, we have

$$\log\left(\frac{A+B+C}{3}\right) \geq \frac{\log A + \log B + \log C}{3} =: R.$$

Here we take

$$A := a^a b^b, B := b^b c^c, C = c^c a^a.$$

Now the function $f(x) := 2x \log x$ is convex on $]0,\infty[$, since $f''(x) = 2/x \geq 0$. Hence

$$\begin{aligned} 2a\log a + 2b\log b + 2c\log c &= f(a)+f(b)+f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \\ &= 3f\left(\frac{1}{2}\right) = -3\log 2. \end{aligned}$$

Hence $3R \geq -3\log 2$, equivalently $R \geq \log(1/2)$ from which we deduce that $A+B+C \geq 3/2$. In other words

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

4855. *Proposed by Ivan Hadinata.*

Find all pairs of positive integers (a, b) such that $a^b - b^a = a - b$.

Solution to problem 4855 Crux Math. 49 (5) 2023, 323

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We claim that all solutions $(a, b) \in \mathbb{N} \times \mathbb{N}$ are given by

$$\boxed{(1, v), (u, 1), (t, t), (2, 3), (3, 2)}$$

where $u, v, t \in \mathbb{N} := \{1, 2, \cdots\}$ can be arbitrarily chosen.

It is easily seen that these are solutions. Now let (a, b) be a solution. Then (b, a) is a solution, too. If $b = a$, or if $b = 1$, then nothing remains to be shown. So we may assume that $a > b > 1$. Let $\log x$ be the natural logarithm. Now the function $f : x \mapsto x/\log x$ is strictly increasing for $x \geq e$ and strictly decreasing for $1 < x \leq e$ with $\min_{x>0} f(x) = e$. So if $a > b \geq 3 > e$,

$$\frac{a}{\log a} > \frac{b}{\log b}$$

or equivalently,

$$b^a > a^b.$$

Hence $a^b - b^a < 0$, but $a - b > 0$. So this case, where $a > b \geq 3$, does not occur. So it remains to consider the case $a > b = 2$. If $a = 3$, then we actually have the solution $(3, 2)$. If $a \geq 4$, then

$$\frac{a}{\log a} \geq \frac{4}{\log 4} = \frac{2}{\log 2},$$

and so

$$a^2 - 2^a \leq 0 < a - 2.$$

Thus this case $a \geq 4 > 2 = b$ does not occur, either. As all cases have been considered, we obtain the assertion.

4854. *Proposed by Michel Bataille.*

Let n be a positive integer and let $\theta_k = \frac{k\pi}{n+1}$. For $r, s \in \{1, 2, \ldots, n\}$, evaluate

$$\sum_{j=1}^{n} (\sin\theta_{jr} + \sin\theta_{js})^2.$$

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We prove that for $1 \leq r, s \leq n$,

$$S := \sum_{j=1}^{n} \left(\sin\left(j\frac{r\pi}{n+1}\right) + \sin\left(j\frac{s\pi}{n+1}\right)\right)^2 = \begin{cases} n+1 & \text{if } r \neq s \\ 2(n+1) & \text{if } r = s \end{cases}$$

We first show that

(93) $$\sum_{j=1}^{n} \cos\left(j\frac{2\rho\pi}{n+1}\right) = \begin{cases} -1 & \text{if } \rho \in \mathbb{Z} \setminus (n+1)\mathbb{Z} \\ n & \text{if } \rho \in (n+1)\mathbb{Z} \end{cases}$$

and that for odd $\rho \in \mathbb{Z}$

(94) $$\sum_{j=1}^{n} \cos\left(j\frac{\rho\pi}{n+1}\right) = 0.$$

To see this, we will use that $\cos x = \text{Re}(e^{ix})$, and that

(95) $$\sum_{j=1}^{n} e^{ijt} = e^{it}\sum_{j=0}^{n-1} e^{ijt} = e^{it}\frac{1-e^{int}}{1-e^{it}} = \frac{e^{it} - e^{i(n+1)t}}{1-e^{it}}.$$

Now put $t = 2\rho\pi/(n+1)$ whenever $\rho \in \mathbb{Z} \setminus (n+1)\mathbb{Z}$. The latter guarantees that the denominator does not vanish. Hence

$$\sum_{j=1}^{n} e^{ij\frac{2\rho\pi}{n+1}} = \frac{e^{i\frac{2\rho\pi}{n+1}} - 1}{1 - e^{i\frac{2\rho\pi}{n+1}}} = -1.$$

Now if $\rho \in (n+1)\mathbb{Z}$, then,

$$\sum_{j=1}^{n} e^{ij\frac{2\rho\pi}{n+1}} = n.$$

Thus (93) holds. If ρ is odd, then, by putting $t = \rho\pi/(n+1)$ in (95), we obtain

$$\sum_{j=1}^{n} e^{ij\frac{\rho\pi}{n+1}} = \frac{e^{i\frac{\rho\pi}{n+1}} + 1}{1 - e^{i\frac{\rho\pi}{n+1}}} = i\, \cot\left(\frac{1}{2}\,\frac{\rho\pi}{n+1}\right).$$

This is a purely imaginary number, so its real part is 0. This yields (94).
From (93) we easily deduce that for $r \in \{1, 2, \ldots, n\}$

(96) $$\sum_{j=1}^{n} \sin^2\left(j\frac{r\pi}{n+1}\right) = \frac{n+1}{2}.$$

In fact, using that $\sin^2 x = \frac{1-\cos 2x}{2}$, we obtain from (93)

$$\begin{aligned}\sum_{j=1}^{n} \sin^2\left(j\frac{r\pi}{n+1}\right) &= \sum_{j=1}^{n} \frac{1-\cos\left(j\frac{2r\pi}{n+1}\right)}{2} \\ &= \frac{n}{2} - \frac{1}{2}\sum_{j=1}^{n}\cos\left(j\frac{2r\pi}{n+1}\right) \\ &= \frac{n+1}{2}.\end{aligned}$$

We are now ready to calculate the value of S.

- *Case 1* $\boxed{r = s.}$ Then

$$\begin{aligned}S = \sum_{j=1}^{n}\left(\sin\left(j\frac{r\pi}{n+1}\right) + \sin\left(j\frac{r\pi}{n+1}\right)\right)^2 \quad &= \quad 4\sum_{j=1}^{n}\sin^2\left(j\frac{r\pi}{n+1}\right) \\ &\underset{(96)}{=} \quad 4\frac{n+1}{2} = 2(n+1).\end{aligned}$$

- *Case 2* $\boxed{r \neq s.}$ Since $r, s \in \{1, 2, \dots, n\}$, r and s do not belong to $(n+1)\mathbb{Z}$. Note that due to $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$,

(97) $$(\sin x + \sin y)^2 = \sin^2 x + \sin^2 y + \cos(x-y) - \cos(x+y).$$

Hence

$$\begin{aligned}S \quad &= \quad \sum_{j=1}^{n}\sin^2\left(j\frac{r\pi}{n+1}\right) + \sum_{j=1}^{n}\sin^2\left(j\frac{s\pi}{n+1}\right) + \sum_{j=1}^{n}\cos\left(j\pi\frac{r-s}{n+1}\right) - \sum_{j=1}^{n}\cos\left(j\pi\frac{r+s}{n+1}\right) \\ &\underset{(96)}{=} \quad n+1+\sum_{j=1}^{n}\cos\left(j\pi\frac{r-s}{n+1}\right) - \sum_{j=1}^{n}\cos\left(j\pi\frac{r+s}{n+1}\right) \\ &= \quad n+1+S_1 - S_2.\end{aligned}$$

Several cases have to be analyzed now:

a) $r-s$ is even, say $r-s = 2\rho$, where $\rho \in \mathbb{Z}$. Then $r+s$ is even, too. Since $0 < |r-s| \leq n-1$ and $0 < r+s \leq 2n < 2(n+1)$, we again have two subcases:

a1) $r+s \notin \mathbb{Z}(n+1)$ (equivalently $r+s \neq n+1$): Then by (93),

$$S = n+1+(-1)-(-1) = n+1.$$

a2) $r+s = n+1 \in \mathbb{Z}(n+1)$: Then n is odd, say $n = 2m+1$ for some $m \in \{0, 1, 2, \dots\}$, and so

$$S_2 = \sum_{j=1}^{2m+1}\cos(j\pi) = \underline{(-1)+(+1)} + \cdots + \underline{(-1)+(+1)} + (-1) = -1.$$

Hence

$$S = n+1+(-1)-(-1) = n+1.$$

b) $r-s$ is odd. Then $r+s$ is odd, too. Again we have two subcases:

b1) $r+s \neq n+1$: Then by (94),

$$S = n+1+0-0 = n+1.$$

b2) $r+s = n+1$. Then n is even, say $n = 2m$ with $m \in \{1, 2, \dots\}$, and so

$$S_2 = \sum_{j=1}^{2m}\cos(j\pi) = \underline{(-1)+(+1)} + \cdots + \underline{(-1)+(+1)} = 0.$$

Hence

$$S = n+1+0-0 = n+1.$$

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4844. *Proposed by Seán M. Stewart.*

Suppose n is a positive integer. Show that the value of the improper integral

$$\int_0^\infty \frac{x^{n-1}e^{-x}}{\sqrt{x}}\left(\sum_{k=0}^{n-1}\binom{2k}{k}\frac{x^{-k}}{2^{2k}(n-k-1)!}\right)dx$$

is independent of n.

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For $n \geq 1$, let

$$I_n := \int_0^\infty \frac{x^{n-1}e^{-x}}{\sqrt{x}}\left(\sum_{k=0}^{n-1}\binom{2k}{k}\frac{x^{-k}}{2^{2k}(n-k-1)!}\right)dx.$$

We show that

$$\boxed{I_n = \sqrt{\pi}}.$$

We use the following well-known formulas, where Γ is the Gamma function:

$$\int_0^\infty x^{s-1}e^{-x}\,dx = \Gamma(s), \quad \Gamma(s+1) = s\,\Gamma(s), s > 0 \tag{98}$$

$$\Gamma\left(m+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\cdots\frac{2m-1}{2} = \sqrt{\pi}\,\frac{\prod_{k=1}^m(2k-1)}{2^m} = \frac{(2m)!}{m!4^m}\sqrt{\pi} \tag{99}$$

So, with $m = n-k-1$,

$$\begin{aligned}
I_n &= \sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{2^{2k}(n-k-1)!}\int_0^\infty x^{(n-k-1/2)-1}e^{-x}\,dx \\
&= \sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{2^{2k}(n-k-1)!}\,\Gamma(n-k-1/2) \\
&= \sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{2^{2k}(n-k-1)!}\,\frac{(2(n-k-1))!}{(n-k-1)!4^{n-k-1}}\,\sqrt{\pi} \\
&= \frac{1}{4^{n-1}}\sum_{k=0}^{n-1}\frac{(2k)!}{(k!)^2}\frac{(2(n-k-1))!}{((n-k-1)!)^2}\sqrt{\pi} = \frac{1}{4^{n-1}}\sum_{k=0}^{n-1}\binom{2k}{k}\binom{2(n-k-1)}{n-k-1}\sqrt{\pi}.
\end{aligned}$$

This is related to the coefficient in the Cauchy product of

$$\sum_{n=0}^\infty\binom{2n}{n}x^n = \sum_{n=0}^\infty\binom{-1/2}{n}(-1)^n4^nx^n = \frac{1}{\sqrt{1-4x}},$$

with itself and which converges for $|x| < 1/4$, or if we take $x = y/4$,

$$\sum_{n=0}^\infty\frac{1}{4^n}\binom{2n}{n}y^n = \frac{1}{\sqrt{1-y}}.$$

In fact, for $|y| < 1$,

$$\sum_{m=0}^\infty y^m = \frac{1}{1-y} = \frac{1}{\sqrt{1-y}}\frac{1}{\sqrt{1-y}} = \sum_{m=0}^\infty\left(\sum_{k=0}^m\frac{\binom{2k}{k}}{4^k}\frac{\binom{2(m-k)}{m-k}}{4^{m-k}}\right)y^m$$

The coefficients being unique, we deduce that for every $m = 0, 1, \cdots$

$$\sum_{k=0}^m\frac{\binom{2k}{k}}{4^k}\frac{\binom{2(m-k)}{m-k}}{4^{m-k}} = 1.$$

Hence, with $m = n-1$, we conclude that $I_n = \sqrt{\pi}$.

4850. *Proposed by George Stoica.*

Let R be a finite field of characteristic 2 and $n \geq 2$. Then the sum of all invertible $n \times n$ matrices over R is the $n \times n$ zero matrix over R.

Solution to problem 4850 Crux Math. 49 (5) 2023, 274, first version [19]

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Let $n \geq 2$. We show that for *any* finite field the sum S of all invertible $n \times n$ matrices is the $n \times n$ zero matrix O_n.

For $n \geq 1$, let $\mathcal{M}_n$ be the set of all $n \times n$ matrices and let $\mathcal{U}_n$ be the set of all invertible $n \times n$ matrices. Since the field has only a finite number of elements, $\mathcal{U}_n$ has only a finite number of elements. So $S := \sum_{U \in \mathcal{U}_n} U$ is a well defined element in $\mathcal{M}_n$. We will show that for every $\tilde{U} \in \mathcal{U}_n$,

$$S \cdot \tilde{U} = S.$$

Fix an invertible matrix $\tilde{U} \in \mathcal{U}_n$ and consider the map

$$\iota : \begin{cases} \mathcal{M}_n & \to \mathcal{M}_n \\ X & \mapsto X \cdot \tilde{U} \end{cases}.$$

Then ι is a bijection of $\mathcal{M}_n$ onto itself. The inverse is given by $\iota^{-1}(Y) = Y \cdot \tilde{U}^{-1}$, since

$$\iota \circ \iota^{-1}(Y) = \iota(Y \cdot \tilde{U}^{-1}) = (Y \cdot \tilde{U}^{-1}) \cdot \tilde{U} = Y$$

and

$$\iota^{-1} \circ \iota(X) = \iota^{-1}(X \cdot \tilde{U}) = (X \cdot \tilde{U}) \cdot \tilde{U}^{-1} = X.$$

Moreover, and this is the main point here, ι maps $\mathcal{U}_n$ bijectively onto itself. Thus (and here we have not yet used that $n \neq 1$)

$$S = \sum_{U \in \mathcal{U}_n} \iota(U) = \iota\Big(\sum_{U \in \mathcal{U}_n} U\Big) = \iota(S) = S \cdot \tilde{U}. \tag{100}$$

Now we use that $n \geq 2$. Take for $\tilde{U}$ and $1 \leq i < j \leq n$ the elementary matrices

$$E_{ij} = (\vec{e}_1, \ldots, \underbrace{\vec{e}_j}_{i\text{-th col}}, \ldots, \underbrace{\vec{e}_i}_{j\text{-th col}}, \ldots, \vec{e}_n),$$

which interchange for $X \cdot E_{ij}$ the i-th and j-th column of X. Thus $S \cdot E_{ij} = S$ implies that all the columns of S are the same. Say $S = (\vec{s}, \ldots, \vec{s})$. Next we consider the matrix

$$E = \begin{pmatrix} 1 & 1 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{pmatrix}.$$

Note that the action $X \cdot E$ of E on a matrix X is to replace the second column of X by the sum of the first and second column. Since E is invertible, we obtain from (100) that $S \cdot E = S$ and so

$$\vec{s} + \vec{s} = \vec{s}.$$

Hence $\vec{s} = \vec{0}$. Consequently $S = O_n$.

Remark We may also consider the case $n = 1$. Note that the smallest field is given by $\mathbb{F}_2 := \{0, 1\}$, with $1 \neq 0$, where 0 is the neutral element for addition and 1 the one for multiplication. This necessarily has characteristic 2. Here $S = 1$. If the finite field is not field-isomorphic to $\mathbb{F}_2$, it has more than two elements, and so there is an (invertible) element u different from 1. Now by (100), $S = Su$, hence $S(1 - u) = 0$. Since $1 - u \neq 0$, hence invertible,we conclude that $S = 0$.

[19] This was tacitly replaced by another problem later on.

4835. *Proposed by George Stoica.*

Prove that the four complex numbers z_i, $i = 1, \ldots, 4$, are the consecutive vertices of a cyclic quadrilateral (or are collinear) in the complex plane if and only if the number $\dfrac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$ is real.

Solution to problem 4835 Crux Math. 49 (4) 2023, 213

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This is a standard result/exercise in old monographs on function theory/complex analysis and is for instance in [65, p. 70]

Using a not so sophisticated wording, we will show that four distinct points z_j $(j = 1, \ldots, 4)$ in the plane belong to a circle or a line if and only if their cross-ratio (bi-rapport, Doppelverhältnis)

$$DV(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_1 - z_4} \Big/ \frac{z_3 - z_2}{z_3 - z_4}$$

is a real number.

In particular, being real, will be independent of the "order" of the points on the circle, respectively line.

Our proof will be done in the extended complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (also called the one-point compactification of $\mathbb{C}$). Let us recall some terminology here. If L is a line in $\mathbb{C}$, then $L \cup \{\infty\}$ is called an extended line. As usual we call the elements of the set of circles and extended lines in $\widehat{\mathbb{C}}$ "generalized circles".

We also use an extension of the definition of the cross-ratio to points in $\widehat{\mathbb{C}}$. This is done by taking limits. For instance

$$D(z_1, z_2, z_3, \infty) = \frac{z_1 - z_2}{z_3 - z_2}. \tag{101}$$

Finally, let us recall the following results:

i) There is a unique linear-fractional map (or in modern terminology, a Möbius transform) $T(z) := (az + b)/(cz + d)$, $ad - bc \neq 0$, viewed as map from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ mapping three distinct points z_2, z_3, z_4 in $\widehat{\mathbb{C}}$ to $0, 1, \infty$, namely $T(z) = DV(z, z_2, z_3, z_4)$.

ii) The cross ratio is invariant under linear-fractional maps:

$$DV(T(z_1), T(z_2), T(z_3), T(z_4)) = DV(z_1, z_2, z_3, z_4).$$

Note that the latter is an immediate consequence of i).

iii) The class of generalized circles is invariant under Möbius transforms.

Now we are ready to confirm the statement above:

Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$, consider the map $S(z) := DV(z, z_2, z_3, z_4)$. Suppose that these z_j belong to a generalized circle E. Now S maps E to the extended real line $\mathbb{R} \cup \{\infty\}$, since $z_2 \to 0$, $z_3 \to 1$ and $z_4 \to \infty$. In particular, $w_j := S(z_j) \in \mathbb{R} \cup \{\infty\}$ for $j = 1, \ldots, 4$. Since $DV(w_1, w_2, w_3, w_4)$ is real, the invariance result shows that $DV(z_1, z_2, z_3, z_4)$ is real.

Conversely, suppose that $DV(z_1, z_2, z_3, z_4)$ is real. Note that $S(z_j) \in \{0, 1, \infty\} \subseteq \mathbb{R} \cup \{\infty\}$ for $j = 2, 3, 4$. Now the image of the extended real line by the inverse Möbius transform S^{-1} is a generalized circle, E. Of course E contains the points z_2, z_3 and z_4. But, by (101), and the assumption, we have

$$S(z_1) = DV(S(z_1), S(z_2), S(z_3), S(z_4)) = DV(z_1, z_2, z_3, z_4) \in \mathbb{R}.$$

Hence $z_1 = S^{-1}(S(z_1)) \in E$. In other words, all the z_j belong either to a circle or a line.

This can be shortened, without the explicit use of the cross ratio. Actually, just iii) is relevant here: Consider the Möbius transform

$$M(z) := \frac{z - z_4}{z - z_2}\,\frac{z_3 - z_2}{z_3 - z_4}.$$

Then z_4, z_3, z_2 are mapped to $0, 1, \infty$, and so the (unique) generalized circle E determined by z_4, z_3, z_2 is mapped to the extended real line. Thus the point z_1 belongs to E if and only if $M(z_1) \in \mathbb{R}$. In other words, all the z_j belong either to a circle or a line if and only if $\frac{z_1-z_4}{z_1-z_2}\,\frac{z_3-z_2}{z_3-z_4} \in \mathbb{R}$.

4836. *Proposed by Mohammad Bakkar.*

Prove the following formula:

$$\frac{\pi^3}{32} = \prod_{n=1, 2n+1 \notin \mathcal{P}}^{\infty} \frac{4n(n+1)}{(2n+1)^2},$$

where $\mathcal{P}$ is the set of prime numbers.

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We first calculate the missing part

$$P := \prod_{\substack{n=1\\2n+1\in\mathcal{P}}}^{\infty} \frac{4n(n+1)}{(2n+1)^2}.$$

Put $p := 2n+1$. Then $n = (p-1)/2$ and so, in view of the Euler formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-2}},$$

we have

$$P = \prod_{\substack{p\in\mathcal{P}\\p\neq 2}} \frac{p^2-1}{p^2} = \frac{4}{3}\,\frac{6}{\pi^2} = \frac{8}{\pi^2}.$$

To calculate

$$R := \prod_{n=1}^{\infty} \frac{4n(n+1)}{(2n+1)^2},$$

we use partial products and Stirling's formula $\lim\limits_{n\to\infty} \dfrac{n^n e^{-n}\sqrt{2\pi n}}{n!} = 1$.

$$\begin{aligned}
P_N := \prod_{n=1}^{N} \frac{4n(n+1)}{(2n+1)^2} &= \frac{4^N\ N!(N+1)!}{\left(\frac{(2N+1)!}{\prod_{n=1}^{N}(2n)}\right)^2} = \frac{4^N N!(N+1)!}{(2N+1)!^2}\,\frac{\left(2^N N!\right)^2}{1}\\
&= \frac{4^{2N} N!^4(N+1)}{(2N+1)!^2}\\
&\sim \frac{4^{2N}\ N^{4N} e^{-4N} 4\pi^2 N^2(N+1)}{(2N+1)^{4N+2} e^{-4N-2}\ 2\pi(2N+1)}\\
&= \pi e^2\, \frac{(2N)^{4N} N^2(N+1)}{(2N+1)^{4N}(2N+1)^3}\\
&= 2\pi e^2\, \frac{1}{\left[\left(1+\frac{1}{2N}\right)^{2N}\right]^2}\,\frac{N^2(N+1)}{(2N+1)^3}\\
&\to 2\pi e^2\, \frac{1}{e^2}\,\frac{1}{8} = \frac{\pi}{4}.
\end{aligned}$$

Hence $\displaystyle\prod_{\substack{n=1\\2n+1\notin\mathcal{P}}}^{\infty} \frac{4n(n+1)}{(2n+1)^2} = \frac{\pi/4}{8/\pi^2} = \frac{\pi^3}{32}.$

A second way to derive the value of P is as follows:

For $z \in \mathbb{C}$ we have

$$\begin{aligned}
\sin(\pi z) &= \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}} \ \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} \\
&= \pi z(1-z)e^z \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)e^{\frac{z}{n+1}} \ \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} \\
&= \pi z(1-z)e^z \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)\left(1+\frac{z}{n}\right)e^{\frac{z}{n+1}-\frac{z}{n}} \\
&= \pi z(1-z)e^z \ e^{\sum_{n=1}^{\infty}\left(\frac{z}{n+1}-\frac{z}{n}\right)} \prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)\left(1+\frac{z}{n}\right) \\
&= \pi z(1-z)\prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)\left(1+\frac{z}{n}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
P &= \prod_{n=1}^{\infty}\frac{4n(n+1)}{(2n+1)^2} = \prod_{n=1}^{\infty}\frac{2n}{2n+1}\frac{2n+2}{2n+1} \\
&= \prod_{n=1}^{\infty}\frac{1}{1+\frac{1}{2n}} \ \frac{1}{1-\frac{1}{2(n+1)}} = \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2(n+1)}\right)} \\
&= \left.\frac{\pi z(1-z)}{\sin(\pi z)}\right|_{z=1/2} \\
&= \frac{\pi}{4}.
\end{aligned}$$

4828. *Soumis par Narendra Bhandari.*

Démontrer que

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec(x+y)\sec(x-y)}{\sec x \sec y} dxdy = \sum_{n=0}^{\infty} (-1)^n/(2n+1)^2.$$

Solution to problem 4828 Crux Math. 49 (3) 2023, 157

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Let

$$I := \int_0^{\pi/4} \underbrace{\int_0^{\pi/4} \frac{\cos x \;\cos y}{\cos(x+y)\;\cos(x-y)}\, dy}_{:=I(x)}\, dx.$$

Now fix the variable x. Since

$$\cos(x+y)\;\cos(x-y) = \cos^2 y - \sin^2 x,$$

we obtain

$$\begin{aligned} I(x) &= \cos x \int_0^{\pi/4} \frac{\cos y}{(1-\sin^2 x) - \sin^2 y}\, dy \\ u := \sin y \quad &= \cos x \int_0^{\sqrt{2}/2} \frac{du}{\cos^2 x - u^2} \\ &= \frac{1}{2}\left(\log(\cos x + u) - \log(\cos x - u)\right)\Big|_{u=0}^{\sqrt{2}/2} \\ &= \frac{1}{2}\log\left(\frac{\cos x + 1/\sqrt{2}}{\cos x - 1/\sqrt{2}}\right). \end{aligned}$$

Hence (using Fubini),

$$I = \frac{1}{2}\int_0^{\pi/4} \log\left(\frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}\right)\, dx. \tag{102}$$

The value of this integral is known to be the Catalan number C (see formula (18) in [66]). An independent proof is below: using that $\cos a + \cos b = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$ and $\cos(a-b) = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$, we obtain

$$\begin{aligned} \log\left(\frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}\right) &= \log\left(\frac{\cos x + \cos \pi/4}{\cos x - \cos \pi/4}\right) \\ &= -\log\tan\left(\frac{x+\pi/4}{2}\right) - \log\tan\left(\frac{-x+\pi/4}{2}\right). \end{aligned}$$

A change of the variable $x + \pi/4 = 2y$, respectively $-x + \pi/4 = 2y$, and a standard integral representation of C yields

$$I = -\frac{1}{2}\int_{\pi/8}^{\pi/4} \log\tan y\;(2dy) - \frac{1}{2}\int_0^{\pi/8} \log\tan y\;(2dy) = -\int_0^{\pi/4} \log\tan y\; dy = C.$$

A proof of this standard representation can be given for instance by using power series or Fourier series:

$$h(z) := \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}.$$

Its Taylor coefficients belong to ℓ^2 and so the associated Fourier series

$$h^*(e^{it}) := \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{i(2n+1)t}$$

converges in the $L^2(]0,\pi[)$-norm to

$$h(e^{it}) = \frac{1}{2}\log(i\cot(t/2)) = i\,\frac{\pi}{4} - \frac{1}{2}\log\tan(t/2)$$

(Actually the series $h^*(e^{it})$ converges pointwise for $z = e^{it}$ with $0 < t < \pi$ by the Abel-Dirichlet rule, but we do not need this.)

Taking real parts, and using that $\int\sum = \sum\int$ (note that Fourier series converge in the L^2-norm, hence in the L^1 norm), we may conclude that

$$-\int_0^{\pi/4} \log\tan y\, dy = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{1}{2n+1}\cos(2n+1)t\, dt = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)^2}.$$

4830. *Proposed by Goran Conar.*

Let $a_i \in (0, \frac{1}{2})$, $i \in \{1, 2, \dots, n\}$ be real numbers such that $\sum_{i=1}^{n} a_i = 1$. Prove that the following inequalities hold:

$$n\sqrt{\frac{n-1}{n+1}} \leq \sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_i}} < (n+1)\sqrt{\frac{n-1}{n+1}}.$$

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First we claim that on $[0, 1/2]$ the function $f(x) = \sqrt{\dfrac{1-x}{1+x}}$ is convex. In fact,

$$f'(x) = -\frac{1}{\sqrt{\frac{1-x}{1+x}}(1+x)^2}$$

and

$$f''(x) = \frac{1-2x}{(1-x)(x+1)^3\sqrt{\frac{1-x}{1+x}}} \geq 0.$$

Since the graph of a convex function lies below the secant determined by $(a, f(a)), (b, f(b))$, we obtain that $f(x) \leq 1 - 2(1 - 3^{-1/2})x$, where $a = 0$ and $b = 1/2$. Since $1 - 3^{-1/2} \geq 1/3$, we deduce that for $0 \leq x \leq 1/2$

$$f(x) \leq 1 - (2/3)x$$

, and so

$$\sum_{i=1}^{n} f(a_i) \leq n - (2/3)\sum_{i=1}^{n} a_i = n - 2/3.$$

But for $n \geq 2$, we have

$$n - 2/3 < (n+1)\sqrt{\frac{n-1}{n+1}} = \sqrt{n^2-1},$$

since

$$n^2 - 1 - (n - 2/3)^2 = 4/3n - 13/9 \geq 8/3 - 13/9 = 11/9 > 0.$$

This upper bound in the problem appears to be artificial. We did not see a way to derive this in a natural way. To prove the reverse inequality, we use Jensen's inequality and obtain

$$\frac{1}{n}\sum_{i=1}^{n} f(a_i) \geq f\left(\frac{\sum_{i=1}^{n} a_i}{n}\right) = f(1/n).$$

Hence

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_i}} \geq n\sqrt{\frac{1-\frac{1}{n}}{1+\frac{1}{n}}} = n\sqrt{\frac{n-1}{n+1}}.$$

4826. *Proposed by Paul Bracken.*

Let H_n is the n-th harmonic number $H_n = \sum_{k=1}^{n} 1/k$. Evaluate the following sum in closed form

$$S = \sum_{k=1}^{\infty} \frac{H_k}{k(k+1)(k+2)}.$$

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We claim that

$$\boxed{S = \frac{\pi^2}{12} - \frac{1}{2}}.$$

Just write

$$\begin{aligned}\frac{H_k}{k(k+1)(k+2)} &= \frac{1}{2}\left(H_k\Big(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\Big)\right)\\ &= \frac{1}{2}\left(\frac{H_k}{k(k+1)} - \frac{H_{k+1}}{(k+1)(k+2)} + \frac{1}{(k+1)^2(k+2)}\right).\end{aligned}$$

Now

$$\frac{1}{(k+1)^2(k+2)} = \frac{1}{k+2} - \frac{k+1-1}{(k+1)^2} = \left(\frac{1}{k+2} - \frac{1}{k+1}\right) + \frac{1}{(k+1)^2}.$$

Since the Cesaro means of the sequences $(1/k)$ converge to 0, that is $H_k/k \to 0$, we conclude that

$$S = \frac{1}{2}\frac{H_1}{2} - \frac{1}{2}\frac{1}{1+1} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k+1)^2} = \frac{\pi^2}{12} - \frac{1}{2}.$$

4825. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Calculate

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)}.$$

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We prove that

$$\boxed{I := \sum_{n=1}^{\infty} \frac{O_n}{n(n+1)} = \log 4.}$$

First we note that

$$\frac{O_n}{n(n+1)} = O_n \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{O_n}{n} - \frac{O_{n+1}}{n+1} + \frac{1}{(2n+1)(n+1)}.$$

Since the Cesaro means of the null sequence $(1/(2n+1))$ converge to 0, we obtain

$$I = \frac{O_1}{1} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)(n+1)} = 1 + 2\log 2 - 1 = \log 4.$$

The value of the series $S := \sum_{n=1}^{\infty} \frac{1}{(2n+1)(n+1)}$ can be determined as follows:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{(2n+1)(n+1)} &= \sum_{n=1}^{N} \Big(\frac{2}{2n+1} - \frac{1}{n+1}\Big) \\
\text{splitting into even an odd} &= \sum_{n=1}^{N} \Big(\frac{1}{2n+1} - \frac{1}{2n+1}\Big) + \sum_{n=1}^{N} \Big(\frac{1}{2n+1} - \frac{1}{2n}\Big) + \sum_{n=N+1}^{2N+1} \frac{1}{n} \\
&= -1 + \sum_{n=1}^{2N+1} (-1)^{n+1} \frac{1}{n} + \sum_{n=N+1}^{2N+1} \frac{1}{n} \\
&\underset{N\to\infty}{\longrightarrow} -1 + \log 2 + \log 2.
\end{aligned}$$

Note that the well-known assertion $\lim_{N\to\infty} \sum_{n=N+1}^{2N} \frac{1}{n} = \log 2$ is a direct consequence of the fact that the Euler-Mascheroni constant γ is given by

$$\gamma = \lim(H_n - \log n),$$

where $H_n := \sum_{i=1}^{n} \frac{1}{i}$, since

$$H_{2N} - H_N = (H_{2N} - \log(2N) - \gamma) + (\log N + \gamma - H_N) + \log 2 \to \log 2.$$

4822. *Proposed by Anton Mosunov.*

The n-th Chebyshev polynomial of the first kind is defined by means of the recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{for } n \geq 2.$$

Prove that for all $n \geq 2$,

$$\frac{1}{3} < \int_1^{+\infty} \frac{dx}{T_n(x)^{2/n}} < \frac{1}{3}\sqrt[n]{4}.$$

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Substituting $x = \cosh t$ we obtain $T_n(\cosh t) = \cosh(nt)$. In particular, T_n has no zeros on $[1, \infty[$. Hence

$$\begin{aligned} I := \int_1^\infty \frac{dx}{T_n(x)^{2/n}} &= \int_0^\infty \frac{\sinh t}{\big(\cosh(nt)\big)^{2/n}}\, dt = \int_0^\infty \frac{e^t - e^{-t}}{2\left(\dfrac{e^{nt}+e^{-nt}}{2}\right)^{2/n}}\, dt \\ &= 2^{-1+2/n} \int_0^\infty \frac{1-e^{-2t}}{e^t\big(1+e^{-2nt}\big)^{2/n}}\, dt. \end{aligned}$$

Hence

$$\begin{aligned} I &< 2^{-1+2/n} \int_0^\infty \frac{1-e^{-2t}}{e^t}\, dt = 2^{-1+2/n}\left[-e^{-t} + \frac{1}{3}e^{-3t}\right]_0^\infty \\ &= 2^{-1+2/n}\frac{2}{3} = \frac{1}{3}\sqrt[n]{4}. \end{aligned}$$

Moreover

$$I > 2^{-1+2/n} \int_0^\infty \frac{1-e^{-2t}}{e^t(1+1)^{2/n}}\, dt = 2^{-1} \int_0^\infty \frac{1-e^{-2t}}{e^t}\, dt = 2^{-1}\frac{2}{3} = \frac{1}{3}.$$

4816. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $a, b, k \geq 0$. Calculate

$$\lim_{n\to\infty} \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2 x^{2n}}\, \mathrm{d}x.$$

Solution to problem 4816 Crux Math. 49 (2) 2023, 101

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We show that for $a, b, k \geq 0$ (k not necessary an integer)

$$\boxed{I_n := \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2 x^{2n}}\, dx \overset{n\to\infty}{\longrightarrow} \sqrt{b} + \frac{\sqrt{a}}{k+1/2}.}$$

Write

$$f_n(x) = x^{k-1/2}\sqrt{a + bn^2 x^{2n+1}}.$$

If $a = 0$, then

$$I_n = \int_0^1 \sqrt{b} n x^{n+k} dx = \frac{n\sqrt{b}}{n+k+1} \to \sqrt{b}.$$

For $a > 0$, let

$$d_n(x) := x^{k-1/2}\Big(\sqrt{a + bn^2 x^{2n+1}} - \sqrt{bn^2 x^{2n+1}}\Big).$$

Then

$$0 \leq d_n(x) = x^{k-1/2}\, \frac{a}{\sqrt{a + bn^2 x^{2n+1}} + \sqrt{b} n x^{n+1/2}} \leq \frac{a}{\sqrt{a}} x^{k-1/2}.$$

Hence d_n is dominated by an $L^1[0,1]$ function and so, by using that $nx^n \to 0$ for $0 < x < 1$,

$$\lim_n \int_0^1 d_n(x) dx = \int_0^1 \lim_n d_n(x) dx = \int_0^1 \sqrt{a} x^{k-1/2} = \frac{\sqrt{a}}{k+1/2}.$$

Consequently,

$$\begin{aligned}
\int_0^1 f_n(x) dx &= \int_0^1 d_n(x) dx + \sqrt{b} \int_0^1 n x^{k-1/2} x^{n+1/2} dx \\
&= \int_0^1 d_n(x) dx + \sqrt{b}\, \frac{n}{k+n+1} \\
&\underset{n\to\infty}{\longrightarrow} \frac{\sqrt{a}}{k+1/2} + \sqrt{b}.
\end{aligned}$$

4819. *Proposed by Daniel Sitaru.*

Let $f : [0,1] \to [0,1]$ be a continuous function and $0 < a \le b < 1$.
Prove that:

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \ge \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt + \int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

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Note that the harmonic mean $x_0 := 2ab/(a+b)$ is less than or equal to the arithmetic mean $y_0 := (a+b)/2$. We show that the inequality holds for arbitrary x_0, y_0 with $0 < x_0 < y_0 < 1$. So let F be that primitive of f on $[0,1]$ with $F(0) = 0$. We shall prove that

$$\boxed{2\int_{x_0}^{y_0} tf(t)\,dt \ge (F(y_0) - F(x_0))(F(y_0) + F(x_0))},$$

from which the desired inequality immediately follows. By partial integration,

$$2\int_{x_0}^{y_0} t\,f(t)\,dt = 2\int_{x_0}^{y_0} tF'(t)\,dt = 2(y_0F(y_0) - x_0F(x_0)) - 2\int_{x_0}^{y_0} F(t)dt. \tag{103}$$

For $0 \le x, y \le 1$, put

$$H(x,y) := 2yF(y) - 2xF(x) - 2\int_x^y F(t)dt - (F(y)^2 - F(x)^2).$$

We have to show that $H(x_0, y_0) \ge 0$. Since $0 \le f \le 1$, $F(x) \le \int_0^x 1\,dt = x$. Hence

$$\frac{\partial H}{\partial x}(x,y) = -2\big(F(x) + xf(x)\big) + 2F(x) + 2F(x)f(x) = 2\big(F(x) - x\big)f(x) \le 0.$$

Consequently, by using that $H(y,y) = 0$, we obtain $\xi \in]x_0, y_0[$ with

$$H(x_0,y_0) = H(x_0,y_0) - H(y_0,y_0) = \underbrace{\frac{\partial H}{\partial x}(\xi, y_0)}_{\le 0}\underbrace{(x_0 - y_0)}_{\le 0} \ge 0.$$

4817. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $abc = 1$. Prove that the following inequality holds

$$\frac{a^7 + a^3 + bc}{a + bc + 1} + \frac{b^7 + b^3 + ca}{b + ca + 1} + \frac{c^7 + c^3 + ab}{c + ab + 1} \geq 3\,.$$

When does equality occur?

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Let $E :=]0,\infty[\times]0,\infty[\times]0,\infty[$ and let $H : E \to]0,\infty[$ be given by

$$H(a,b,c) = \frac{a^7 + a^3 + bc}{a + bc + 1} + \frac{b^7 + b^3 + ca}{b + ca + 1} + \frac{c^7 + c^3 + ab}{c + ab + 1}.$$

Put $L := \{(a,b,c) \in E : abc = 1\}$. To be shown is that $\inf_L H = 3$ and that this lower bound is obtained exactly at $(1,1,1)$. To this end, consider for $x > 0$ the function

$$f(x) := \frac{x^7 + x^3 + x^{-1}}{x + x^{-1} + 1} = \frac{x^8 + x^4 + 1}{x^2 + x + 1} = x^6 - x^5 + x^3 - x + 1.$$

Then f is convex on $[0,\infty[$. In fact,

$$f'(x) = 6x^5 - 5x^4 + 3x^2 - 1 \text{ and } f''(x) = 30x^4 - 20x^3 + 6x = 2x(15x^3 - 10x^2 + 3).$$

Now $f''(x) = 2x\big(5x^2(3x-2)+3\big)$. Then, clearly, $f''(x) \geq 0$ if $x \geq 2/3$. Since

$$\max_{[0,2/3]} x^2(2-3x) = 32/3^5 \leq 3/5,$$

we deduce that $f''(x) \geq 0$ on $[0,2/3]$, too. Due to Jensen's inequality, for $(a,b,c) \in L$

$$H(a,b,c) \quad = \quad f(a) + f(b) + f(c) = 3\,\frac{f(a)+f(b)+f(c)}{3} \geq 3\,f\Big(\frac{a+b+c}{3}\Big)$$

Since f is convex for $x \geq 0$, $f(x) \geq f(1) + f'(1)(x-1) = 1 + 3(x-1) = -2 + 3x$. Why we take evaluation at 1? Because it works! It is an a posteriori choice, since the minimal value is taken at $(a,b,c) = (1,1,1)$. Thus we obtain the estimate

$$H(a,b,c) \geq 3\,(-2 + (a+b+c)) = -6 + 3\left(a + b + \frac{1}{ab}\right).$$

We can even avoid Jensen's inequality:

$$H(a,b,c) = f(a) + f(b) + f(c) \geq (-2+3a) + (-2+3b) + (-2+3c) = -6 + 3(a+b+c).$$

Since $a + b + \frac{1}{ab} \geq 3$ (see below) we deduce that for $abc = 1$ we have $H(a,b,c) \geq -6 + 9 = 3$. As $H(1,1,1) = 3$, we are done.

The inequality $g(a,b) := a + b + \dfrac{1}{ab} \geq 3$ is well known. It can for instance be shown by using differential calculus:

$$g_a(a,b) = 1 - \frac{1}{ab^2} = 0 \iff ab^2 = 1 \text{ and } g_b(a,b) = 1 - \frac{1}{ba^2} = 0 \iff ba^2 = 1.$$

In other words, $ab(a-b) = 0$. Hence $a = b = 1$ is the only stationary point. Thus $g(1,1) = 3$ is the minimum, since the limit of g at the boundary $ab = 0$ is ∞.

4811. *Proposed by Nguyen Viet Hung.*

Find all positive integers n such that $\sqrt{n^3+1}+\sqrt{n+2}$ is a positive integer.

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We show that $n=2$ is the only solution. In fact $\sqrt{2^3+1}+\sqrt{2+2}=3+2=5$. Now, for $x,y\geq 0$, one has $\sqrt{x}+\sqrt{y}\in\mathbb{N}$ if and only if x and y are perfect squares. To see this, just note that

$$\sqrt{x}+\sqrt{y}=\frac{x-y}{\sqrt{x}-\sqrt{y}}$$

implies that $\sqrt{x}+\sqrt{y}\in\mathbb{Q}$ if and only $\sqrt{x}-\sqrt{y}\in\mathbb{Q}$ and so, by adding (respectively substracting), $\sqrt{x}$ and $\sqrt{y}$ are rational. Thus $\sqrt{x}=p/q$ for some $p,q\in\mathbb{N}$ with no common divisor. Hence $x^2=p^2/q^2\in\mathbb{N}$, and so $q=1$.

Due to a classical result by L. Euler, the Diophantine equation $n^3+1=m^2$ has in $\mathbb{N}=\{0,1,2,\dots\}$ only the solutions $(m,n)=(1,0)$ and $(m,n)=(3,2)$ (see for instance [67], a reference provided to the first author by Amol Sasane). Thus $n=2$ is the only positive integer also satisfying $\sqrt{n+2}\in\mathbb{N}$.

4810. *Proposed by Goran Conar.*

Let $a_1, a_2, \dots, a_n > 0$ be real numbers such that $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$, $n > 1$. Prove that

$$\frac{a_2^2 + a_3^2 + \cdots + a_n^2}{(a_2 + a_3 + \cdots + a_n)^3} + \frac{a_1^2 + a_3^2 + \cdots + a_n^2}{(a_1 + a_3 + \cdots + a_n)^3} + \cdots + \frac{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}{(a_1 + a_2 + \cdots + a_{n-1})^3} \geq \frac{n\sqrt{n}}{(n-1)^2}.$$

Solution to problem 4810 Crux Math. 49 (1) 2023, 45

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We first show that whenever $\sum_{j=1}^n a_j^2 = 1$, then

(104)
$$\frac{\sum_{\substack{j=1\\ j\neq i}}^n a_j^2}{\Big(\sum_{\substack{j=1\\ j\neq i}}^n a_j\Big)^3} \geq \frac{1}{\sqrt{1-a_i^2}}\,\frac{\sqrt{n-1}}{(n-1)^2}.$$

In fact, using Cauchy-Schwarz, we immediately obtain

$$\frac{\sum_{\substack{j=1\\ j\neq i}}^n a_j^2}{\Big(\sum_{\substack{j=1\\ j\neq i}}^n a_j\Big)^3} \geq \frac{\sum_{\substack{j=1\\ j\neq i}}^n a_j^2}{\Big(\big(\sum_{\substack{j=1\\ j\neq i}}^n a_j^2\big)(n-1)\Big)^{3/2}} = \frac{1}{\sqrt{1-a_i^2}}\,\frac{\sqrt{n-1}}{(n-1)^2}.$$

Next we prove that whenever $\sum_{j=1}^n a_j^2 = 1$, then

(105)
$$\sum_{i=1}^n \frac{1}{\sqrt{1-a_i^2}} \geq n\sqrt{\frac{n}{n-1}}.$$

In fact, consider the convex function $f(x) = \frac{1}{\sqrt{1-x}}$. By Jensen's inequality (or one of the possible defintions of convexity), if $\sum_{j=1}^n t_j = 1$ where $(0 \leq t_j \leq 1)$, then

$$f\Big(\sum_{j=1}^n t_j x_j\Big) \leq \sum_{j=1}^n t_j f(x_j).$$

Here we choose $x_i = a_i^2$, and $t_j = 1/n$. Note that $\frac{\sum_{i=1}^n a_i^2}{n} = 1/n$. Hence

$$\sum_{i=1}^n \frac{1}{\sqrt{1-a_i^2}} = n\sum_{i=1}^n \frac{1}{n} f(x_i) \geq n f\Big(\frac{1}{n}\sum_{i=1}^n x_j\Big) = \frac{n}{\sqrt{1-\frac{1}{n}}} = n\sqrt{\frac{n}{n-1}}.$$

Now putting (104) and (105) together yields

$$\sum_{i=1}^n \frac{\sum_{\substack{j=1\\ j\neq i}}^n a_j^2}{\Big(\sum_{\substack{j=1\\ j\neq i}}^n a_j\Big)^3} \geq \frac{\sqrt{n-1}}{(n-1)^2}\, n\sqrt{\frac{n}{n-1}} = \frac{n\sqrt{n}}{(n-1)^2}.$$

4803. *Proposed by Nguyen Viet Hung.*

Find all non-negative integers a, b, c and pairs (p, q) of prime numbers satisfying

$$p^{2a} + q^{2b} = (2c+1)^2.$$

Solution to problem 4803 Crux Math. 49 (1) 2023, 44

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It turns out that the triple $(3, 4, 5)$ satisfying $3^2 + 4^2 = 5^2$ is relevant here. Only one solution to the problem with $p \leq q$ exists: $p = 2, q = 3$ and $a = 2, b = 1, c = 2$. To sum up:

$$\boxed{2^{2\cdot 2} + 3^{2\cdot 1} = (2 \cdot 2 + 1)^2}$$

To see this, we use of course the well known parametrizations of the solutions to $A^2 + B^2 = C^2$, which are given by

$$(*)\ A = 2mn, B = m^2 - n^2 \text{ and } C = m^2 + n^2,\ m, n \in \mathbb{N}.$$

The conditions to be dealt with are

$$i)\ 2mn = p^a,\ ii)\ m^2 - n^2 = q^b \text{ and } iii)\ m^2 + n^2 = 2c + 1.$$

• First we note that $(a, b) = (0, 0)$ is not admissible as $1 + 1 = 2$ is even. Now if $b = 0$ and $a \neq 0$, then by i) p necessarily must be an even prime, that is $p = 2$. Hence

$$2^{2a} + 1 = (2c + 1)^2.$$

By (*), $1 = m^2 - n^2$ and $2^{2a} = 2mn$. Consequently m and n are powers of 2. Hence $m^2 - n^2$ is an even number; and not 1. Thus $ab \neq 0$.

• So let $ab > 0$. Since p is prime, m and n can only be powers of 2 by (i). Due to iii), telling us that $m^2 + n^2$ is an odd number, not both m and n can be proper powers of 2. Since $m \geq n$ (by ii)), we necessarily have $n = 1$ and $m = 2^x$ with $x \neq 0$. By ii),

$$q^b = m^2 - 1 = (2^x)^2 - 1 = (2^x - 1)(2^x + 1).$$

This implies that $q \neq 2$ (as the right hand side is odd). Since the difference of the factors is 2, $q \geq 3$ cannot divide both factors. Thus we can only have that the factor $2^x - 1$ equals 1.

Hence $x = 1$ and $q^b = 3$, yielding $b = 1$ and $q = 3$. Finally by i), $p^a = 2mn = 2 \cdot 2^1 \cdot 1 = 2^2$. So $p = 2$ and $a = 2$. Finally, $c = 2$ as $3^2 + 4^2 = 5^2$.

4809. *Proposed by Daniel Sitaru.*

Let $a, b > 0$. Find

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \left(\int_0^1 \frac{x^k}{ax+b} dx \right)^{-1} \left(\int_0^1 \frac{x^k}{bx+a} dx \right)^{-1}.$$

Solution to problem 4809 Crux Math. 49 (1) 2023, 45

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We show, more generally, that whenever $f, g : [0,1] \to [0, \infty[$ are continuous and $f(1)g(1) \neq 0$, then

$$\boxed{\lim_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\,dx \right)^{-1} \left(\int_0^1 x^k g(x)\,dx \right)^{-1} = \frac{1}{3}\,\frac{1}{f(1)}\frac{1}{g(1)}.}$$

Hence the limit in the problem is $(a+b)^2/3$.

Proof Let $M := \max\{|f(x)| : 0 \le x \le 1\}$. Given ε with $0 < \varepsilon < \frac{1}{2}\min\{f(1), g(1)\}$, choose $\delta > 0$ so that $|f(x) - f(1)| \le \varepsilon$ for $\delta \le x \le 1$. Moreover, let n_0 be so large that $\delta^{k+1} \le \varepsilon/(2M)$ for $k \ge n_0$. Then

$$\begin{aligned}
\left| \int_0^1 x^k f(x)\,dx - \frac{1}{k+1} f(1) \right| &= \left| \int_0^1 x^k (f(x) - f(1))\,dx \right| \\
&\le 2M \int_0^\delta x^k + \int_\delta^1 x^k |f(x) - f(1)|\,dx \\
&\le 2M \frac{\delta^{k+1}}{k+1} + \varepsilon \int_0^1 x^k dx \\
&\le \frac{\varepsilon}{k+1} + \frac{\varepsilon}{k+1}.
\end{aligned}$$

Therefore

$$\left(\frac{1}{k+1} f(1) + \frac{2\varepsilon}{k+1} \right)^{-1} \le \left(\int_0^1 x^k f(x)\,dx \right)^{-1} \le \left(\frac{1}{k+1} f(1) - \frac{2\varepsilon}{k+1} \right)^{-1}.$$

We conclude that

$$\sum_{k=n_0}^{n} \frac{k+1}{f(1)+2\varepsilon}\,\frac{k+1}{g(1)+2\varepsilon} \le \sum_{k=n_0}^{n} \left(\int_0^1 x^k f(x)\,dx \right)^{-1} \left(\int_0^1 x^k g(x)\,dx \right)^{-1} \le \sum_{k=n_0}^{n} \frac{k+1}{f(1)-2\varepsilon}\,\frac{k+1}{g(1)-2\varepsilon}.$$

Hence, by using that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$, we deduce that

$$\limsup_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\,dx \right)^{-1} \left(\int_0^1 x^k g(x)\,dx \right)^{-1} \le \frac{1}{3}\,\frac{1}{g(1)-2\varepsilon}\,\frac{1}{f(1)-2\varepsilon}$$

and

$$\liminf_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\,dx \right)^{-1} \left(\int_0^1 x^k g(x)\,dx \right)^{-1} \ge \frac{1}{3}\,\frac{1}{f(1)+2\varepsilon}\,\frac{1}{g(1)+2\varepsilon}.$$

from which we conclude that

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{n=1}^{n} \left(\int_0^1 x^k f(x)\,dx \right)^{-1} \left(\int_0^1 x^k g(x)\,dx \right)^{-1} = \frac{1}{3}\,\frac{1}{f(1)}\frac{1}{g(1)}.$$

Remark The lower estimate show that the limit is infinite if $f(1)g(1) = 0$.

4805. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $ab + bc + ca = 4abc$. Prove

$$\frac{1}{\sqrt[a]{a}} + \frac{1}{\sqrt[b]{b}} + \frac{1}{\sqrt[c]{c}} \geq 4\sqrt[3]{\frac{4}{3}}.$$

Solution to problem 4805 Crux Math. 49 (1) 2023, 44

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First we note that $ab + bc + ca = 4abc$ is equivalent to

$$(*) \qquad \ell(a, b, c) := \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 4.$$

If $a = b = c$, then this condition is satisfied if $a = 3/4$. Let

$$g(a, b, c) = a^{-1/a} + b^{-1/b} + c^{-1/c}.$$

It suffices to show that the minimum M of g under condition (*) is obtained for $a = b = c$. Note that $M := g(3/4, 3/4, 3/4) = 3(3/4)^{-4/3} = 4(4/3)^{1/3} \sim 4.402$.

The gradient of the Lagrange function

$$H(a, b, c, \lambda) = g(a, b, c) + \lambda(\ell(a, b, c) - 4)$$

is zero if

$$\lambda = a^{-1/a}\,(1 - \log a) = b^{-1/b}\,(1 - \log b) = c^{-1/c}\,(1 - \log c).$$

Since the function $x \mapsto x^{-1/x}\,(1 - \log x)$ is strictly decreasing on $]0, \infty[$, the only solution is where $a = b = c$. The existence of the minimum is shown as follows (note that

$$E := \{(a, b, c) : a, b, c > 0, \ell(a, b, c) = 4\}$$

is not compact. Condition (*) implies that $a, b, c \geq 1/4$. Let $L := \inf_E g$. Then

$$L \geq 3 \min_{[1/4,\infty[} x^{-1/x} = 3e^{-1/e} \geq 3 \times 0.692 = 2.076.$$

If this infimum is not taken on E, then there is $a_n \to \infty$ (or $b_n \to \infty$, or $c_n \to \infty$) such that $(a_n, b_n, c_n) \in E$ and $g(a_n, b_n, c_n) \to L$. In particular $a_n^{-1/a_n} \to 1$. We may assume that $b_n \to b_0$ and $c_n \to c_0$ (since otherwise $b_n \to \infty$ and so $c_n \to 1/4$, as well as $L = 1 + 1 + 4^4 > M$, a contradiction). Hence $L = \inf_{E'}(1 + b^{-1/b} + c^{-1/c})$, where

$$E' = \{(b, c) : b, c > 0, 1/b + 1/c = 3\}.$$

In particular, $b \geq 1/3$. Thus (by using Lagrange again, yielding $x = 2/3$)

$$L = \inf_{]1/3,\infty[} 1 + x^{-1/x} + \left(\frac{3x-1}{x}\right)^{\frac{3x-1}{x}} \overset{x=2/3}{=} 1 + 2(3/2)^{3/2} \sim 4.674 > M.$$

A contradiction. Consequently $(a_n, b_n, c_n) \to (\alpha, \beta, \gamma) \in E$ and so the infimum is a minimum. Hence

$$g(a, b, c) \geq g(\alpha, \beta, \gamma) = L = M.$$

Here is a second proof, based on the article [68] (which unfortunately contains many typos (poor proofreading? Poor referee job?). The function $f(x) := x^x$ is convex. Let $T_u(x) := f'(u)(x - u) + f(u)$ be the tangent to the graph of f at the point $(u, f(u))$. Then $f(x) \geq T_u(x)$. Next, let $x_1 = 1/a$, $x_2 = 1/b$ and $x_3 = 1/c$. Then with $u := S = (x_1 + x_2 + x_3)/3$,

$$\sum_{j=1}^{3} f(x_j) \geq \sum_{j=1}^{3} T_S(x_j) = \sum_{j=1}^{3} (f'(S)(x_j - S) + f(S)) = f'(S) \sum_{j=1}^{3} (x_j - S) + 3f(S)$$

$$= f'(S)\sum_{j=1}^{3} x_j - 3Sf'(S) + 3f(S) = 3f(S).$$

Since $S = (1/a + 1/b + 1/c)/3 = 4/3$, we obtain with $1/a + 1/b + 1/c = 4$ that

$$(1/a)^{1/a} + (1/b)^{1/b} + (1/c)^{1/c} = \sum_{j=1}^{3} f(x_j) \geq 3f(4/3) = 3(4/3)^{4/3} = 4(4/3)^{1/3}.$$

4801. *Proposed by Michel Bataille.*

Find all functions $f:(0,\infty)\to\mathbb{R}$ such that

$$f\left(x+\frac{1}{y}\right)=yf(xy+y)$$

for all $x,y>0$.

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We show that all solutions are given by $f(x)=\dfrac{C}{1+x}$ for $C\in\mathbb{R}$.

- It is straightforward to check that these are solutions:

$$f\left(x+\frac{1}{y}\right)=\frac{c}{1+x+\frac{1}{y}}=\frac{cy}{y+yx+1}=yf(xy+y).$$

- Suppose that $f:]0,\infty[\to\mathbb{R}$ is a solution. Let $y=\frac{1}{1+x}$. Then

$$f(2x+1)=f(x+\frac{1}{y})=yf(y(1+x))=\frac{1}{1+x}f(1). \tag{106}$$

Next, let $y=\frac{1}{x}$. Hence, by using (106),

$$f(2x)=f(x+\frac{1}{y})=yf(y(1+x))=\frac{1}{x}f(1+\frac{1}{x})=\frac{1}{x}f(1+2\frac{1}{2x})\overset{(106)}{=}\frac{1}{x}\frac{f(1)}{1+\frac{1}{2x}}=\frac{2f(1)}{2x+1}.$$

Now let $X:=2x$ and $C:=2f(1)$. Then $f(X)=\dfrac{2f(1)}{X+1}=\dfrac{C}{1+X}$.

4772. *Proposed by Mihaela Berindeanu.*

Find all functions $f:(0,\infty)\longrightarrow(0,\infty)$ such that $f(kx+f(y))=\dfrac{y}{k}\cdot f(xy+1)$ for all $x,y\in(0,\infty)$, where $k>0$ is a real and fixed parameter.

Solution to problem 4772 Crux Math. 48 (8) 2022, 483

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For $k=1$, this problem was given for instance in the Middle European Mathematical Olympiad (MEMO) in 2012 in Switzerland (see [70] and [69]) and we follow those published solutions.

We claim that for $a>0$, all solutions $f:\mathbb{R}^+\to\mathbb{R}^+$ of [20]

$$f(ax+f(y))=\frac{y}{a}\,f(xy+1) \tag{107}$$

are given by $f(x)=a/x$. First, it is straightforward to see that this is a solution. Now we proceed as in [70, 69]. Let f be a solution.

Step 1 Consider for $y>0$, $y\neq a$, the auxiliary function

$$g(y):=\frac{a-yf(y)}{a-y}$$

(this function is formally obtained by solving in $\mathbb{R}\times\mathbb{R}^+$ the equation $ax+f(y)=xy+1$, which gives $x=x_y=\frac{1-f(y)}{a-y}$ for $y\neq a$, and so $ax+f(y)=\frac{a-yf(y)}{a-y}=g(y)$. It will turn out that $x=-1/y$ and $g\equiv 0$).

Now for every $y>0$ with $y\neq a$ and $x_y>0$, we have that $g(y)\leq 0$, since otherwise f is well-defined at $g(y)>0$ and so $f(g(y))=\frac{y}{a}f(g(y))$, yielding that $y=a$, a contradiction.

Step 2

Case 1 If there would exist $y_0>1$ such that $f(y_0)<a/y_0$, then with $x_0:=1-\frac{1}{y_0}>0$ we have $x_0y_0+1=y_0$,

$$u_0:=ax_0+f(y_0)=a-\frac{a}{y_0}+f(y_0)<a,$$

and

$$f(u_0)=f(ax_0+f(y_0))=\frac{y_0}{a}\,f(y_0)<1.$$

Then $x_{u_0}:=\frac{1-f(u_0)}{a-u_0}>0$ and so

$$g(u_0)=ax_{u_0}+f(u_0)=x_{u_0}u_0+1>0.$$

But by Step 1, $g(u_0)\leq 0$, a contradiction.

Case 2 If there would exist $y_1>1$ such that $f(y_1)>a/y_1$, then by the same reasoning as above, with $x_1:=1-\frac{1}{y_1}$ and

$$u_1:=ax_1+f(y_1)>a,$$

we have $f(u_1)>1$ and so $g(u_1)>0$, again. A contradiction.

We conclude that $f(y)=a/y$ for every $y>1$. To deal with the remaining case, take $x=1/a$ and $0<y\leq 1$. Then by (107),

$$f(1+f(y))=\frac{y}{a}f\left(\frac{y}{a}+1\right). \tag{108}$$

[20] We prefer to use the letter a instead of k, as for us k always belongs to $\mathbb{N}$.

As both $1 + f(y)$ and $\frac{y}{a} + 1$ are bigger than 1, we deduce from (108) that

$$\frac{a}{1 + f(y)} = \frac{y}{a} \frac{a}{\frac{y}{a} + 1} = \frac{ay}{y + a}.$$

Hence $f(y) = a/y$.

4779. *Proposed by Marian Ursărescu.*

Let $0 < a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and with $f(a) = f(b)$. Prove that there exist distinct $c_1, c_2 \in (a, b)$ such that

$$\sqrt{b}f'(c_1) + \sqrt{a}f'(c_2) = 0.$$

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If f is constant, then $f' \equiv 0$ and we may choose any numbers $a < c_1 < c_2 < b$ to satisfy

$$\sqrt{b}f'(c_1)_+\sqrt{a}f'(c_2) = 0. \tag{109}$$

Otherwise, f takes its distinct extremal values on $[a, b]$. We may assume that $M := \max_{[a,b]} f > f(a)$ (if not, $M = f(a)$ and so $\min_{[a,b]} f < f(a)$ and we consider $-f$). Say $M = f(x_0)$ for some $x_0 \in]a, b[$. Then $f'(x_0) = 0$, and due to continuity of f', there are $a < x_1 < x_2 \leq x_0$ with $f'(x) > 0$ for $x \in]x_1, x_2[$, but $f'(x_2) = 0$; we may choose

$$x_2 = \inf\{t \leq x_0 : f' \equiv 0 \text{ on } [t, x_0]\}.$$

By a similar argument, there are $x_0 \leq y_2 < y_1$ such that $f'(y_2) = 0$, but $f'(x) < 0$ for $x \in]y_2, y_1[$. By the intermediate value theorem for continuous functions, here for f', there exists a small $\varepsilon > 0$ such that f' takes every value from $[0, \varepsilon]$ on $]x_1, x_2]$ and every value from $[-\varepsilon, 0]$ on $[y_2, y_1[$. Now choose $c_1 \in]x_1, x_2[$ so that $\frac{\sqrt{b}}{\sqrt{a}}f'(c_1) \in]0, \varepsilon[$ (this is possible since $\lim_{x \nearrow x_2} f'(x) = 0$). Hence there exists $c_2 \in]y_2, y_1[$ with

$$f'(c_2) = -\frac{\sqrt{b}}{\sqrt{a}}f'(c_1).$$

Thus $\sqrt{b}f'(c_1)_+\sqrt{a}f'(c_2) = 0$ and $c_1 < c_2$.

Remark I do not see the role played by the special coefficients $\sqrt{a}$ and $\sqrt{b}$. The whole works for any $0 < s_1 < s_2 < \infty$.

4771. *Proposed by Michel Bataille.*

Let I be an open interval containing 0 and 1 and let $f : I \to \mathbb{R}$ be a differentiable, strictly increasing, convex function. If $f'(1) < 2f(1)$, prove that there exist positive real numbers a, b such that

$$\int_0^1 (f(x))^{2n+1}\,dx \sim a \cdot \frac{b^n}{n} \qquad \text{as } n \to \infty$$

and express a and b as a function of $f(1)$ and $f'(1)$.

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The problem is a bit ambiguous, due to an undefined $\sim$ symbol. Let

$$L_n := \int_0^1 f(x)^{2n+1}\,dx \quad \text{and} \quad R_n = a \cdot \frac{b^n}{n}$$

Is it $L_n - R_n \to 0$? Or $L_n/R_n \to 1$? Or $cL_n \le R_n \le CL_n$ for almost every n and some positive constants c, C? Note that, a priori, it is not even clear that $L_n > 0$.

We are going to show the following:

Example 16.

$$\boxed{\lim_{n\to\infty} \frac{n+1}{f(1)^{2n+1}} \int_0^1 f(x)^{2n+1}\,dx = \frac{f(1)}{2f'(1)}.}$$

Hence, with $a := \dfrac{f(1)^2}{2f'(1)}$ and $b = f(1)^2$ we get that $L_n/R_n \to 1$.

Proof. Since f is assumed to be increasing, we see that $f'(x) \ge 0$ for $0 \le x \le 1$. To exclude that for some points $x_0 \in]0,1]$, $f'(x_0) = 0$, we need the convexity [21] of f: in fact, let T be the tangent to the graph of f at $(x_0, f(x_0))$; then $T(x) = f(x_0) + f'(x_0)(x - x_0)$. The convexity of f implies that the graph of f lies above T. In particular, if $f'(x_0) = 0$, then, due to f being strictly increasing, $f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon)$ would contradict this fact. We conclude that $f'(x) > 0$ for every $x \in]0,1]$.

To calculate our limit, we let $0 < s < 1$ and write the integral $\dfrac{n+1}{f(1)^{2n+1}} L_n$ as $I_n(s) + J_n(s)$, where

$$I_n(s) = \frac{n+1}{f(1)^{2n+1}} \int_0^s f(x)^{2n+1}\,dx \quad \text{and} \quad J_n(s) = \frac{n+1}{f(1)^{2n+1}} \int_s^1 f(x)^{2n+1}\,dx.$$

Claim 1 There is a function $h(s)$ with $0 < h(s) < 1$, such that

$$\frac{f(1)}{2f'(1)}\Big(1 - h^{2n+2}(s)\Big) \le J_n(s) \le \frac{f(1)}{2f'(s)}. \tag{110}$$

To see this, note that f convex and C^1 imply that f' is increasing (by the way, a fact equivalent to f being convex). By the mean-value theorem, and for $s < x \le 1$, there is $c_x \in]s,1[$ with $f'(c_x) = \frac{f(1)-f(x)}{1-x}$. Hence

$$f'(s) \le f'(c_x) \le f'(1)$$

and so

$$f'(s) \le \frac{f(1) - f(x)}{1 - x} \le f'(1).$$

[21] Note that f merely being strictly increasing, does not exclude the existence of zeros of f': $f(x) = (x - 1/2)^3$.

In other words

$$f(1) - f'(1) + f'(1)x \le f(x) \le f(1) - f'(s) + f'(s)x. \tag{111}$$

Now for $f(x) = Ax + B$ with $A \neq 0$ we have

$$\int_s^1 (Ax+B)^{2n+1}\, dx = \frac{(A+B)^{2n+2} - (As+B)^{2n+2}}{A(2n+2)}.$$

Applying this to (111) yields

$$\begin{aligned}
\frac{n+1}{f(1)^{2n+1}} \int_s^1 f(x)^{2n+1}\, dx &\le \frac{n+1}{f(1)^{2n+1}} \frac{(f'(s) + f(1) - f'(s))^{2n+2} - (f'(s)s + f(1) - f'(s))^{2n+2}}{f'(s)(2n+2)} \\
&= \frac{1}{2f'(s)} \frac{f(1)^{2n+2} - (f'(s)s + f(1) - f'(s))^{2n+2}}{f(1)^{2n+1}} \\
&= \frac{f(1)}{2f'(s)} \left(1 - \left(1 - \frac{f'(s)}{f(1)}(1-s)\right)^{2n+2}\right) \\
&\le \frac{f(1)}{2f'(s)}
\end{aligned}$$

because $0 \le 1 - \frac{f'(s)}{f(1)}(1-s) < 1$ for $s \in [s_1, 1]$. Similarily,

$$\begin{aligned}
\frac{n+1}{f(1)^{2n+1}} \int_s^1 f(x)^{2n+1}\, dx &\ge \frac{n+1}{f(1)^{2n+1}} \frac{(f'(1) + f(1) - f'(1))^{2n+2} - (f'(1)s + f(1) - f'(1))^{2n+2}}{f'(1)(2n+2)} \\
&= \frac{1}{2f'(1)} \frac{f(1)^{2n+2} - (f'(1)s + f(1) - f'(1))^{2n+2}}{f(1)^{2n+1}} \\
&= \frac{f(1)}{2f'(1)} \left(1 - \left(1 - \frac{f'(1)}{f(1)}(1-s)\right)^{2n+2}\right) \\
&=: \frac{f(1)}{2f'(1)} \left(1 - h(s)^{2n+2}\right),
\end{aligned}$$

with $h(s) := 1 - \frac{f'(1)}{f(1)}(1-s)$. Note that $0 < h(s) < 1$ for $s \in [s_2, 1]$.

This finishes the proof of Claim 1.

Claim 2 $\lim_{n\to\infty} I_n(s) = 0$ for every $0 < s < 1$.

To this end, we need to show that $\max_{[0,1]} |f| = f(1)$ and that the maximum is *only* obtained at 1 (note that f may take negative values). In fact, since f is increasing, $f(0) \le f(x) \le f(1)$ for every $x \in [0,1]$. If $f(0) \ge 0$, nothing has to be proven. So let $f(0) < 0$. Then, by the mean value theorem on $[0,1]$ there is $0 < c_x < 1$ such that

$$f(x) = f(0) + f'(c_x)x \le f(0) + f'(1)x \le f(0) + f'(1)$$

(note that f' is increasing). Using that $0 \le f'(1) < 2f(1)$ [22], we obtain $f(1) < f(0) + 2f(1)$. Hence $f(0) > -f(1)$. As f is strictly increasing, we also have $f(0) < f(1)$, and so $|f(0)| < f(1)$. Moreover, $|f(x)| \neq f(1)$ for any $x \in [0,1[$.

We conclude that

$$|I_n(s)| = \left| \frac{n+1}{f(1)^{2n+1}} \int_0^s f(x)^{2n+1} \right| \le (n+1)s \left(\frac{\max_{[0,s]} |f(x)|}{f(1)}\right)^{2n+1} =: (n+1)M^{2n+1},$$

where $0 < M = M(s) < 1$. As $\sum_{n=1}^{\infty} (n+1)M^{2n+1}$ converges, $I_n(s) \to 0$ as $n \to \infty$.

We are now ready to determine the limit of $\frac{n+1}{f(1)^{2n+1}} \int_0^1 f(x)^{2n+1}\, dx$. To this end, fix $\varepsilon > 0$ and choose $s_3 = s_3(\varepsilon) \in]0,1[$ so that for all $s \in [s_3, 1]$

$$\left| \frac{f(1)}{2f'(s)} - \frac{f(1)}{2f'(1)} \right| < \varepsilon.$$

[22] It is only here that we use this assumption.

Now for $s_0 := \max\{s_1, s_2, s_3\}$, depending on ε, we obtain from Claim 1 that

$$\frac{f(1)}{2f'(1)}\Big(1 - h^{2n+2}(s_0)\Big) \leq J_n(s_0) \leq \frac{f(1)}{2f'(s_0)} \leq \frac{f(1)}{2f'(1)} + \varepsilon.$$

Since $0 < h(s_0) < 1$, there is $n_0 = n_0(\varepsilon, s_0)$ such that

$$0 < h(s_0)^{2n+2} < \varepsilon \text{ for all } n \geq n_0.$$

Thus, for $n \geq n_0$

$$\frac{f(1)}{2f'(1)}(1 - \varepsilon) \leq J_n(s_0) \leq \frac{f(1)}{2f'(1)} + \varepsilon.$$

By Claim 2, there is $n_1 \geq n_0$ (depending on ε) such that $|I_n(s_0)| < \varepsilon$ for $n \geq n_1$. We conclude that for these $n \geq n_1$

$$\frac{n+1}{f(1)^{2n+1}} L_n = I_n(s_0) + J_n(s_0) \begin{cases} \leq & \varepsilon + \frac{f(1)}{2f'(1)} + \varepsilon \\ \geq & -\varepsilon + \frac{f(1)}{2f'(1)}(1 - \varepsilon). \end{cases}$$

Hence

$$\left| \frac{n+1}{f(1)^{2n+1}} L_n - \frac{f(1)}{2f'(1)} \right| \leq \max\left\{2\varepsilon, \varepsilon\left(1 + \frac{f(1)}{2f'(1)}\right)\right\}.$$

□

Remark The function $f(x) = x - 1/2$ shows that the assertion may fail if $f'(1) = 2f(1)$, since in this case $L_n = 0$. On the other hand, it may hold, too if $f'(1) = 2f(1)$. In fact, if $f(x) = e^{2x}$, then $f'(1) = 2f(1)$ and

$$L_n = \frac{e^{4n+2} - 1}{4n+2} \quad \text{and} \quad R_n = \frac{e^4}{4e^2} \cdot \frac{e^{4n}}{n} = \frac{e^{4n+2}}{4n},$$

nevertheless $L_n/R_n \to 1$. What is the reason for this? Well, an analysis of the proof shows that the condition $f'(1) < 2f(1)$ can be replaced by the assumption that the maximum of $|f|$ is *only* obtained at 1. This makes the class of functions with the wished assymptotic behavior of the integrals $\int_0^1 f(x)^{2n+1}\, dx$ much larger.

4780. *Proposed by Florică Anastase.*

Let $0 < a < b, m = \frac{a+b}{2}$ and $f\colon [a,b] \to \mathbb{R}$ differentiable with derivative continuous on $[a,b]$ such that $f(m) = 0$. Prove that

$$2a^3 \int_{-a}^{a} (f'(x))^2\, dx \geq 3\left(\int_{-a}^{a} f(x)\, dx\right)^2.$$

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The assertion is not compatible with the hypotheses. So we prove the following two results:

Example 17. *Let $a > 0$ and $f \in C^1[-a,a]$. If $f(0) = 0$, then*

$$\int_{-a}^{a} (f'(x))^2\, dx \geq \frac{3}{2a^3}\left(\int_{-a}^{a} f(x)\, dx\right)^2.$$

Example 18. *Let $0 < a, b < \infty$ and $f \in C^1[a,b]$. If $f((a+b)/2) = 0$, then, with $C = \frac{12}{(b-a)^3}$,*

$$\int_{a}^{b} (f'(x))^2\, dx \geq C\left(\int_{a}^{b} f(x)\, dx\right)^2.$$

Proof of Example 1. Let p be a polynomial. Then, using The Cauchy-Schwarz inequality

$$I := \left(\int_0^a (f'p)(x)\, dx\right)^2 \leq \left(\int_0^a (f'(x))^2\, dx\right)\left(\int_0^a p(x)^2\, dx\right)$$

Using partial integration,

$$I = \left((f(x)p(x)\Big|_0^a - \int_0^a f(x)p'(x)\, dx\right)^2$$

Now choose $p(x) = x - a$. Then $\int_0^a p(x)^2\, dx = \frac{1}{3}(x-a)^3\Big|_0^a = \frac{1}{3}a^3$. Hence, by noticing that $p(a) = f(0) = 0$,

$$I = \left(\int_0^a f(x)\, dx\right)^2 \leq \left(\int_0^a (f'(x))^2\, dx\right)\frac{1}{3}a^3$$

If we choose $p(x) = x + a$, then $p(-a) = 0$, and we similarily obtain the appropriate estimation for $\int_{-a}^0 f(x)dx$. Hence, using that $(x+y)^2 \leq 2(x^2+y^2)$,

$$\left(\int_{-a}^{a} f(x)\, dx\right)^2 \leq \frac{2}{3}a^3 \int_{-a}^{a} (f'(x))^2\, dx$$

□

Proof of Example 2. Just use the affine transformation ϕ given by $\phi(x) = x + \frac{a+b}{2}$. Then $\phi(-\frac{b-a}{2}) = a$ and $\phi(\frac{b-a}{2}) = b$, as well as $\phi(0) = \frac{a+b}{2}$. Let $c := (b-a)/2$. Hence, with $F(t) := f(\phi(t))$ for $-c \leq t \leq c$ we obtain

$$\int_a^b (f'(x))^2\, dx = \int_{-c}^{c} (F'(t))^2\, dt \geq \frac{3}{2c^3}\left(\int_{-c}^{c} F(t)\, dt\right)^2 = \frac{12}{(b-a)^3}\left(\int_a^b f(x)\, dx\right)^2.$$

□

Of course Example 1 is a special case of Example 2. Is C best possible? Let

$$q(x) = \begin{cases} \dfrac{(x-a)^2}{2} - \dfrac{(b-a)^2}{8} & \text{if } a \le x \le (a+b)/2 \\ \dfrac{(x-b)^2}{2} - \dfrac{(a-b)^2}{8} & \text{if } (a+b)/2 \le x \le b. \end{cases}$$

Then q is continuous on $[a,b]$, $q((a+b)/2) = 0$ and

$$\int_a^b (q'(x))^2\,dx = \frac{12}{(b-a)^3}\left(\int_a^b q(x)\,dx\right)^2.$$

Unfortunately, q is not C^1. How to modify?

4777. *Proposed by Goran Conar, modified by the Editorial Board.*

Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \geq 1$ such that $\sum_{i=1}^n \frac{1}{x_i} = 1$. Prove

$$\frac{n}{1/2+n^2} < \sum_{i=1}^n \frac{1}{\frac{1}{2}+x_i^2} < \frac{2}{3}.$$

Solution to problem 4777 Crux Math. 48 (8) 2022, 484

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The assertion is not correct. In fact, let $\mathbf{x} := (x_1, \ldots, x_n)$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$,

$$S := \Big\{(x_1, \ldots, x_n) \in (\mathbb{R}^+)^n : \sum_{i=1}^n \frac{1}{x_i} = 1\Big\},$$

and

$$f(\mathbf{x}) := \sum_{i=1}^n \frac{1}{\frac{1}{2}+x_i^2}.$$

We prove that for $n \geq 2$,

$$\boxed{\frac{n}{1/2+n^2} = \min_{\mathbf{x}\in S} f(\mathbf{x}) < \sup_{\mathbf{x}\in S} f(\mathbf{x}) = \frac{2}{3},}$$

and that for $n = 1$, $S = \{1\}$ and so

$$\boxed{f(x) = \frac{1}{\frac{1}{2}+x^2} = f(1) = \frac{2}{3}.}$$

Proof Wlog $n \geq 2$. First we note that $\sum_{i=1}^n 1/x_i = 1$ for $x_i \in \mathbb{R}^+$ implies that $x_i \geq 1$ for every i. Now $\max\limits_{1\leq x<\infty} \dfrac{x}{1+2x^2} = \dfrac{1}{3}$, since the function is decreasing on $[1, \infty[$. Hence, for $\mathbf{x} \in S$,

$$\sum_{i=1}^n \frac{1}{\frac{1}{2}+x_i^2} = \sum_{i=1}^n \frac{x_i}{1+2x_i^2}\frac{2}{x_i} < \frac{2}{3}\sum_{i=1}^n \frac{1}{x_i} = \frac{2}{3},$$

since for $n \geq 2$, no x_i can be 1. If for $k > n$

$$\mathbf{x}_k = \Big(x_1^{(k)}, \ldots, x_n^{(k)}\Big) := \Big(\frac{1}{1-(n-1)/k}, k, \ldots, k\Big),$$

then $\sum_{i=1}^n (1/x_i^{(k)}) = 1$, $\mathbf{x}_k \to (1, \infty, \ldots, \infty)$ and $f(\mathbf{x}_k) \to 2/3$. Hence $\sup_S f = 2/3$.

To prove the assertion on the minimum, we use Lagrange. It is preferable to work with the new variable $y_j := 1/x_j$ (to get a compact definition set, guarantying the existence of the global extrema). So let

$$S' = \Big\{(y_1, \ldots, y_n) \in \mathbb{R}^n,\ y_j \geq 0 : \sum_{j=1}^n y_j = 1\Big\}$$

and

$$g(y_1, \ldots, y_n) := \sum_{i=1}^n \frac{y_i^2}{1+\frac{1}{2}y_i^2}.$$

Then S' is compact and $\inf f_S = \inf g_{S'} = \min g_{S'} =: m$. Say $g(\mathbf{x}') = m$ for some $\mathbf{x}' \in S'$. In order to apply Lagrange, we need to show that $\mathbf{x}'$ is an interior point of S' (in symbols, $\mathbf{x}' \in (S')^\circ$). Let $\mathbf{y}' := (1/n, \ldots, 1/n)$. Then $\mathbf{y}' \in (S')^\circ$. Now on $\partial S'$ at least one of the coordinates of these points $\mathbf{y} := (y_1, \ldots, y_n) \in \partial S'$ is 0. Say, $y_n = 0$. But then $\sum_{i=1}^{n-1} y_i = 1$ and (via induction on n, starting with the trivial case of one-tuples)

$$g(\mathbf{y}) \geq \frac{n-1}{1/2 + (n-1)^2} > \frac{n}{1/2 + n^2} = g(\mathbf{y}').$$

Hence the absolute minimum of g on S' does not belong to the boundary.

By Lagrange's theorem, there exists $\lambda \in \mathbb{R}$ and $(y_1, \ldots, y_n) \in S'$ such that

$$\nabla\Big(g(y_1, \ldots, y_n) + \lambda(1 - \sum_{i=1}^{n} y_i)\Big) = \mathbf{0}.$$

That is, for every $i \in \{1, \ldots, n\}$,

$$\lambda = \frac{2y_i}{(1 + \frac{1}{2}y_i^2)^2}. \tag{112}$$

Unfortunately, the function $y \mapsto q(y) := \frac{2y}{(1+\frac{1}{2}y^2)^2}$ is not injective on $[0, 1]$ (note that the derivative vanishes at $y = \pm\sqrt{2/3}$). So we must discuss several cases (see figure 18):

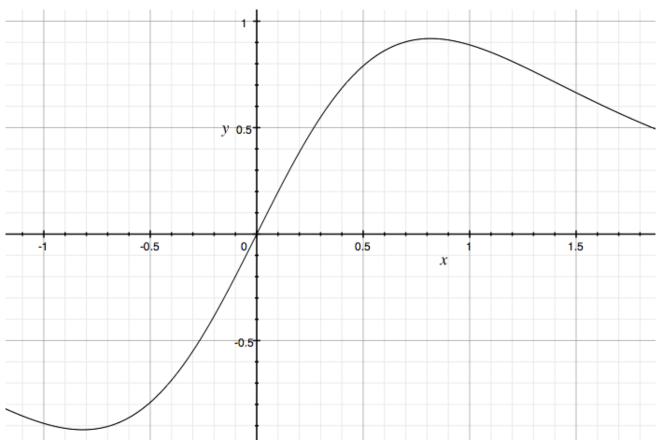

FIGURE 18. Non injectivity of q on $[0, 1]$

(i) If $8/9 = q(1) \leq \lambda < \max_{[0,1]} q$, then the equation $q(y) = \lambda$ has two solutions $0 < y_1, y_2 \leq 1$.

(ii) If $\lambda = \max_{[0,1]} q$ or if $0 \leq \lambda < q(1) = 8/9$, then the equation $q(y) = \lambda$ has exactly one solution $0 \leq y_0 \leq 1$.

(iii) In all other cases, there is no solution with $y \geq 0$.

We first show that the case (i) does not yield minimal solutions. In fact, for fixed $\lambda \in [q(1), \max_{[0,1]} q[$, equation (112) has 2^n solutions of the form $P := (\underbrace{a, \ldots, a}_{k\text{-times}}, \underbrace{b, \ldots, b}_{(n-k)\text{-times}})$ and their permutations, where $k = 0, \ldots, n$ and $0 \leq a \leq b \leq 1$. Note that

$$q(1/n) = \frac{8n^3}{(1 + 2n^2)^2} \leq q(1/2) < q(1) = \frac{8}{9} < q(\sqrt{2/3}). \tag{113}$$

Hence $1/n \leq 1/2 < \min\{a, b\}$ (see figure 18).

Let $A := (1/n, \ldots, 1/n)$. Then $A \in S'$. Since the function $y \mapsto y^2/(1 + \frac{1}{2}y^2)$ is increasing on $[0, \infty[$, we deduce that

$$g(P) = k\frac{a^2}{1 + \frac{1}{2}a^2} + (n-k)\frac{b^2}{1 + \frac{1}{2}b^2} > g(A),$$

so P does not yield a minimum. Thus only the second case occurs. That is, we need to consider only a solution of (112) of the form $(y_1, \ldots, y_n) = (a, \ldots, a)$ with $0 < a \leq 1$. Using the constraint condition $\sum_{i=1}^{n} y_i = 1$, we obtain that $a = 1/n$, hence $(y_1, \ldots, y_n) = (1/n, \ldots, 1/n)$.

Consequently, $\mathbf{x}' = (1/n, \ldots, 1/n)$ is the unique point where g takes its absolute minimum on S'. We conclude that

$$\min g_{S'} = \frac{n}{\frac{1}{2} + n^2}.$$

For completeness, we observe that $M := \max_{S'} g$ necessarily is obtained on the boundary of S' (for instance, $M = g(1, 0, \ldots, 0) = 2/3$), as Lagrange only yields a single stationary point of the Lagrange function in $(S')^\circ$.

A second way to see that case (i) does not occur goes as follows:

We first show that the case (i) does not yield minimal solutions. In fact, for fixed $\lambda \in [q(1), \max_{[0,1]} q[$, equation (112) has 2^n solutions of the form $P := (\underbrace{a, \ldots, a}_{k\text{-times}}, \underbrace{b, \ldots, b}_{(n-k)\text{-times}})$ and their permutations, where $k = 0, \ldots, n$ and $0 \leq a \leq b \leq 1$. Note that $q(1/2) = (8/9)^2$ and that $n \geq 2$. Thus

$$q(1/n) \leq q(1/2) < q(1) \leq \lambda < q(\sqrt{2/3}). \tag{114}$$

Hence $1/n \leq 1/2 < \min\{a, b\} = a$ (see figure 18). Since for such a point $P = (y_1, \ldots, y_n)$ we have

$$\sum_{i=1}^{n} y_i = ka + (n-k)b > k\frac{1}{2} + (n-k)\frac{1}{2} = \frac{n}{2} \geq 1,$$

P does not belong to S'; that is such a solution of the system (112) of equations does not satisfy the constraint $P \in S'$.

4763. *Proposed by William Weakley.*

Let K be a field and let S be a nonempty subset of K that is closed under subtraction.

a) For all K and S, characterize the functions $f : S \to K$ such that

$$f(x)f(y) = f(x-y) \text{ for all } x, y \in S.$$

b) As K and S vary, what finite cardinalities can the set of such functions have?

Partial Solution to problem 4763 Crux Math. 48 (7) 2022, 421

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Here we give our thoughts on this not very precisely formulated problem.

First we note that $S \subseteq K$ necessarily is an additive subgroup of the field K. Note that $\{0,1\} \subseteq K$. In particular $0 = x - x \in S$ and with $x \in S$ we have $-x = 0 - x \in S$.

If

$$(FE) \qquad f(x-y) = f(x)f(y) \text{ for all } x, y \in S$$

then we get the following:

(1) $y = x \Longrightarrow f(0) = f(x)^2$

(2) $y = 0 \Longrightarrow f(x) = f(x)f(0) \Longrightarrow f(x)(1 - f(0)) = 0$

Case 1 There exists $x_0 \in S$ with $f(x_0) = 0$. Then, by (1), $f(0) = 0$ and so $f(x) = 0$ for all $x \in S$.

Case 2 f has no zeros. Then (2) implies that $f(0) = 1$.

We claim that $f(2x) = 1$ for every $x \in S$ (note that $\mathbb{Z}S \subseteq S$).

In fact, $f(x) = f(2x - x) = f(2x)f(x)$, hence $f(2x) = 1$.

We conclude that for $S = \mathbb{R}$ e.g., the constant function $f(y) = 1$ is the only solution, as every $y \in \mathbb{R}$ writes as $y = 2x$ for some x.

Next we show that f is even and that $f(x) \in \{-1, 1\}$. In fact, by (FE), for $x = 0$,

$$f(-y) = f(0)f(y) = f(y) \text{ for every } y \in S.$$

Hence $1 = f(2u) = f(u - (-u)) = f(u)f(-u) = f(u)^2$ for any $u \in S$.

If $S = \mathbb{Z}$, then we have three solutions: $f \equiv 0$, $f \equiv 1$ but also

$$f(n) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd.} \end{cases}$$

In fact by the claim above, $f(2m) = 1$ for every $m \in \mathbb{Z}$. Now let $\sigma := f(1)$. We already know that $\sigma = \pm 1$. Now for every $m \in \mathbb{Z}$,

$$\sigma = f(1) = f((2m+1) - 2m) = f(2m+1)f(2m) = f(2m+1).$$

Let $P := P_f := \{x \in S : f(x) = 1\}$ and $R := \{x \in S : f(x) = -1\}$. Then P is a subgroup of S since $x, y \in P$ implies that $x - y \in P$, because $f(x-y) = f(x)f(y) = 1 \cdot 1 = 1$.

As shown above, $2S \subseteq P \subseteq S$ and $2S$ is a subgroup of S. Here $S = 2S$ if and only if all the translation operators $\tau_x : S \to S, y \mapsto x - y$ have a fixed point.

Also note that R has the following property:

$$(PR) \qquad (R - R) \subseteq P \text{ and } (R - P) \cup (P - R) \subseteq R.$$

Conversely, if P is a proper subgroup of S and $R := S \setminus P$ such that (PR) holds, then the function g given by

$$g(x) = \begin{cases} 1 & \text{if } x \in P \\ -1 & \text{if } x \in R \end{cases},$$

satisfies the functional equation (FE) $g(x-y) = g(x)g(y)$ for $x, y \in S$.

Note that P may be strictly bigger than $2S$: in fact, let $K = \mathbb{C}$, $S := \mathbb{Z} + i\mathbb{Z}$, $P = 2\mathbb{Z} + i\mathbb{Z}$ and $R = S \setminus P$. Then S, P, R satisfy (PR), but $P := 2S$ does not satisfy (PR).

If $S = K$ is a field of characteristic 2, then $P_f = R = S$ (note that $1 = -1$), and so only the constant functions 1 and 0 satisfy (FE).

4747. *Proposed by Stanescu Florin.*

Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 f(x) + f(y)) = f(f(x^3)) + y$$

for all $x, y \in \mathbb{R}$.

Solution to problem 4747 Crux Math. 48 (2022), 282 by

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We claim that all solutions of the functional equation

$$f(x^2 f(x) + f(y)) = f(f(x^3))) + y,\ \ x, y \in \mathbb{R}$$

are given by $f(x) = x$ and $f(x) = -x$.

Claim1 f is injective:
Put $x = 0$. Then

(115)
$$f(f(y)) = f(f(0)) + y.$$

Now if $f(y_1) = f(y_2)$, then by (115)

$$f(f(y_1)) = f(f(0)) + y_1 \text{ and } f(f(y_2)) = f(f(0)) + y_2$$

Hence $y_1 = y_2$.

Claim 2 f is surjective:
Let $w \in \mathbb{R}$. Then, by (115),

$$w = f(f(0)) + (w - f(f(0))) = f(f(w - f(f(0))).$$

Claim 3 $f(0) = 0$:
Take $y = 0$: then $f(x^2 f(x) + f(0)) = f(f(x^3))$. Since f is bijective, we conclude that $x^2 f(x) + f(0) = f(x^3)$. Now put $x = 1$: then $1^2\ f(1) + f(0) = f(1)$. Hence $f(0) = 0$.

Claim 4 $f \circ f = id$ (that is, f is an involution).
This follows from (115).
Hence our equation becomes

(116)
$$f(x^2 f(x) + f(y)) = x^3 + y\ \ (x, y \in \mathbb{R}).$$

In particular, for $y = 0$,

(117)
$$f(x^2 f(x)) = x^3 \text{ or equivalenty } x^2 f(x) = f(x^3).$$

Claim 5 f is additive:
In fact, the surjectivity of f and $x \mapsto x^3$ now imply that $x \mapsto x^2 f(x)$ is surjective, too. Hence

$$f(\underbrace{x^2 f(x)}_{=a} + \underbrace{f(y)}_{=b}) = x^3 + y = f(x^2 f(x)) + y = f(a) + f(b)$$

yields the additivity of f.
Claim 6 $f(-x) = -f(x)$.
Just use that with $f(0) = 0$ and f additive,

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x).$$

Claim 7 Let $f(a) = 1$. Then $a = \pm 1$.

Recall that by (5) and (6), $f(a+b) = f(a) + f(b)$ for $a, b \in \mathbb{R}$ and $f(mx) = mf(x)$ for every $m \in \mathbb{Z}$. Hence, by (117), for $x = a + b$,

$$(a+b)^2 f(a+b) = f((a+b)^3).$$

Expansion yields:

$$\begin{aligned}
(a^2+b^2+2ab)(f(a)+f(b)) &= f(a^3)+3f(a^2b)+3f(ab^2)+f(b^3) \iff \\
\underline{a^2f(a)}+b^2f(a)+2abf(a)+a^2f(b)+\underline{b^2f(b)}+2abf(b) &= \underline{f(a^3)}+3f(a^2b)+3f(ab^2)+\underline{f(b^3)} \iff \\
b^2f(a)+2abf(a)+a^2f(b)+2abf(b) &= 3f(a^2b)+3f(ab^2).
\end{aligned}$$

- Let $b = 1$ and note that $a = f(1)$. Then

$$1 + 2a + a^3 + 2a^2 = 3f(a^2) + 3 = 3a^3 + 3 \iff$$

$$2a^3 - 2a^2 - 2a + 2 = 0 \iff a^2(a-1) - (a-1) = 0 \iff (a-1)(a^2-1) = 0 \iff a \in \{-1, 1\}.$$

Claim 8 If the additive function f satisfies $x^2 f(x)) = f(x^3)$, then $f(x) = f(1)\,x$. To see this, we consider four cases:

- Let $a = 2$, $f(1) = \pm 1$ and $b = x$. Then

$$\boxed{\pm x^2 \pm 4x - 4f(x) + 2xf(x) - 3f(x^2) = 0}. \tag{118}$$

- Let $a = 1$, $f(1) = \pm 1$ and $b = x$. Then,

$$\boxed{\pm x^2 \pm 2x - 2f(x) + 2xf(x) - 3f(x^2) = 0}. \tag{119}$$

Calculating (118)-(119), yields $\pm 2x - 2f(x) = 0$. Hence $f(x) = \pm x = f(1)x$.

One can also prove Claim 8 without using Claim 7, and then deducing Claim 7 from Claim 8 if additionally we assume that f is an involution.

In

$$b^2f(a) + 2abf(a) + a^2f(b) + 2abf(b) = 3f(a^2b) + 3f(ab^2). \tag{120}$$

choose $a = 1$, resp. $a = 2$ and $b = x$. Then $f(2) = f(2 \cdot 1) = 2f(1)$ and so

$$x^2f(1) + 2f(1)x + f(x) + 2xf(x) - 3f(x) - 3f(x^2) = 0 \tag{121}$$

$$2x^2f(1) + 8f(1)x + 4f(x) + 4xf(x) - 12f(x) - 6f(x^2) = 0 \tag{122}$$

Hence, by calculating (121)-$\frac{1}{2}$(122), we obtain

$$-2f(1)x - f(1)x + 3f(x) = 0. \tag{123}$$

Hence $f(x) = f(1)x$. Using (117), that is $f(x^2 f(x)) = x^3$, we have

$$f(1)x^2 f(1)x = x^3.$$

Hence $f(1)^2 = 1$ and so $f(1) = \pm 1$.

4657. *Proposed by George Stoica.*

Let us consider the equation $f(x) + f(2x) = 0$, $x \in \mathbb{R}$.

(i) Prove that, if f is continuous at 0, then $f(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Construct a function f, discontinuous at every $x \in \mathbb{R}$, that solves the given equation.

Solution to problem 4657 Crux Math. 47 (2021), 301 by

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a) Suppose that the function f satisfies $f(x) + f(2x) \equiv 0$ on $\mathbb{R}$. Then the continuity of f at $x = 0$ implies that $f \equiv 0$. In fact, fix $x \in \mathbb{R} \setminus \{0\}$. By induction, $f(x/2^n) = (-1)^n f(x)$. By taking limits, the continuity at 0 implies that for n even we get $f(0) = f(x)$ and for n odd, we get $f(0) = -f(x)$. Hence $2f(x) = f(0) - f(0) = 0$, and so $f \equiv 0$.

b) Define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(0) = 0$, $f(x) = 1$ if $1 \le x < 2$ and x rational, $f(x) = -1$ if $1 < x < 2$ and x irrational. If $n \in \mathbb{N}$ and $2^n \le x < 2^{n+1}$, put $f(x) = (-1)^n f(x/2^n)$. If $\frac{1}{2^{n+1}} \le x < \frac{1}{2^n}$, put $f(x) = (-1)^n f(2^n x)$. If $x < 0$, then let $f(x) = f(-x)$. Then f is discontinuous everywhere and, by construction, $f(x) + f(2x) = 0$.

c) All solutions to $f(x) + f(2x) \equiv 0$ on $\mathbb{R}$:
Let $g : [-2, -1[\cup [1, 2[\to \mathbb{R}$ be an arbitrary function. Put

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ (-1)^n g(x/2^n) & \text{if } 2^n \le |x| < 2^{n+1} \\ (-1)^n g(2^n x) & \text{if } \frac{1}{2^{n+1}} \le |x| < \frac{1}{2^n}. \end{cases}$$

This functional equation and its companion $f(x) = f(2x)$ appear multiple times, see [71, 72, 73, 74, 75, 76, 77, 78, 79, 80]

4636. *Proposed by Mihaela Berindeanu.*

Solve the following equation over the set of real numbers:

$$(3^x + 7)^{\log_4 3} - (4^x - 7)^{\log_3 4} = 4^x - 3^x - 14.$$

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Raymond Mortini, Rudolf Rupp

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The equation

$$(3^x + 7)^{\log_4 3} - (4^x - 7)^{\log_3 4} = 4^x - 3^x - 14$$

has on $\mathbb{R}$ the unique solution $x = 2$. In fact, first note that $a := \log_4 3 = \frac{\log 3}{\log 4} > 0$ and $\log_3 4 = 1/a$. Then with $A := 3^x + 7$ and $B := 4^x - 7$ we have to solve $A^a - B^{1/a} = B - A$ or equivalently,

$$A^a + A = (B^{1/a})^a + B^{1/a}.$$

Since the function $x \mapsto x^a + x$ is strictly increasing, we deduce that $A = B^{1/a}$. In other words, $3^x + 7 = (4^x - 7)^{1/a}$, or equivalently

$$\log 4 \log(4^x - 7) = \log 3 \log(3^x + 7). \tag{124}$$

The curve $y(x) = \log 4 \log(4^x - 7) - \log 3 \log(3^x + 7)$ is defined for $x > \log 7/\log 4 := x_0$ with $\lim_{x\to x_0} y(x) = \infty$ and its derivative

$$y'(x) = \log^2 4 \frac{1}{1 - 7x^{-4}} - \log^2 3 \frac{1}{1 + 7x^{-3}}$$

is strictly decreasing with $\lim_{y\to x_0} y'(x) = \infty$ and $\lim_{x\to\infty} y'(x) = (\log^2 4 - \log^2 3)$. Note that the asymptote at infinite is the line $y = (\log^2 4 - \log^2 3)x$. In particular, $y' > 0$ and so the curve is strictly increasing and its unique zero is $x_1 = 2$ (observe that $\log(4)\log(9) = 4\log(2)\log(3) = \log(3)\log(16)$, so (124) holds).

4634. *Proposed by George Stoica.*

Let $\sum_{n=1}^{\infty} a_n < \infty$ for $a_n > 0$, $n = 1, 2, \ldots$ Find $\lim_{n\to\infty} n \cdot \sqrt[n]{a_1 \cdots a_n}$.

Solution to problem 4634 Crux Math. 47 (2021), 200 by

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For $a_n > 0$, let $G_n := n(a_1 \cdots a_n)^{1/n}$. Then $\lim_{n\to\infty} G_n = 0$ whenever $\sum_{n=1}^{\infty} a_n$ is convergent. In fact, given $\varepsilon > 0$, choose N so big that $\sum_{n=N}^{\infty} a_n < \varepsilon$. Due to the arithmetic-geometric inequality, for $n > N$,

$$\begin{aligned} G_n &= (a_1 \cdots a_N)^{1/n} \, \frac{n}{n-N} \Big((n-N)(a_{N+1} \cdots a_n)^{1/(n-N}\Big)^{\frac{n-N}{n}} (n-N)^{1-\frac{n-N}{n}} \\ &\le \sigma_n \Big(\sum_{j=N+1}^{n} a_j \Big)^{\frac{n-N}{n}}, \end{aligned}$$

where

$$\sigma_n := (a_1 \cdots a_N)^{1/n} \, \frac{n}{n-N} (n-N)^{N/n}.$$

Since $\lim_n \sigma_n = 1$, we have $\limsup_n G_n \le \limsup_n \varepsilon^{\frac{n-N}{n}} = \varepsilon$, from which we deduce that $G_n \to 0$.

4615. *Proposed by Anthony Garcia.*

Let f be a twice differentiable function on $[0,1]$ such that $\int_0^1 f(x)dx = \frac{f(1)}{2}$. Prove that

$$\int_0^1 (f''(x))^2 dx \geq 30(f(0))^2.$$

Solution to problem 4615 Crux Math. 47 (2021), 301 by

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If p is a polynomial, we have (due to Cauchy-Schwarz)

$$\left|\int_0^1 f''pdx\right|^2 \leq \left(\int_0^1 (f'')^2 dx\right)\left(\int_0^1 p^2 dx\right).$$

Now, by using twice integration by parts,

$$\int f''pdx = (f'+c)p - \left((f+cx+c')p' - \int (f+cx+c')p''dx\right)$$

Now let $p(x) = x(x-1)$. Evaluation at the end-points and using the hypothesis that $\int_0^1 fdx = f(1)/2$, yields

$$\int_0^1 f''pdx = -f(0).$$

Since $\int_0^1 p^2 dx = \int_0^1 (x^4 + x^2 - 2x^3)dx = 1/30$, we deduce that

$$\int_0^1 (f'')^2 dx \geq 30f(0)^2.$$

Equality is given if $f'' = p$ and $f(1) = 2\int_0^1 fdx$; for instance if

$$f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - \frac{1}{30}.$$

Here $f(1) = -7/60$.

Generalizations appear in [9].

6. EMS Newsletter

Aufgabe Q68, EMS Newsletter 25 (1997), 27.

Q.68 *A function f satisfies the equation $f(x+1)+f(x-1)=\sqrt{2}\cdot f(x)$ for all real x. Prove that this function is periodic.*

First Solution (Raymond Mortini, Luxembourg, Université de Metz, Département de Mathématiques)

–Ed. *It is the first time that I received a submission in German language, and astoundingly it came to me from France. The overtures of friendship between Germans and Frenchmen aforementioned appear to work. So, it is fortunate that Mathematics can amplify such advances; hence this solution is presented true to the original.*

Behauptung. Es sei f eine Funktion auf $\mathbb{R}$ welche der Bedeinung genügt:

$$f(x+1)+f(x-1)=\sqrt{2}f(x)$$

Dann hat f die Periode 8.

Beweis. Es sei $x\in\mathbb{R}$ beliebig aber fest gewählt. Dan ergeben sich aus der Voraussetzung die folgenden Gleichungen:

$$f(x+8)=\sqrt{2}f(x+7)-f(x+6)=\sqrt{2}[-f(x+5)+\sqrt{2}f(x+6)]-f(x+6)=-\sqrt{2}f(x+5)+f(x+6)=-f(x+4).$$

Damit ergibt sich sofort die Behauptung $f(x+8)=-f(x+4)=-(-f(x))=f(x)$.

Bemerkung. Alle Lösungen der obigen Funktionalgleichung haben die Form

$$f(x+n)=r(x)\cdot sin(\theta(x)+n\cdot\tfrac{\pi}{4})\text{ für }x\in[0,1[,\quad n\in\mathbb{Z},$$

wobei $r(x)>0$ und $\theta(x)$ beliebige Funktionen sind. ∎

Also solved by Dr. J N Lillington.

7. Math. Gazette

109.H (Toyesh Prakash Sharma)

Evaluate

$$\left(\int_0^\infty e^{-x^2} \cos(\ln x)\, dx\right)^2 + \left(\int_0^\infty e^{-x^2} \sin(\ln x)\, dx\right)^2.$$

Solution to problem 109.H, Math. Gazette 109, Issue 575 (2024), p. 354

by Raymond Mortini and Rudolf Rupp

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109.B (Seán Stewart)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \operatorname{sech}^2\left(\frac{\pi}{2^{n+1}}\right) = \frac{4}{\pi^2} - \operatorname{cosech}^2\left(\frac{\pi}{2}\right).$$

Solution to problem 109.B, Math. Gazette 109, Issue 574 (2024), p. 169

by Raymond Mortini and Rudolf Rupp

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108.H (Mark Hennings)
Prove the following:

(a) $$\int_0^\pi \frac{x(\pi - x)}{\sin x} dx = 7\zeta(3),$$

(b) $$\int_{-\infty}^{\infty} \tan^{-1}(e^x) \tan^{-1}(e^{-x}) dx = \frac{7}{4}\zeta(3).$$

Solution to problem 108.H, Math. Gazette 108, Issue 572 (2024), p. 364

by Raymond Mortini and Rudolf Rupp

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(a) We first consider the Fourier series of the function

$$g(x) = \begin{cases} x(\pi - x) & \text{if } 0 \le x \le \pi \\ x(\pi + x) & \text{if } -\pi \le x \le 0 \end{cases}$$

(the odd extension of $x(\pi - x)$), and extend it 2π-periodically. Then

$$g(x) = \frac{8}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(nx)}{n^3}.$$

Hence

$$\begin{aligned} I_1 := \int_0^\pi \frac{x(\pi - x)}{\sin x} dx &= \frac{8}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^3} \int_0^\pi \frac{\sin(nx)}{\sin x} dx \\ &= \frac{8}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^3} \pi = 8 \frac{7}{8} \xi(3) = 7\zeta(3). \end{aligned}$$

Note that the integrand coincides with $U_{n-1}(\cos x)$, where U_n is the Chebyshev polynomial of the second kind. To see that for odd n,

$$J_n := \int_0^\pi \frac{\sin(nx)}{\sin x} dx = \pi$$

(a well known result), it suffices to show that $J_n = J_{n+2}$. This holds, though, since

$$\begin{aligned} J_{n+2} = \int_0^\pi \frac{\sin(nx)\cos 2x + \cos(nx)\sin 2x}{\sin x} dx &= \int_0^\pi \left(\frac{\sin(nx)}{\sin x} (1 - 2\sin^2 x) + 2\cos nx \cos x \right) dx \\ &= J_n + 2 \int_0^\pi (-\sin(nx)\sin x + \cos(nx)\cos x)\, dx \\ &= J_n + 2 \int_0^\pi \cos(n+1)x \, dx \\ &= J_n. \end{aligned}$$

(b) We shall work with suitable variable substitutions to regain the integral in (a). We use $\arctan x$ instead of the ambiguous notation $\tan^{-1} x$. Note that $\arctan(1/y) + \arctan y = \pi/2$ for $y > 0$.

$$\begin{aligned}
I_2 := \int_{-\infty}^{\infty} \arctan(e^x)\arctan(e^{-x})dx \quad & \underset{x=\log y}{\overset{e^x=y}{=}} \quad \int_0^{\infty} \frac{\arctan y \ \arctan \frac{1}{y}}{y} dy \\
& \underset{\arctan y = s}{\overset{y=\tan s}{=}} \quad \int_0^{\pi/2} \frac{s(\frac{\pi}{2}-s)}{\tan s} \frac{1}{\cos^2 s} ds \\
& = \quad 2\int_0^{\pi/2} \frac{s(\frac{\pi}{2}-s)}{\sin(2s)} ds \\
& \overset{2s=t}{=} \quad \int_0^{\pi} \frac{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)}{\sin t} dt \\
& = \quad \frac{1}{4} I_1.
\end{aligned}$$

108.G (Sean M. Stewart)

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$ where n is a positive integer. Evaluate:

$$\lim_{n\to\infty} n\left(\prod_{k=1}^{n} l_k\right)^{\frac{2}{n}}.$$

Solution to problem 108.G, Math. Gazette 108, Issue 572 (2024), p. 364

by Raymond Mortini and Rudolf Rupp

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We suppose that the l_k in the integrand should be the I_k, that is

$$I_k := \int_0^{\pi/2} (\sin x)^k dx.$$

So let

$$S_n := n\left(\prod_{k=1}^{n} I_k\right)^{\frac{2}{n}}.$$

We claim that

$$\boxed{\lim_n S_n = \frac{\pi}{2}e}.$$

We shall use that

$$I_k = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}.$$

Now S_n is a telescopic product; hence

$$S_n = n\left(\frac{\sqrt{\pi}}{2}\right)^2 \; \frac{\Gamma(1)}{\big(\Gamma(\frac{n}{2}+1)\big)^{2/n}}.$$

Using (the non-discrete) Stirling formula, which tells us that

$$\lim_{x\to\infty} \frac{\Gamma(x)}{\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x} = 1,$$

we obtain with $\Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2}\Gamma\left(\frac{n}{2}\right)$ that

$$S_n \quad\sim\quad \frac{\pi}{4}\frac{n}{\left(\frac{n}{2}\sqrt{\frac{2\pi}{n/2}}\left(\frac{n/2}{e}\right)^{n/2}\right)^{2/n}} = \frac{\pi}{4}\frac{n}{\left(\frac{n}{2}\right)^{2/n}\left(\frac{4\pi}{n}\right)^{1/n}\frac{n}{2e}} \to \frac{\pi}{2}e.$$

108.F (Peter Shiu)

Let $z = x + iy$ and $w = u + iv$ be complex numbers satisfying $z^2 + w^2 = r^2$, where $r > 0$. Show that if (x, y) runs over an ellipse with foci $\pm r$, then (u, v) runs over the same ellipse.

Solution to problem 108.F, Math. Gazette 108, Issue 572 (2024), p. 364

by Raymond Mortini and Rudolf Rupp

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This is entirely trivial: Note that an ellipse with focii $\pm r$, $r > 0$, in the z-plane is given by $|z-r| + |z+r| = 2c$ for some positive constant c. Hence, as $|z^2 - r^2| = |w^2|$, respectively $|w^2 - r^2| = |z^2|$, and through squaring,

$$\begin{aligned} |z-r|^2 + |z+r|^2 + 2|z-r|\ |z+r| &= 2|z|^2 + 2r^2 + 2|z^2 - r^2| \\ &= 2|z|^2 + 2r^2 + 2|w|^2 \end{aligned}$$

As the latter is symmetric in z and w, we obtain

$$(|z-r| + |z+r|)^2 = (|w-r| + |w+r|)^2.$$

As the terms are positive, it follows that $|z-r| + |z+r| = |w-r| + |w+r|$.

108.E (Ovidiu Gabriel Dinu)

Find all positive integers such that each of n, $n + 2$, $n + 6$, $n + 8$ and $n + 14$ is a prime number.

Solution to problem 108.E, Math. Gazette 108, Issue 572 (2024), p. 364

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We claim that only $n = 5$ yields the desired prime quintuplets.

Suppose that $n, n+2, n+6, n+8, n+14$ are prime numbers. Say $n = a_0 + \sum_{k=1}^{m} a_k 10^k$, with $0 \leq a_k \leq 9$. Obviously $a_0 \in \{1,3,7,9\}$ or $a_0 = 5$ and all $a_k = 0$ for $k \geq 1$.

Case 1 $a_0 = 1$. Then $n+14$ is not prime since $n+14 = 5 + (a_1+1)10 + \sum_{k=1}^{m} a_k 10^k$ is divisible by 5 [23].

Case 2 $a_0 = 3$. Then $n+2$ is not prime since $n+2 = 5 + \sum_{k=1}^{m} a_k 10^k$ is divisible by 5.

Case 3 $a_0 = 7$. Then $n+8$ is not prime since $n+8 = 5 + (a_1+1)10 + \sum_{k=1}^{m} a_k 10^k$ is divisible by 5.

Case 4 $a_0 = 9$. Then $n+6$ is not prime since $n+6 = 5 + (a_1+1)10 + \sum_{k=1}^{m} a_k 10^k$ is divisible by 5.

So it remains $a_0 = 5$: $(5, 7, 11, 13, 19)$, the only prime quintuple of this form.

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[23]Note that $a_1 + 1$ may be equal to 10; that does not alter the divisibility property, though.

108.D (Toyesh Prakash Sharma)

(a) Show that $\int_0^\infty (1 - e^{-2x}) \dfrac{\sin^2 x}{x^3} = \dfrac{\pi}{2}$

(b) Show that $\int_{-\infty}^{\infty} \cos^2(\tan x) \dfrac{\sin^2 x}{x^2} dx = \dfrac{\pi}{2}\left(1 + \dfrac{1}{e^2}\right) = \int_{-\infty}^{\infty} \sin^2(\tan x) \dfrac{\cos^2 x}{x^2} dx.$

Solution to problem 108.D, Math. Gazette 108, Issue 571 (2024), p. 167

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We give for (a) two proofs.

(a1) This is done via the Laplace transform $F(s) := \mathcal{L}[f](s) := \int_0^\infty f(t)e^{-st}dt$ for certain admissible functions f. First note that $\dfrac{1-e^{-2x}}{x} = \displaystyle\int_{s=0}^2 e^{-sx}ds$. Then, due to the convergence of the integrals, and Fubini's theorem,

$$\begin{aligned} I := \int_0^\infty (1-e^{-2x})\frac{\sin^2 x}{x^3}dx &= \int_0^\infty \frac{1-e^{-2x}}{x}\,\frac{\sin^2 x}{x^2}dx = \int_{x=0}^\infty \int_{s=0}^2 e^{-sx}\frac{\sin^2 x}{x^2}dsdx \\ &= \int_{s=0}^2 \left(\int_{x=0}^\infty e^{-sx}\,\frac{\sin^2 x}{x^2}\,dx\right) ds. \end{aligned}$$

Starting with $\mathcal{L}[\sin^2 t](s) = \mathcal{L}[\frac{1-\cos(2t)}{2}](s) = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4}$ and using twice the formula

$$\mathcal{L}[f(t)/t](s) = \int_s^\infty F(\sigma)d\sigma,$$

we see that the Laplace transform of the function $S(x) := \frac{\sin^2 x}{x^2}$ is given by

$$\frac{s\log s}{2} - \frac{s\log(s^2+4)}{4} + \underbrace{\arctan(\frac{2}{s})}_{=\frac{\pi}{2}-\arctan\frac{s}{2}}\ .$$

Hence

$$I = \left[\frac{s\pi}{2} + \frac{s^2\log s}{4} - \frac{s^2+4}{8}\log(s^2+4) + \frac{1}{2} - s\arctan\frac{s}{2} + \log\left(\frac{s^2}{4}+1\right)\right]_0^2 = \frac{\pi}{2}.$$

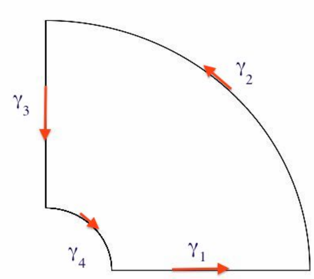
(a2) Note that

$$I = \frac{1}{2}\int_0^\infty (1-e^{-2x})\frac{1-\cos(2x)}{x^3}dx.$$

So we need to apply the residue theorem to the function

$$f(z) := \frac{(1-e^{-2z})(1-e^{2iz})}{z^3}$$

and the contour



$$\Gamma(r,R)=\gamma_1(r,R)\oplus\gamma_2(R)\oplus\gamma_3(r,R)\oplus\gamma_4(r,R),$$

where $\gamma_1(n)=[r,R]$, $\gamma_2(R)=Re^{it}, 0\le t\le\pi/2$, $\gamma_3^-(r)=it$, $r\le t\le R$, $\gamma_4^-(r,R)=re^{it}$, $0\le t\le\pi/2$.

Observe that $\int_{\Gamma(r,R)} f(z)dz=0$. Since

$$\begin{aligned}\int_{\gamma_3^-(r,R)} f(z)dz &= \int_r^R f(it)idt=\int_r^R \frac{(1-e^{-2it})(1-e^{-2t})}{(-i)t^3}idt\\ &= -\overline{\int_{\gamma_1(r,R)} f(z)dz},\end{aligned}$$

we have

$$\int_{\gamma_1(r,R)} f(z)dz+\int_{\gamma_3(r,R)} f(z)dz=2\int_{\gamma_1(r,R)}\mathrm{Re}\, f(z)dz=4\int_r^R(1-e^{-2x})\frac{\sin^2 x}{x^3}dx.$$

Now $I(R):=\displaystyle\int_{\gamma_2(R)} f(z)dz\to 0$ as $R\to\infty$. In fact,

$$|I(R)| = \left|\int_0^{\pi/2} f(Re^{it})iRe^{it}dt\right|\le\int_0^{\pi/2}\frac{(1+e^{-2R\cos t})(1+e^{-2R\sin t})}{R^2}dt\le\frac{4\,\frac{\pi}{2}}{R^2}.$$

Moreover, $J(r):=\displaystyle\int_{\gamma_4(r)} f(z)dz\to-2\pi$ as $r\to 0$. In fact, since the Taylor expansion of $zf(z)=\frac{(1-e^{-2z})(1-e^{2iz})}{z^2}$ at the origin equals

$$-4i+(4+4i)z-4z^2+\mathcal{O}(z^3),$$

we have

$$J(r) = -\int_0^{\pi/2}\frac{(1-e^{-2re^{it}})(1-e^{2ire^{it}})}{r^2e^{3it}}ie^{it}dt \overset{r\to 0}{\longrightarrow} -(-4i)i\int_0^{\pi/2}1\,dt=-2\pi$$

(note that $\lim_{r\to 0}\int=\int\lim_{r\to 0}$ since the integrand is uniformly continuous for $(r,t)\in\,]0,1]\times[0,2\pi]$.)

Now

$$\begin{aligned}0 &= \lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma(r,R)} f(z)dz\\ &= \lim_{\substack{R\to\infty\\ r\to 0}}\left(\int_{\gamma_1(r,R)} f(z)dz+\int_{\gamma_3(r,R)} f(z)dz\right)+\lim_{R\to\infty}\int_{\gamma_2(R)} f(z)dz+\lim_{r\to 0}\int_{\gamma_4(r)} f(z)dz\\ &= 4I+0-2\pi.\end{aligned}$$

Consequently, $I=\pi/2$.

(b) We first observe that due to the majorant $1/x^2$ for $|x|\ge 1$, the integral exists as Lebesgue integal as well as improper Riemann integral (note the discontinuity points at $-\pi/2+k\pi, k\in\mathbb{Z}$).

Since $f(x):=\cos^2(\tan x)$ is π-periodic and even, we may use the Lobashevski integral formula

$$\int_0^\infty f(x)\frac{\sin^2 x}{x^2}dx=\int_0^{\pi/2} f(x)dx$$

(see e.g. [86] [24]) to conclude that

$$\begin{aligned}\int_{-\infty}^\infty\cos^2(\tan x)\frac{\sin^2 x}{x^2}dx &= 2\int_0^{\pi/2}\cos^2(\tan x)dx\\ &\underset{dx=\frac{ds}{1+s^2}}{\overset{s:=\tan x}{=}} 2\int_{s=0}^\infty\frac{\cos^2 s}{1+s^2}ds=\int_{s=0}^\infty\frac{1+\cos(2s)}{1+s^2}ds\\ &= \arctan s\Big|_0^\infty+\frac{1}{2}\int_{-\infty}^\infty\frac{\cos(2s)}{1+s^2}ds\\ &= \frac{\pi}{2}+\pi i\,\mathrm{Res}\left(\frac{e^{2iz}}{1+z^2};z=i\right)\\ &= \frac{\pi}{2}+\pi i\frac{e^{-2}}{2i}=\frac{\pi}{2}\left(1+\frac{1}{e^2}\right).\end{aligned}$$

[24] The proof there is also valid for the case of all even Riemann-integrable π-periodic functions.

Here we have applied the residue theorem to evaluate the classical integral $\int_{-\infty}^{\infty} \frac{\cos(2s)}{1+s^2} ds$, which used to be an exercise in many complex analysis courses (see e.g. [87, p. 210]).

Now we have the following identities:

$$\int_{-\infty}^{\infty} \sin^2(\tan x) \frac{\cos^2 x}{x^2} dx - \int_{-\infty}^{\infty} \cos^2(\tan x) \frac{\sin^2 x}{x^2} dx$$
$$= \int_{-\infty}^{\infty} \left(\sin^2(\tan x) \frac{\cos^2 x}{x^2} + \sin^2(\tan x) \frac{\sin^2 x}{x^2} \right) dx - \int_{-\infty}^{\infty} \left(\sin^2(\tan x) \frac{\sin^2 x}{x^2} + \cos^2(\tan x) \frac{\sin^2 x}{x^2} \right) dx$$
$$= \int_{-\infty}^{\infty} \frac{\sin^2(\tan x)}{x^2} dx - \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx.$$

The latter difference, though, vanishes. In fact, since $\sum \int = \int \sum$ (note that all terms are positive),we obtain in view of the classical formula (see e.g. [86]),

$$\frac{1}{\sin^2 x} = \sum_{k=-\infty}^{\infty} \frac{1}{(k\pi + x)^2}$$

that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin^2(\tan x)}{x^2} dx &= \sum_{k=-\infty}^{\infty} \int_{-\frac{\pi}{2}+k\pi}^{\frac{\pi}{2}+k\pi} \frac{\sin^2(\tan x)}{x^2} dx \\
&\overset{x=-\frac{\pi}{2}+k\pi+u}{=} \sum_{k=-\infty}^{\infty} \int_0^{\pi} \frac{\sin^2(\tan(u-\frac{\pi}{2}))}{(-\frac{\pi}{2}+k\pi+u)^2} du \\
&= \int_0^{\pi} \sin^2(-\cot u) \sum_{k=-\infty}^{\infty} \frac{1}{(k\pi + (u-\frac{\pi}{2}))^2} \, du \\
&= \int_0^{\pi} \sin^2(-\cot u) \frac{1}{\sin^2(u-\frac{\pi}{2})} du = \int_0^{\pi} \frac{\sin^2(-\cot u)}{\cos^2 u} \, du \\
&\underset{du=\sin^2 u ds}{\overset{-\cot u = s}{=}} \int_{-\infty}^{\infty} \frac{\sin^2 s}{\cos^2 u} \, \sin^2 u \, ds = \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} \, ds.
\end{aligned}$$

Amazing! Hence the second formula

$$\int_{-\infty}^{\infty} \frac{\sin^2(\tan x) \, \cos^2 x}{x^2} dx = \frac{\pi}{2}\left(1 + \frac{1}{e^2}\right)$$

holds, too.

108.C (George Stoica)

Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ with the property that, for any $a > 0$, the function $x \mapsto f(x) f(a - x)$ is constant on the interval $[0, a]$.

Solution to problem 108.C, Math. Gazette 108, Issue 571 (2024), p. 167

by Raymond Mortini

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We claim that all solutions f continuous on $[0, \infty[$ are given by

$$\boxed{\lambda\, e^{\beta x}, \text{ where } \lambda, \beta \in \mathbb{R}},$$

or equivalently

$$\boxed{f \equiv 0 \text{ or } \pm e^{\alpha+\beta x}, \text{ where } \alpha, \beta \in \mathbb{R}.}$$

To verify this, we first note that, trivially, all these functions are solutions, since for $0 \leq a \leq x$

$$\lambda e^{\beta x} \cdot \lambda e^{\beta(a-x)} = \lambda^2 e^{\beta a},$$

Conversely, let f be a solution to the problem, that is, $f(x) \cdot f(a-x) =: c(a)$ is independent of x whenever $0 \leq x \leq a$, and this for any $a > 0$.

• Let $x = 0$. Then $f(0)f(a) = c(a)$. Now let $x = a/2$. Then $f(a/2)f(a/2) = c(a)$, too. Hence, for every $a > 0$,

$$f(a/2)^2 = f(0)f(a). \tag{125}$$

Thus f has everywhere the sign of $f(0)$ or $f \equiv 0$ if $f(0) = 0$. Since f is a solution if and only λf is a solution, we may assume wlog that $f(0) = 1$. Hence, by (125), $f > 0$ on $[0, \infty[$.

• Let $F(x) := \log f(x)$. Then, for any $a > 0$, F satisfies on $[0, a]$ the functional equation

$$F(x) + F(a-x) = C(a) \tag{126}$$

for some (continuous) function $C(a)$.

Now, by (125), $C(a) = 2F(a/2) = F(0) + F(a) = F(a)$. In particular, for $b = a/2$,

$$2F(b) = F(2b) \tag{127}$$

Moreover, if we take $x = a/3$,

$$F(a/3) + F((2/3)a) = C(a) = F(a).$$

Thus, for $b = a/3$,

$$0 = F(b) + F(2b) - F(3b) = F(b) + 2F(b) - F(3b),$$

and so

$$3F(b) = F(3b). \tag{128}$$

We conclude that for every $x > 0$, and $n, k \in \mathbb{N} := \{0, 1, 2, \dots\}$,

$$F\left(\frac{2^n}{3^k}x\right) = \frac{2^n}{3^k}F(x),$$

since, by induction, $F(2^n x) = 2^n F(x)$ and $F(x/3^k) = \frac{1}{3^k}F(x)$. Now replace x by $2^n x$.

Since the set $\{\frac{2^n}{3^k} : n, k \in \mathbb{N}\}$ is dense in $[0, \infty[$ (see e.g. [16, p. 1879]), continuity of F in $[0, \infty[$ yields that $F(\mu x) = \mu F(x)$ for every $\mu > 0$ and $x > 0$. Therefore, for $x = 1$, $F(\mu) = \mu F(1)$. Consequently, with $\beta := F(1)$,

$$f(\mu) = e^{\beta\mu}.$$

The following list is ordered according to the "chronological" appearance in this text.

References

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