Möbius Estimates

11684 [2013, 76]. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France, and Rudolf Rupp, Georg-Simon-Ohm Hochschule Nürnberg, Nuremberg, Germany.* For complex a and z, let

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad \rho(a, z) = \frac{|a - z|}{|1 - \overline{a}z|}.$$

(a) Show that whenever $-1 < a, b < 1$,

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\rho(a, b)$$

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2.$$

(b) For complex α, β with $|\alpha| = |\beta| = 1$, let

$$m(z) = m_{a,b,\alpha,\beta}(z) = |\alpha\phi_a(z) - \beta\phi_b(z)|.$$

Determine the maximum and minimum, taken over z with $|z| = 1$, of $m(z)$.

Solution by the proposers.
(b) Observe that ϕ_a is its own inverse. Let $c = (b - a)/(1 - a\overline{b})$ and let

$$\lambda = -\frac{1 - a\overline{b}}{1 - \overline{a}b}.$$

Since ϕ_b is a bijection of the unit circle onto itself,

$$\max_{|z|=1} \left|\alpha\phi_a(z) - \beta\phi_b(z)\right| = \max_{|z|=1} \left|\alpha\overline{\beta}\phi_a\big(\phi_b(z)\big) - z\right| = \max_{|z|=1} \left|\alpha\overline{\beta}\lambda\phi_c(z) - z\right|.$$

The same identities hold when the maximum is replaced by the minimum. Put $\gamma = \alpha\overline{\beta}\lambda$, and let $-\pi < \arg\gamma \le \pi$. For $|z| = 1$, let $H(z) = |\gamma\phi_c(z) - z|$. We have

$$H(z) = \left|\gamma\,\frac{z(c\overline{z} - 1)}{1 - \overline{c}z} - z\right| = \left|\gamma\,\frac{1 - c\overline{z}}{1 - \overline{c}z} - 1\right| = \left|\gamma\,\frac{w}{\overline{w}} + 1\right|,$$

where $w = 1 - c\overline{z} = 1 - c/z$. As z moves around the unit circle, w moves around the circle $|w - 1| = |c|$. Write $w = |w|e^{i\theta}$. Note that θ varies on the interval $[-\theta_m, \theta_m]$, where $|\theta_m| < \pi/2$ and $\sin\theta_m = |c| = \rho(a, b)$. Now

$$H(z) = \left|\gamma e^{2i\theta} + 1\right| = 2\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right|.$$

Hence

$$\max_{|z|=1} H(z) = 2\max\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a, b)\right\} \qquad (*)$$

and

$$\min_{|z|=1} H(z) = 2\min\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a, b)\right\}.$$

(a) Specialize $(*)$ by taking $a, b \in (-1, 1)$ and $\alpha = \beta = 1$, so that $\gamma = -1$. By the maximum principle, the maximum on the disk is achieved on the boundary, so

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\max\left\{|\sin\theta| : |\theta| \le \arcsin\rho(a, b)\right\} = 2\rho(a, b).$$

For the other part of **(a)**, instead specialize $(*)$ by taking $a, b \in (-1, 1)$ and $\alpha = 1$, $\beta = -1$, so that $\gamma = 1$. This gives

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2\max\left\{|\cos\theta| : |\theta| \le \arcsin\rho(a, b)\right\} = 2.$$

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