13.9 The Pochhammer curve

In this final section we deal with an interesting curve: the Pochhammer contour. First we need the result that the winding number is invariant under homotopies.

Proposition 13.84. *Let $X \subseteq \mathbb{C}$ be a subset. Suppose that for $j = 0, 1$, $f_j : \mathbb{T} \to X$ are two continuous maps that are homotopic in X. Let $\Gamma_j = f_j(\mathbb{T})$ be the associated closed curves. Then, for every $a \in \mathbb{C} \setminus X$,*

$$n(\Gamma_1, a) = n(\Gamma_2, a).$$

Proof. Let $H : \mathbb{T} \times [0,1] \to X$ be a homotopy with $H(\xi, 0) = f_0(\xi)$ and $H(\xi, 1) = f_1(\xi)$ for every $\xi \in \mathbb{T}$. Put $f_t(\xi) = H(\xi, t)$ and $\Gamma_t = f_t(\mathbb{T})$. Then all the Γ_t are closed curves and, for every $a \in \mathbb{C} \setminus X$, the map $t \mapsto n(\Gamma_t, a)$ is continuous. In fact, if $t \to t_0$, then the uniform continuity of H implies that $||f_t - f_{t_0}||_\infty \to 0$. Thus, by Corollary 6.39, $n(\Gamma_t, a) \to n(\Gamma_{t_0}, a)$. As $n(\Gamma_t, a)$ is an integer-valued map, the connectivity of $[0,1]$ implies that $n(\Gamma_t, a)$ is constant. □

Corollary 13.85. *If γ is a null-homotopic closed curve in the domain $D \subseteq \mathbb{C}$, then γ is null-homologous in D.*

Proof. Since for constant maps f, $n(f, a) = 0$ for every $a \in \mathbb{C} \setminus D$, we deduce from Proposition 13.84 that $n(\gamma, a) = 0$, too. Hence γ is null-homologous in D. □

Corollary 12.11 and Theorem 13.83 show that for $D = \mathbb{C} \setminus \{0\}$ and each simply connected domain $D \subseteq \mathbb{C}$ a closed curve Γ in D is null-homologous if and only if Γ is null-homotopic in D. The following example of the Pochhammer contour shows that this no longer holds for $D = \mathbb{C} \setminus \{0, 2\}$ [311].

Example 13.86. *Let γ_1 be the unit circle $e^{2\pi i s}$, $0 \leq s \leq 1$, and γ_2 the circle $2 - e^{2\pi i s}$, $0 \leq s \leq 1$. Then the Pochhammer curve γ_0 in $D = \mathbb{C} \setminus \{0, 2\}$, given by*

$$\gamma_0 = \gamma_1 \oplus \gamma_2 \oplus \gamma_1^- \oplus \gamma_2^-,$$

is null-homologous, but not null-homotopic in D. Equivalently, $\gamma_1 \oplus \gamma_2$ is not D-homotopic to $\gamma_2 \oplus \gamma_1$.

See figure 75, where $\Gamma_1 = \gamma_1([0,1]), \Gamma_2 = \gamma_2([0,1]), \Gamma_3 = \gamma_3([0,1]) := \gamma_1^-([0,1])$, and $\Gamma_4 = \gamma_4([0,1]) := \gamma_2^-([0,1])$. The Pochhammer contour γ is a curve which surrounds counterclockwise the point 0, then counterclockwise the point 2, then clockwise the point 0, and finally clockwise the point 2 (see figure 75).

Proof. Let $\Gamma_0 = \gamma_0([0,4])$, where we use the parametrization

$$\gamma_0(s) = \begin{cases} \gamma_1(s) & \text{if } 0 \leq s \leq 1 \\ \gamma_2(s-1) & \text{if } 1 \leq s \leq 2 \\ \gamma_3(s-2) & \text{if } 2 \leq s \leq 3 \\ \gamma_4(s-3) & \text{if } 3 \leq s \leq 4. \end{cases}$$

[311] We thank B. Burckel for several E-mail exchanges on this curve.

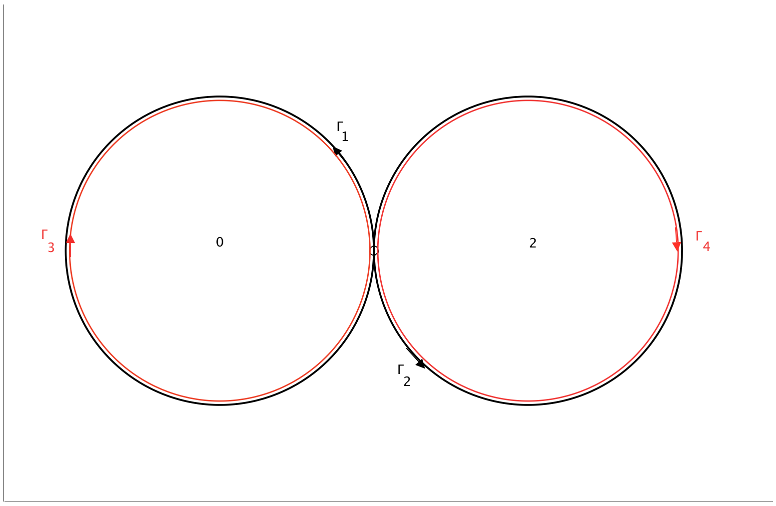

Figure 75: The Pochhammer curve: null-homologous but not null-homotopic in the twice punctured domain $\mathbb{C} \setminus \{0, 2\}$.

Since $n(\gamma_0, 0) = n(\gamma_0, 2) = 0$, we see that $n(\gamma_0, a) = 0$ for every $a \in \mathbb{C} \setminus \Gamma_0$, and so γ_0 is null-homologous in D. Now suppose that γ_0 is null-homotopic in $D = \mathbb{C} \setminus \{0, 2\}$. Then, by Corollary 13.64(13.64) there exists a $(1, 1)$-homotopy $H : [0, 4] \times [0, 1] \to D$ of closed curves such that $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = 1 \in D$ for every $s \in [0, 4]$.

Due to Corollary 7.343 (7.343) [312], $H = e^h$ for some $h \in C([0, 4] \times [0, 1], \mathbb{C})$ (note that by Example 7.338, the convex set $K = [0, 4] \times [0, 1]$ is contractible).

Let log denote the principal branch of the logarithm on $\mathbb{C}\setminus\,]-\infty, 0]$. Since 2 is not in the image of H, $\log 2 + 2k\pi i$ does not belong to the image of h.

Now for $(s, t) = (0, 0)$ we have $1 = \gamma_0(0) = H(0, 0) = e^{h(0,0)}$. So we may assume that $h(0, 0) = 0$.

Claim 1 The function $h(\cdot, 0)$ necessarily is given by

$$h(s, 0) = \begin{cases} 2\pi i s & \text{if } 0 \leq s \leq 1 \\ \log(2 - e^{2\pi i s}) + 2\pi i & \text{if } 1 \leq s \leq 2 \\ 2\pi i(3 - s) & \text{if } 2 \leq s \leq 3 \\ \log\left(2 - e^{2\pi i(4-s)}\right) & \text{if } 3 \leq s \leq 4. \end{cases}$$

(see figure 76).

For the proof, we distinguish four cases.

Case 1. If $0 \leq s \leq 1$, $H(s, 0) = e^{h(s,0)}$ and $H(s, 0) = \gamma_0(s) = e^{2\pi i s}$ imply that $h(s, 0) = 2\pi i s + 2k\pi i$ for some k independent of $s \in [0, 1]$. Since $h(0, 0) = 0$ we conclude that $h(s, 0) = 2\pi i s$ for $s \in [0, 1]$.

Case 2. Now let $s \in [1, 2]$ and put $L(s) = \log(2 - e^{2\pi i s}) + 2\pi i$. Note that this is well defined, since all values of $2 - e^{2\pi i s}$ are contained in the right-half plane.

Now $H(s, 0) = e^{h(s,0)}$, $H(s, 0) = \gamma_0(s) = 2 - e^{2\pi i s}$ and $e^{L(s)} = 2 - e^{2\pi i s}$ imply that $h(s, 0) = L(s) + 2k\pi i$ for some $k \in \mathbb{Z}$. Now, by the first case, $h(1, 0) = 2\pi i$. Hence $k = 0$.

[312] One may also use Borsuk's Theorem 12.5, a more advanced tool.

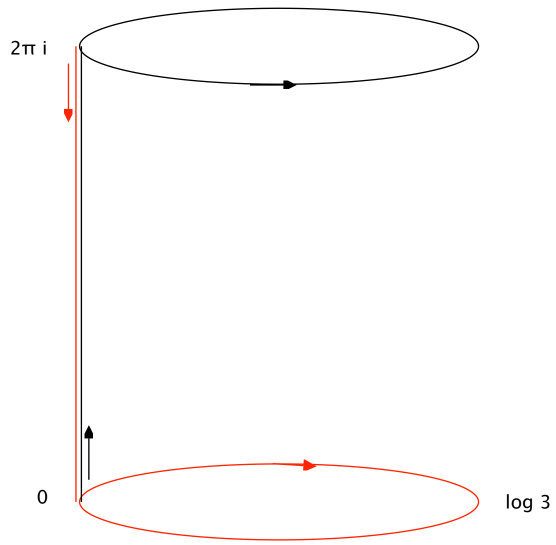

Figure 76: The logarithm of the Pochhammer curve in the domain $\mathbb{C}\setminus\{\log 2+2k\pi i : k\in\mathbb{Z}\}$.

Case 3. If $s\in[2,3]$, then $\gamma_0(s)=\gamma_3(s-2)=\gamma_1^-(s-2)$. Now a parametrization of γ_1^- is given by $\theta:[0,1]\mapsto e^{2\pi i(1-\theta)}$. So $\gamma_0(s)=e^{2\pi i(3-s)}$. Now $\gamma_0(s)=H(s,0)=e^{h(s,0)}$ implies $h(s,0)=2\pi i(3-s)+2k\pi i$ for some $k\in\mathbb{Z}$. But by Case 2, $h(2,0)=2\pi i$. Thus $k=0$.

Case 4. Let $s\in[3,4]$, and put $\tilde{L}(s)=\log(2-e^{2\pi i(4-s)})$. Note that this is well defined, since all values of $2-e^{2\pi i(4-s)}$ are contained in the right-half plane.

Next note that $\gamma_0(s)=\gamma_4(s-3)=\gamma_2^-(s-3)$. Now a parametrization of γ_2^- is given by $\theta:[0,1]\mapsto 2-e^{2\pi i(1-\theta)}$. Hence $\gamma_0(s)=2-e^{2\pi i(4-s)}$. Now $\gamma_0(s)=H(s,0)=e^{h(s,0)}$ implies $h(s,0)=\tilde{L}(s)+2k\pi i$ for some $k\in\mathbb{Z}$. By Case 3, $h(3,0)=0$. As $\tilde{L}(3)=0$, we deduce again that $k=0$. This proves Claim 1.

Now we use that H is a $(1,1)$-loophomotopy. This implies, in particular, that $e^{h(0,t)}=H(0,t)=1$ for every $t\in[0,1]$. Hence $h(0,t)=2k\pi i$ for some $k\in\mathbb{Z}$. Since $h(0,0)=0$, we deduce that $k=0$ and so $e^{h(s,1)}=1$ implies that $h(s,1)=0$. We conclude that h is a $(0,0)$-homotopy in $\mathbb{C}\setminus\{\log 2+2k\pi i : k\in\mathbb{Z}\}$ between the curve $h_0(s):=h(s,0)$ and the constant 0. As the winding numbers for curves are invariant under loophomotopies (Proposition 13.84), $n(h_0,\log 2)$ would coincide with $n(\delta_0,\log 2)=0$, an obvious contradiction to the fact that $n(h_0,\log 2)=-1$ (see figure 76).

□