

A constructive proof of the Nullstellensatz for subalgebras of $A(K)$

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Abstract

We present a simple constructive proof of the Nullstellensatz for subalgebras of $A(K)$.

Introduction

Let k be any field and let $R = k[x_1, \dots, x_n]$ denote the ring of all polynomials in variables x_1, \dots, x_n . One version of Hilbert's famous Nullstellensatz tells us that for any polynomials $P_1, \dots, P_m \in R$ which do not have a common zero, there exist polynomials $Q_1, \dots, Q_m \in R$ such that $\sum_{i=1}^m Q_i P_i$ has no zeros at all. (For a simple proof see P. Smith [7].) If additionally k is assumed to be algebraically closed, then the polynomials Q_1, \dots, Q_m can be chosen so that $\sum_{i=1}^m Q_i P_i$ is invertible in R , hence is a constant.

It is the aim of this short note to prove in an elementary and constructive way a similar result for subalgebras of $A(K)$, where $A(K)$ is the Banach algebra of all continuous complex valued functions on a compact set K of the plane \mathbb{C} which are analytic in the interior K^0 of K .

It is a well known result of R. Arens [3] that, if f_1, \dots, f_m are functions in $A(K)$ with no common zero, then there exists $g_1, \dots, g_m \in A(K)$ such that $\sum_{i=1}^m g_i f_i = 1$, hence the Nullstellensatz holds in $A(K)$. His proof, however, far from being elementary, cannot be considered as constructive, since it depends on Gelfand theory. Technically rather involved is also Carleson's constructive proof [1] of the Nullstellensatz for the disk algebra $A(\overline{\mathbb{D}})$; moreover, it is not elementary since it uses some form of the Hahn-Banach

extension theorem and Riesz's representation theorem for bounded linear functionals on $C(\overline{D})$. The same can be said about P.J. Cohen's proof [2]. The first really elementary proof of the Nullstellensatz for various subalgebras of $A(\overline{D})$, which includes Wiener's algebra W^+ of absolutely convergent Taylor series on \overline{D} , was given by von Renteln [6].

To achieve our goal mentioned above, we give an appropriate modification of v.Renteln's methods. One of the main features — if viewed with respect to Arens', Carleson's and Cohen's results — will be the fact that our algebras may not be topologically complete.

Remarks. 1. The methods of this paper are used by the second author to determine the Bass stable rank of all inversionally closed algebras mentioned below.

2. Certain refinements of v.Renteln's techniques may be applied in the context of a uniform algebra A to study the relationship between the algebraic properties of A and the geometric properties of the underlying Banach space A (see [5]).

Main results

Let $C(\partial K)$ resp. $R(\partial K)$ denote the Banach algebras of all continuous complex valued functions on ∂K , resp. of all those functions in $C(\partial K)$ which can be approximated uniformly by rational functions with poles off ∂K . Let $\|f\|_K = \sup_{z \in K} |f(z)|$.

A subalgebra A of $A(K)$ is called *stable* if A contains the polynomials and if $\frac{f-f(z_0)}{z-z_0} \in A$ whenever $f \in A$ and $z_0 \in K^0$. Many of the well known examples of subalgebras of $A(K)$ have this property, e.g.,

$$W^+ = \left\{ \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n| < \infty \right\};$$

$$A^\infty(K) = \{f \in C(K) : f \text{ analytic in } K^0, \forall n \exists g \in C(K) : g|_{K^0} = f^{(n)}|_{K^0}\};$$

$$\Lambda_\alpha(K) = \{f \in C(K) : f \text{ analytic in } K^0 \text{ and satisfies a Hölder-Lipschitz condition on } K \text{ of order } \alpha\}$$

Theorem. Let K be a compact subset of the plane such that $R(\partial K) = C(\partial K)$ and let A be a stable subalgebra of $A(K)$. Then the following Nullstellensatz holds in A :

If f_1, \dots, f_N are functions in A that have no common zero in K , then there exist functions $g_1, \dots, g_N \in A$ such that

$$\sum_{i=1}^N g_i f_i \text{ has no zero in } K.$$

Remark 3. It should be noted that our assumption on K is rather mild, since by the Hartogs-Rosenthal theorem $R(\partial K) = C(\partial K)$ if ∂K has two-dimensional Lebesgue measure zero. By the way, a constructive proof of this theorem is known (see [4]).

Proof. Let f_1, \dots, f_N be functions in A with no common zero. Then $\sum_{i=1}^N |f_i| \geq \delta > 0$ on K . Let $g_k = \bar{f}_k / (\sum_{i=1}^N |f_i|^2)$. Of course we can assume that no generator f_i is the zero function. Then $g_k \in C(\partial K)$ and $\sum g_k f_k = 1$ on ∂K . Since $C(\partial K) = R(\partial K)$, we can choose rational functions r_k with poles off ∂K such that

$$\|g_k - r_k\|_{\partial K} \leq \frac{1}{2N} \|f_k\|_{\partial K}^{-1}.$$

Then we obtain the following estimates on ∂K :

$$\left| \sum_1^N r_k f_k \right| \geq \left| \sum_1^N g_k f_k \right| - \sum_1^N \|r_k - g_k\|_{\partial K} |f_k| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Let $r_k = p_k/q_k$, where p_k and q_k are polynomials such that q_k has no zeros on ∂K . If we define $q = \prod_1^N q_k$, then $f := \sum_1^N (q r_k) f_k$ belongs to the ideal $I = (f_1, \dots, f_N)$ generated by the functions f_k . Moreover, the function f has no zeros on ∂K . The identity theorem for analytic functions now implies that f has only finitely many zeros in K^0 (note that K^0 may be disconnected). Let z_1, \dots, z_n be these zeros, incl. multiplicities.

Since $\sum |f_i| \geq \delta > 0$ on K , there exists for every z_j ($j = 1, \dots, n$) a generator $h_j \in \{f_1, \dots, f_N\}$ such that $h_j(z_j) \neq 0$.

By using the stability of A and n -times successively the formulae

$$\frac{f(z)}{z - z_j} = \frac{1}{h_j(z_j)} \left[\frac{f(z)}{z - z_j} \cdot h_j(z) - \frac{h_j(z) - h_j(z_j)}{z - z_j} \cdot f(z) \right] \in I$$

we can get rid of all the zeros of f ; the result is a function $h \in I$ without zeros in K . \square

In order to obtain the sharper version of Hilbert's Nullstellensatz, namely, the existence of polynomials Q_i such that $\sum_{i=1}^m Q_i P_i$ is invertible in R , hence a constant, it is required that the underlying field k is algebraically closed. In our case of subalgebras of $A(K)$ we have to assume that A is "inversionally closed" defined as follows.

Definition. A subalgebra A of $A(K)$ is called *inversionally closed* if a function $f \in A$ is invertible in A if and only if it has no zero in K .

Obviously, every invertible function in A has no zeros in K . What we require here, is the converse. But on the other hand it is easy to see that this requirement is also necessary in order that the following version of the Nullstellensatz holds.

Corollary 1 *Let A be an inversionally closed, stable subalgebra of $A(K)$ with $R(\partial K) = C(\partial K)$. Then the ideal generated by the functions $f_1, \dots, f_N \in A$ equals the whole algebra if and only if the functions f_1, \dots, f_N have no common zero in K .*

Corollary 2 *Under the assumptions on A of Corollary 1 an ideal I in A is maximal if and only if it has the form*

$$I = M(z_0) = \{f \in A : f(z_0) = 0\}$$

for some $z_0 \in K$.

The proofs are straightforward.

References

1. L. Carleson: On bounded analytic functions and closure problems. Ark. Mat. 2 (1954), 283-291.
2. P.J. Cohen: A note on constructive methods in Banach algebras. Proc. Amer. Math. Soc. 12 (1961), 159-163.
3. T.W. Gamelin: Uniform algebras. Chelsea Publishing Company, New York, 1984.
4. F. Hartogs, A. Rosenthal: "Über Folgen analytischer Funktionen. Math. Ann. 104 (1931), 606-610.
5. R. Mortini, M. von Renteln: Strong extreme points and ideals in uniform algebras. To appear in Archiv Math.
6. M. von Renteln: A simple constructive proof of an analogue of the Corona theorem. Proc. Amer. Math. Soc. 83 (1981), 299-303.
7. P. Smith: Solution to problem No. 6508. Amer. Math. Monthly (1987), 697.

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