When is the product of two planar harmonic mappings harmonic?

Raymond Mortini

ABSTRACT. We determine all complex-valued harmonic functions u and v defined on a planar domain for which uv, respectively $u^2 - v^2$ is harmonic

Key words and phrases. Harmonic mappings

2000 Mathematics Subject Classification. Primary 31A05. Secondary 30A99.

RESUMEN. Se determinan todas las funciones armónicas de valor complejo u y v definidas en un dominio plano para las cuales uv, respectivamente $u^2 - v^2$, es armónica

In a recent problem in the *American Mathematical Monthly* [Ha] W. HAYMAN asked to determine when the product of two non-constant, real valued harmonic functions u and v in a domain $\Omega \subseteq \mathbb{C}$ is harmonic. This appears also as exercice 1 in ([Ru], p. 249). The answer was the following: If uv is harmonic in Ω, then u has an harmonic conjugate $\tilde{u}$

on Ω and there are constants $\alpha, \beta \in \mathbb{R}$ such that

$$v = \alpha \tilde{u} + \beta.$$

For a proof see [AMM]. We will give here a description of those *complex valued* harmonic functions u and v for which uv is harmonic. Our tools will be the $\partial \overline{\partial}$–calculus.

It is well known [FL] that a function $u : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$–differentiable at z_0 if and only if u can be represented in the form

$$u(z) = u(z_0) + A(z - z_0) + B(\overline{z} - \overline{z_0}) + \mathcal{O}(z - z_0),$$

where $\mathcal{O}(w)$ is a function satisfying $\lim_{w \to 0} \mathcal{O}(w)/w = 0$. The complex numbers A and B are called the *Wirtinger derivatives* of u at z_0 and are denoted by $A = \dfrac{d}{dz}u(z_0) = \partial u(z_0) = u_z(z_0)$ and $B = \dfrac{d}{d\overline{z}}u(z_0) = \overline{\partial}u(z_0) = u_{\overline{z}}(z_0)$. If u_x and u_y are the usual partial derivatives of the function $(x, y) \mapsto u(x + iy)$, where $z = x + iy$, then $u_z = \dfrac{1}{2}(u_x - iu_y)$ and $u_{\overline{z}} = \dfrac{1}{2}(u_x + iu_y)$.

By the Cauchy-Riemann equations, a differentiable function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $f_{\overline{z}} \equiv 0$. In that case, its derivative f' coincides with f_z , respectively with f_x. The Laplacian $\Delta u = u_{xx} + u_{yy}$ for a C^2 function $u : \mathbb{C} \to \mathbb{C}$ can be written as $4u_{z\overline{z}}$. A complex-valued function $u : \mathbb{C} \to \mathbb{C}$ is harmonic if $\Delta u = 0$.

We shall now give an answer to the problem when the product of two complex valued harmonic functions, also called harmonic mappings, is harmonic. For the solution we need to define what we understand by normalized $\mathbb{C}$–harmonic conjugates for complex valued harmonic functions. So let $u = a + ib$ be harmonic in a domain $D \subseteq \mathbb{C}$. Every harmonic mapping v in D making $u + iv$ holomorphic, is called a $\mathbb{C}$–harmonic conjugate of u. Of course it need not exist. A sufficient condition for the existence is that D is simply connected. In fact, in that case, just choose $v = \tilde{a} + i\tilde{b}$, where $\tilde{a}$ and $\tilde{b}$ are (real-valued) harmonic conjugates of a and b respectively.

The normalized $\mathbb{C}$–harmonic conjugate of $u = a + ib$ is, by definition, that harmonic mapping $\tilde{u}$ given by $\tilde{u} = \tilde{a} + i\tilde{b}$, whenever the real-valued

harmonic conjugates of a and b exist. We tacitely assume that $\tilde{u}(z_0) = 0$ for a fixed point z_0. It is clear that if v_1 and v_2 are two $\mathbb{C}$–harmonic conjugates of u, then $v_2 - v_1$ is holomorphic. Thus, in the case the normalized $\mathbb{C}$–harmonic conjugate $\tilde{u}$ of u exists, any $\mathbb{C}$–harmonic conjugate v of u is given by $v = \tilde{u} + h$ for some holomorphic function h. For example Im $z + f(z)$ will be a $\mathbb{C}$–harmonic conjugate of $u(z) = $ Re z for every $f \in H(\mathbb{C})$, the normalized $\mathbb{C}$-harmonic conjugate will be Im z. Let us point out that for non-simply connected domains, the existence of a $\mathbb{C}$–harmonic conjugate does not imply the existence of the normalized $\mathbb{C}$–harmonic conjugate. In fact, let $u(z) = z + \log|z|$ and $v(z) = i \log|z|$. Then v is a $\mathbb{C}$–harmonic conjugate of u in $\mathbb{C} \smallsetminus \{0\}$, but the real part of u does not have a harmonic conjugate on $\mathbb{C} \smallsetminus \{0\}$.

Let us recall that if $u = a + ib$ is a harmonic mapping in a simply connected domain D, then there exists two holomorphic functions h and k such that $u = h + \overline{k}$. In fact, observe that $\tilde{\tilde{b}} = -b$ and let $A = (a - \tilde{b})/2$ and $B = (a + \tilde{b})/2$; $h = A + i\tilde{A}$ and $k = B + i\tilde{B}$. Then $u = h + \overline{k}$.

Proposition. *Let u and v be two non constant complex valued harmonic functions on a domain $\Omega \subseteq \mathbb{C}$. Suppose that uv is harmonic. Then either both u and v are holomorphic or antiholomorphic in Ω or u has a normalized $\mathbb{C}$–harmonic conjugate $\tilde{u}$ on Ω such there are constants $\alpha, \beta \in \mathbb{C}$ with*

$$v = \alpha\tilde{u} + \beta. \tag{1}$$

Conversely, let u be a harmonic mapping, $\tilde{u}$ a normalized $\mathbb{C}$–harmonic conjugate of u and let v be as in (1). Then v and uv are harmonic mappings.

Proof. Let $f = u_z$, $F = \overline{u}_z$, $g = v_z$ and $G = \overline{v}_z$. By harmonicity, $f_{\bar{z}} = F_{\bar{z}} = g_{\bar{z}} = G_{\bar{z}} = 0$. Hence f, F, g and G are holomorphic in Ω. Moreover, $u_{\bar{z}} = \overline{(\overline{u}_z)} = \overline{F}$ and $v_{\bar{z}} = \overline{G}$. Since $\Delta = 4 \dfrac{d}{dz} \dfrac{d}{d\bar{z}}$, we get:

$$0 = \frac{1}{4}\Delta(uv) = f\overline{G} + \overline{F}g. \tag{2}$$

• *Case 1.* Suppose that $fFgG \not\equiv 0$. Then on $\Omega \smallsetminus Z(gG)$ we obtain that

$$\frac{f}{g} = -\frac{\overline{F}}{\overline{G}}.$$

Thus $\dfrac{f}{g}$ is holomorphic, as well as anti–holomorphic on $\Omega \smallsetminus Z(gG)$. Hence $\dfrac{f}{g}$ is identically constant, say $\dfrac{f}{g} \equiv -\lambda \neq 0$. We conclude that

$$\text{i)} \quad u_z = -\lambda v_z \qquad \text{and ii)} \qquad u_{\bar{z}} = \lambda v_{\bar{z}}.$$

By ii), and Riemann's theorem on removable singularities, we see that $u - \lambda v$ is holomorphic on Ω and $u + \lambda v$ is anti–holomorphic on Ω. Let $h = u - \lambda v$ and $k = \overline{u + \lambda v}$. Then

$$u = \frac{h + \overline{k}}{2} \quad \text{and} \quad \lambda v = \frac{-h + \overline{k}}{2} \tag{3}$$

Let $u = a + ib$. By (3),

$$a = \frac{1}{2}\,\mathrm{Re}\,(h + k) \quad \text{and} \quad b = \frac{1}{2}\,\mathrm{Re}\,(i(-h + k)).$$

Hence, in Ω, the real part and imaginary part of u have real-valued harmonic conjugates $\tilde{a}$ and $\tilde{b}$ respectively. Then $\tilde{u} = \tilde{a} + i\tilde{b}$ is a normalized $\mathbb{C}$–harmonic conjugate of u.

Subtracting the holomorphic function $u - \lambda v$ from the holomorphic function $u + i\tilde{u}$ shows that $\lambda v + i\tilde{u}$ is holomorphic. Since we have $\overline{\tilde{u}} = \widetilde{\overline{u}}$, we get that $\overline{u} + i\widetilde{\overline{u}}$ is holomorphic, hence $u - i\tilde{u}$ anti–holomorphic. Subtracting this anti–holomorphic function from the anti–holomorphic function $u + \lambda v$, shows that $\lambda v + i\tilde{u}$ is anti–holomorphic. Therefore $\lambda v + i\tilde{u}$ is identically constant. Thus $v = \alpha\tilde{u} + \beta$ for some $\alpha, \beta \in \mathbb{C}$.

• *Case 2.* Suppose that $F \equiv 0$, or equivalently, that u is holomorphic. Then, by (2), $f\overline{G} \equiv 0$. Since u is not constant and $\overline{F} = u_{\bar{z}} \equiv 0$, we get that $f = u_z \not\equiv 0$ and so $\overline{G} \equiv 0$. Thus v is holomorphic. Similarly, if $f \equiv 0$, or equivalently u antiholomorphic, then we conclude that v is antiholomorphic. An analoguous reasoning for $g \equiv 0$ and $G \equiv 0$ yields the assertion.

Let us now prove the converse. Clearly, v defined as in (1) is harmonic. Let $u = \rho + i\sigma$. Then $\tilde{u} = \tilde{\rho} + i\tilde{\sigma}$. Hence

$$u\tilde{u} = (\rho\tilde{\rho} - \sigma\tilde{\sigma}) + i(\sigma\tilde{\rho} + \rho\tilde{\sigma}). \tag{4}$$

The real part of (4) is the difference of two real valued harmonic functions, hence harmonic; the imaginary part of (4) is the real part of the holomorphic function $-i(\sigma + i\tilde{\sigma})(\rho + i\tilde{\rho})$, hence harmonic. Summing up, we conclude that $u(\alpha\tilde{u} + \beta)$ is harmonic.   ☑

As a special case one gets that for a harmonic mapping u the function u^2 is harmonic if and only if u is holomorphic or anti–holomorphic. Of course we cannot replace in (1) the normalized $\mathbb{C}$–harmonic conjugate by an arbitrary $\mathbb{C}$–harmonic conjugate $\tilde{u} + f$, f holomorphic, as the example $u(z) = \text{Re } z$ and $v(z) = \text{Im } z + z$ shows.

As an indirect corollary we obtain the following result.

Corollary. *Let u and v be two non constant complex valued harmonic functions on a domain $\Omega \subseteq \mathbb{C}$. Then $u^2 - v^2$ is harmonic if and only if either both u and v are holomorphic or anti–holomorphic, or u has a normalized $\mathbb{C}$–harmonic conjugate $\tilde{u}$ on Ω such that there are constants $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $\alpha^2 + \beta^2 = 1$ with*

$$v = \alpha u + \beta\tilde{u} + \gamma. \tag{5}$$

Sketch of a Proof. By our previous proposition, the harmonicity of $u^2 - v^2 = (u - v)(u + v)$ implies that $u - v = c_1\widetilde{(u + v)} + c_2$ and $u + v = c_3\widetilde{(u - v)} + c_4$ for some constants $c_j \in \mathbb{C}$. It is now easy to see that the existence of the normalized $\mathbb{C}$–harmonic conjugates of $u + v$ and $u - v$ implies the existence of the normalized $\mathbb{C}$–harmonic conjugates of u and v and that $\widetilde{u \pm v} = \tilde{u} \pm \tilde{v} + c$. Solving the system of equations above for the variables $(v, \tilde{v})$, yields that $v = \alpha u + \beta\tilde{u} + \gamma$, excepted for some special cases, where u and v are both holomorphic or anti–holomorphic. Hence, $u^2 - (\alpha u + \beta\tilde{u})^2$, and so $[(1 - \alpha^2)/\beta^2]u^2 - \tilde{u}^2$ are harmonic mappings. Since $u^2 - \tilde{u}^2$ is harmonic (note that $(u + i\tilde{u})^2 = (u^2 - \tilde{u}^2) + 2iu\tilde{u}$ is holomorphic), subtracting this function from the previous one, yields that $[(1 - \alpha^2)/\beta^2 - 1]u^2$ is harmonic. Thus either u is holomorphic or anti–holomorphic, or the coefficient of u^2 is zero. Therefore $\alpha^2 + \beta^2 = 1$.

It is easily seen that any function of the form (5) makes $u^2 - v^2$ harmonic. ☑

In the real case, we obtain that $u^2 - v^2$ is harmonic if and only if $v = \cos\theta u + \sin\theta\tilde{u} + r$ for some $\theta, r \in \mathbb{R}$ (u, v nonconstant and harmonic).

Quite generally, one could ask for which polynomials $P(z_1, ..., z_n)$ (or even entire functions) of n complex variables the function $P(u_1, \cdots, u_n)$ is harmonic whenever the u_j are harmonic. The "one-variable" case $g \circ f$, where g and f are locally harmonic mappings in $\mathbb{C}$ has been solved by E. REICH in $[\text{R}_1]$ and $[\text{R}_2]$.

I would in particular be interested to know when the product of two real-valued harmonic functions in a domain $\Omega \subseteq \mathbb{R}^n$ is harmonic.

References

[AMM] AMM *Solution of problem 10651*, Amer. Math. Monthly **106** (1999), 782.

[FL]   WOLFGANG FISHER & INGO LIEB, *Funktionentheorie*, Vieweg-Verlag, 1988, Braunschweig.

[Ha]   WALTER HAYMAN, *Problem 10651*, Amer. Math. Monthly **105**, (1998), 271.

[R_1]   EDGAR REICH, *Local decomposition of harmonic mappings*, Complex Variables **9**, (1987), 263-269.

[R_2]   EDGAR REICH, *The composition of harmonic mappings*, Ann. Acad. Sci. Fenn, Ser. Math. **12** (1987), 47-53.

[Ru]   WALTER RUDIN *Real and Complex Analysis*, McGraw-Hill, 3^d edition, 1986

RAYMOND MORTINI, DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE METZ, ILE DU SAULCY, F-57045 METZ, FRANCE
e-mail: mortini@poncelet.univ-metz.fr
http://poncelet.sciences.univ-metz.fr/∼mortini