Simultaneous stabilization in the disk algebra

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Abstract

Let (f_j, g_j) $(j = 1, \ldots, k)$ be k pairs of functions in the disk algebra $A(\overline{\mathbb{D}})$. Assume that f_j and g_j have no common zeros in $\overline{\mathbb{D}}$ $(j = 1, \ldots, k)$. It is shown that in the case $k = 2$ there exist $\alpha, \beta \in A(\overline{\mathbb{D}})$ such that $\alpha f_j + \beta g_j$ $(j = 1, 2)$ are invertible functions in $A(\overline{\mathbb{D}})$, but that the problem has a negative solution for $k \geq 3$.

1 Introduction

One of the most central issues in the discipline of control theory is to find a controller that internally stabilizes a dynamical system by feedback. Motivation and examples of this objective can be found in any elementary textbook on system theory (see for example [4]). For linear time invariant systems this *stabilization problem* was tackled in the very beginning of control theory. It is always possible to find a controller which internally stabilizes a single linear time invariant system. A question of practical significance and which is also directly connected to this one was addressed recently in the control theory literature (Saeks and Murray 1982 [17] and Vidyasagar and Viswanadham 1982 [20]). The problem described there is known as the *simultaneous stabilization problem* and has the following system theory formulation: "given a finite collection of linear time invariant systems, when it is possible to find a single controller which internally stabilizes each of these systems?". This question has not yet received a complete answer and this paper is devoted to it.

1.1 The simultaneous stabilization problem in control theory

An elegant mathematical formulation of the simultaneous stabilization problem, known as the *factorization approach*, was given in [18] and we follow here essentially the same approach.

We use $\mathbb{D}$ to denote the open unit disk in the complex plane and $\overline{\mathbb{D}}$ for the closed unit disk. $\mathbb{R}(s)$ is the set of real rational functions and $\mathbb{R}_{\overline{\mathbb{D}}}(s)$ is the subset of $\mathbb{R}(s)$ whose elements have no poles in $\overline{\mathbb{D}}$. Trivially, $\mathbb{R}_{\overline{\mathbb{D}}}(s)$ is a ring and we denote its set of invertible elements, or units, by $\mathbb{R}_{\overline{\mathbb{D}}}^{-1}(s)$. A pair (f,g) of elements in $\mathbb{R}_{\overline{\mathbb{D}}}(s)$ is a *coprime pair* if f and g have no common non-unit divisors or, alternatively, if they have no common zeros in $\overline{\mathbb{D}}$.

With these notations the simultaneous stabilization problem for k systems can be restated in the following form (see [18] for more details):

> Let (f_i, g_i) be coprime pairs in $\mathbb{R}_{\overline{\mathbb{D}}}(s)$, $i = 1, \ldots, k$. When is it possible to find $\alpha, \beta \in \mathbb{R}_{\overline{\mathbb{D}}}(s)$ such that $\alpha f_i + \beta g_i$ ($i = 1, \ldots, k$) are invertible functions in $\mathbb{R}_{\overline{\mathbb{D}}}(s)$?

After it was first stated in 1982, this problem has been extensively studied in control theory. When $k = 1$ the answer is clear, $\mathbb{R}_{\overline{\mathbb{D}}}(s)$ is an Euclidean ring (see [10]) and hence, if (f,g) is a coprime pair then there exist $\alpha, \beta \in \mathbb{R}_{\overline{\mathbb{D}}}(s)$ such that $\alpha f + \beta g = 1$ so that the problem always has a solution. Moreover, in this case, there exists a parametrization of all the solutions α and β of $\alpha f + \beta g \in \mathbb{R}_{\overline{\mathbb{D}}}^{-1}(s)$ (see [18]).

Unfortunately, when $k = 2$, the problem does not always have a solution and the question was then to find conditions on (f_i, g_i), $i = 1, 2$, for $\alpha f_i + \beta g_i \in \mathbb{R}_{\overline{\mathbb{D}}}^{-1}(s)$ to be solvable for $i = 1, 2$. A first intermediate step was given by Youla, Bongiorno and Lu [23]. Using their result, Vidyasagar and Viswanadan [20] and Saeks and Murray [17] finally found a necessary and sufficient condition under which the problem admits a solution. This condition is known under the name of *parity interlacing property*.

After these results were obtained much effort has been devoted to the search of conditions on the solvability of the problem when $k \geq 3$. These attempts have so far been unsuccessful and most of the literature deals with either necessary or sufficient conditions without giving a complete answer to the problem (see Emre [3], Ghosh [8], Ghosh and Byrnes [7], Wei and Barmish [22], Kwakernaak [12], Maeda and Vidyasagar [13],...).

1.2 The simultaneous stabilization problem in the disk algebra

In this paper we shall study the simultaneous stabilization problem not in its original context of $\mathbb{R}_{\overline{\mathbb{D}}}(s)$, but in the disk algebra. The interest of doing this appears very clearly in [18] ~~(see, e.g., Theorem 13, p. 38), where the simultaneous stabilization problem is first solved in the disk algebra rather than in $\mathbb{R}_{\overline{\mathbb{D}}}(s)$~~.

The disk algebra $A(\overline{\mathbb{D}})$ is the set of continuous functions on the closed unit disk $\overline{\mathbb{D}}$, which are holomorphic in $\mathbb{D}$. Under the usual pointwise operations and under the sup-norm, $A(\overline{\mathbb{D}})$ is a complex, commutative Banach algebra with unit element 1. Moreover, the rational functions with poles off $\overline{\mathbb{D}}$ are dense in $A(\overline{\mathbb{D}})$. We use $A(\overline{\mathbb{D}})^{-1}$ to denote the set of invertible elements of $A(\overline{\mathbb{D}})$.

Recall that a pair (f, g) of functions in $A(\overline{\mathbb{D}})$ is said to be *coprime*, if f and g have no proper common divisors.

The problem studied in this paper is the simultaneous stabilization problem for the disk algebra:

Simultaneous Stabilization Problem: Let (f_j, g_j) be coprime pairs in $A(\overline{\mathbb{D}})$ $(j = 1, \ldots, k)$. When is it possible to find $\alpha, \beta \in A(\overline{\mathbb{D}})$ such that

$$\alpha f_j + \beta g_j \in A(\overline{\mathbb{D}})^{-1} \quad (j = 1, \ldots, k)?$$

It was hoped at some time that these problems always admit a solution for any k. We shall show that this problem always has a solution for $k = 1$ and $k = 2$ whatever the coprime pairs (f_j, g_j) are. For $k \geq 3$, however, we show by a counterexample that the problem has (in general) no solution.

The method of proof can be generalized to a more algebraic setting. These generalizations and their consequences outside control theory will be dealt with in [14].

2 Main results

2.1 Simultaneous stabilization for $k \leq 2$

First of all let us give a useful characterization of coprime pairs in the disk algebra (and notice the analogy with the coprimeness in $\mathbb{R}_{\overline{\mathbb{D}}}(s)$).

Lemma 2.1: *The pair (f,g) of disk algebra functions is coprime if and only if f and g have no common zeros in $\overline{\mathbb{D}}$.*

Proof. This follows immediately from the proof of Lemma 2.3 in [16, p. 53]. □

Note that Lemma 2.1 is in general not true for other algebras of analytic functions, for example, the Banach algebra H^∞ of bounded analytic functions in $\mathbb{D}$.

Lemma 2.1 now gives us the solution for $k = 1$:

Proposition 2.2: *For every coprime pair (f,g) there always exist $\alpha, \beta \in A(\overline{\mathbb{D}})$ such that*

$$\alpha f + \beta g = 1.$$

Proof. Using Lemma 2.1 the functions $f, g \in A(\overline{\mathbb{D}})$ have no common zeros. Then the assertion follows, for example, from [9, p. 88]. □

So far we have found a particular solution for the simultaneous stabilization problem in case $k = 1$. Now we shall give all solutions.

Lemma 2.3: *Let (f,g) be a coprime pair in $A(\overline{\mathbb{D}})$ and assume that $\alpha, \beta \in A(\overline{\mathbb{D}})$ is a fixed solution of*

$$\alpha f + \beta g = 1.$$

Then every solution of $\tilde{\alpha} f + \tilde{\beta} g = 1$ is of the form

$$\tilde{\alpha} = \alpha + hg, \quad \tilde{\beta} = \beta - hf,$$

where h runs through all functions of the disk algebra.

Proof. Just multiply the equation $\tilde{\alpha} f + \tilde{\beta} g = 1$ by α, resp. β. This gives

$$\tilde{\alpha}(\alpha f) + \tilde{\beta}(\alpha g) = \alpha, \quad \text{resp.} \quad \tilde{\beta}(\beta g) + \tilde{\alpha}(\beta f) = \beta.$$

Using the identity $\alpha f + \beta g = 1$, we obtain

$$\tilde{\alpha} = \alpha + (\tilde{\alpha}\beta - \tilde{\beta}\alpha)g, \quad \text{resp.} \quad \tilde{\beta} = \beta - (\tilde{\alpha}\beta - \tilde{\beta}\alpha)f.$$

This yields the assertion if we put $h := \tilde{\alpha}\beta - \tilde{\beta}\alpha$. Note that $h \in A(\overline{\mathbb{D}})$. On the other hand, if we have

$$\tilde{\alpha} = \alpha + hg, \quad \text{resp.} \quad \tilde{\beta} = \beta - hf,$$

then

$$\begin{aligned} 1 &= \alpha f + \beta g = (\tilde{\alpha} - hg)f + (\tilde{\beta} + hf)g \\ &= \tilde{\alpha} f + \tilde{\beta} g. \end{aligned}$$

□

A quick glance at the proof of Lemma 2.3 reveals that its assertion is true in any commutative ring with unit element.

Let us now discuss the case $k = 2$. We first derive a necessary condition in $A(\overline{\mathbb{D}})$ in order the $k = 2$ problem is solvable. Let (f, g) be an arbitrary coprime pair in $A(\overline{\mathbb{D}})$. Assume that for every coprime pair (f_j, g_j) $(j = 1, 2)$ there exist $\alpha, \beta \in A(\overline{\mathbb{D}})$ such that $\alpha f_j + \beta g_j \in A(\overline{\mathbb{D}})^{-1}$ $(j = 1, 2)$. Taking especially the pairs $(1, 0)$ and (f, g), we see that there exist $\alpha \in A(\overline{\mathbb{D}})^{-1}$ and $\beta \in A(\overline{\mathbb{D}})$ such that $\alpha f + \beta g \in A(\overline{\mathbb{D}})^{-1}$. This means that $f + hg \in A(\overline{\mathbb{D}})^{-1}$ for some $h \in A(\overline{\mathbb{D}})$.

It will emerge from the proof of the following theorem that this condition is also sufficient to obtain a positive solution in the case $k = 2$.

Theorem 2.4: *Let (f_1, g_1), (f_2, g_2) be coprime pairs in $A(\overline{\mathbb{D}})$. Then there always exist $\alpha, \beta \in A(\overline{\mathbb{D}})$ such that*

$$\alpha f_j + \beta g_j \quad (j = 1, 2)$$

are invertible functions in $A(\overline{\mathbb{D}})$.

Proof. Using Proposition 2.2 we see that for each pair (f_j, g_j) there exist $\alpha_j, \beta_j \in A(\overline{\mathbb{D}})$ such that

$$\alpha_j f_j + \beta_j g_j = 1 \quad (j = 1, 2).$$

If we replace α_1 by $\alpha_1 + hg_1$ and β_1 by $\beta_1 - hf_1$, the first identity is still satisfied (see Lemma 2.3). Thus it remains to show the existence of $h \in A(\overline{\mathbb{D}})$ such that

$$(\alpha_1 + hg_1)f_2 + (\beta_1 - hf_1)g_2 \in A(\overline{\mathbb{D}})^{-1}.$$

This we rewrite in the form

$$(1) \qquad f + hg \in A(\overline{\mathbb{D}})^{-1},$$

where

$$f := \alpha_1 f_2 + \beta_1 g_2, \quad g := g_1 f_2 - g_2 f_1,$$

or more instructively

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ g_1 & -f_1 \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}.$$

But the determinant of the matrix above is identically -1, so the matrix itself is invertible. Since (f_2, g_2) is coprime, the functions f_2 and g_2 have no common zero in $\overline{\mathbb{D}}$ (Lemma 2.1). Thus f and g have no common zero in $\overline{\mathbb{D}}$. But then it is well known (see [11] or [2, Theorem 1.2, p. 629]) that relation (1) can always be satisfied. Thus each 2-problem has a solution. □

Remark. An analysis of the proof shows that the simultaneous stabilization problem with data $(f_j, g_j) \in R^2$ satisfying $f_j R + g_j R = R$ $(j = 1, 2)$ has always a solution in a commutative ring R with identity element if and only if R has the so-called Bass stable rank 1 (for a definition of this term see, e.g., [2]).

2.2 Simultaneous stabilization for $k \geq 3$

We now shall attack the simultaneous stabilization problem for $k \geq 3$. On the one hand, those problems seem to be highly overdetermined for $k \geq 3$, on the other hand, there is a very rich supply of function pairs α, β satisfying one identity of the problem; see the proof of Theorem 2.4.

Surprisingly, it turns out that there is in general no solution. A mysterious feature of our proof is that it will not give an explicit counterexample. It merely shows the existence of such an example. However, it will have a very simple form.

Theorem 2.5: *There exist three coprime pairs (f_j, g_j) $(j = 1, 2, 3)$ in the disk algebra $A(\overline{\mathbb{D}})$ such that the simultaneous problem*

$$\alpha f_j + \beta g_j \in A(\overline{\mathbb{D}})^{-1} \quad (j = 1, 2, 3)$$

does not admit any solution α, β in $A(\overline{\mathbb{D}})$.

Proof. Let us denote $(f_1, g_1) := (1, 0)$, $(f_2, g_2) := (1, z)$ and $(f_3, g_3) := (n^2 z, 1)$, where n is an integer. We now show that there is an integer n such that the simultaneous problem

$$\alpha f_j + \beta g_j \in A(\overline{\mathbb{D}})^{-1} \quad (j = 1, 2, 3)$$

admits no solution at all.

Assuming the contrary, for every integer n there exist $\alpha_n, \beta_n \in A(\overline{\mathbb{D}})$ such that

$$\begin{aligned} \alpha_n \cdot 1 + \beta_n \cdot 0 &\in A(\overline{\mathbb{D}})^{-1}, \\ \alpha_n \cdot 1 + \beta_n \cdot z &\in A(\overline{\mathbb{D}})^{-1}, \\ n^2 \alpha_n \cdot z + \beta_n \cdot 1 &\in A(\overline{\mathbb{D}})^{-1}. \end{aligned}$$

Since the first line merely states the invertibility of α_n, we rewrite the system as follows:

$$\begin{aligned} 1 + h_n \cdot z &\in A(\overline{\mathbb{D}})^{-1}, \\ n^2 z + h_n &\in A(\overline{\mathbb{D}})^{-1} \end{aligned} \tag{1}$$

(where we have abbreviated $h_n := \beta_n / \alpha_n$). This gives reason to investigate the following auxiliary functions on the open unit disk $\mathbb{D}$:

$$\Phi_n(z) = \frac{n^2 z^2 + z h_n(z)}{1 + z h_n(z)}.$$

Obviously, using (1), these functions are analytic in $\mathbb{D}$, and no function Φ_n has a zero in $\mathbb{D} \setminus \{0\}$.

Moreover, each function Φ_n attains the value $w = 1$ only twice in $\mathbb{D}$, namely at $z = \pm 1/n$. Thus, by the generalized form of Montel's normal family criterion (see [5, Satz 2, p. 60]) the sequence (Φ_n) is normal in $\mathbb{D} \setminus \{0\}$.

Passing to a subsequence, if necessary, we may assume that Φ_n converges uniformly on compact subsets of $\mathbb{D} \setminus \{0\}$. Again, it is well known that there are only two cases:

Case 1: Φ_n tends locally uniformly to infinity, i.e., the functions $1/\Phi_n$ tend uniformly to zero on every compact set in $\mathbb{D} \setminus \{0\}$.

But then, given $\varepsilon > 0$, we have

$$\left|\frac{1 + zh_n(z)}{n^2 z + h_n(z)}\right| \le \varepsilon |z| \quad \left(n \ge n_0(\varepsilon),\ |z| = \frac{1}{2}\right).$$

Abbreviating ψ_n as the analytic function on the left side, we see that

$$|\psi_n(z)| \le \frac{1}{4} \quad \left(n \ge n_1,\ |z| = \frac{1}{2}\right).$$

Moreover,

$$\frac{h_n(z)}{n^2}[\psi_n(z) - z] = \frac{1}{n^2} - z\psi_n(z),$$

which gives the estimation

$$\left|\frac{h_n(z)}{n^2}\right| \le \frac{\frac{1}{n^2} + \frac{\varepsilon}{2}}{\frac{1}{2} - \frac{1}{4}} \quad \left(|z| = \frac{1}{2}\right)$$

for all integers n larger than $n_1 + n_0(\varepsilon)$.

Now the maximum modulus principle implies that the same inequality holds throughout the disk $|z| \le 1/2$, showing that the functions h_n/n^2 tend uniformly to zero therein. By the second identity in (1), we see that all functions u_n,

$$u_n(z) := z + \frac{1}{n^2}\, h_n(z),$$

are invertible in $\mathbb{D}$, but tend uniformly to the identity function in $|z| < 1/2$, which is neither invertible nor identical zero. This contradicts a well known result of Hurwitz. Thus this case cannot occur.

Case 2: Φ_n tends locally uniformly to an analytic function Φ in $\mathbb{D} \setminus \{0\}$.

In this case the functions Φ_n are uniformly bounded on compact subsets of $\mathbb{D} \setminus \{0\}$, say,

$$|\Phi_n(z)| \le M \quad \left(|z| = \frac{1}{2},\ n \in \mathbb{N}\right).$$

Since $\Phi_n(z) = 1 + \dfrac{n^2 z^2 - 1}{1 + zh_n(z)}$ holds, we have

$$\left|\frac{n^2 z^2 - 1}{1 + zh_n(z)}\right| \le M + 1 \quad \left(|z| = \frac{1}{2},\ n \in \mathbb{N}\right),$$

which shows the following inequality:

$$\left|\frac{n^2}{1+zh_n(z)}\right| \le \frac{M+1}{\frac{1}{4}-\frac{1}{n^2}} \quad \left(|z|=\frac{1}{2},\ n \ge 3\right). \tag{2}$$

Since the functions on the left side are all analytic in $\mathbb{D}$, the same inequality holds in the closed disk $|z| \le 1/2$ by the maximum modulus principle. If we evaluate the inequality above at $z = 0$, we obtain

$$n^2 \le \frac{M+1}{\frac{1}{4}-\frac{1}{n^2}} \quad (n \ge 3),$$

which is obviously false for sufficiently large intergers n.

Because for large n we get a contradiction to the solvability of all simultaneous stabilization problems, we are done. □

As a corollary to Theorem 2.5 we can show that it is in general not possible to chose *both* cofactors $\alpha, \beta \in A(\overline{\mathbb{D}})$ in the equation $\alpha f + \beta g = 1$ to be invertible. (Note that it is always possible to chose one cofactor to be invertible, see [11], [2`].)

Corollary 2.6: *There exist two functions $f, g \in A(\overline{\mathbb{D}})$ such that in every representation of the unit element*

$$\alpha f + \beta g = 1$$

at least α or β is not invertible (i.e., has a zero in $\overline{\mathbb{D}}$).

Proof. Let f and g be two functions defined by $f(z) = z$ and $g(z) = 1-n^2z^2$. Then (f, g) is a coprime pair, since there are no common zeros in $\overline{\mathbb{D}}$ (see Lemma 2.1). Assume on the contrary that there exist, for all integers n, functions $u_n, v_n \in A(\overline{\mathbb{D}})$ which are invertible such that

$$v_n = u_n f + g,$$

i.e.,

$$v_n(z) = [u_n(z) - n^2z + n^2z]z + 1 - n^2z^2 \quad (z \in \overline{\mathbb{D}},\ n \in \mathbb{N}).$$

This gives

$$v_n(z) - [u_n(z) - n^2z]z = n^2z^2 + 1 - n^2z^2 = 1.$$

If we put $h_n(z) = u_n(z) - n^2 z$, we obtain the following two identities:

$$
(1) \qquad \begin{aligned} 1 + h_n \cdot z &= v_n, \\ n^2 z + h_n &= u_n. \end{aligned}
$$

This is just the system we dealt with in the proof of Theorem 2.5, where we showed that it is impossible to hold for all integers n. Thus picking a number n where (1) fails to hold gives our assertion. □

Corollary 2.6 answers negatively the question whether the disk algebra has the so-called unit-1-stable range, see [6].

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