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Analytic contractions, nontangential limits, and the index of invariant subspaces

(joint work with Alexandru Aleman and Carl Sundberg)

Let $\mathcal{H}$ be a Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ such that the operator M_ζ of multiplication with the identity function ζ defines a contraction operator. Let k denote the reproducing kernel of $\mathcal{H}$ and let g be a nonzero function in $\mathcal{H}$. It turns out that up to sets of linear Lebesgue measure 0 the set

$$\Delta(\mathcal{H}) = \{z \in \partial\mathbb{D} : \text{ nt-}\limsup_{\lambda \to z}(1 - |\lambda|^2)\frac{||k_\lambda||^2}{|g(\lambda)|^2} < \infty\}$$

is independent of the choice of g. It is one of our results that $\Delta(\mathcal{H})$ is the largest subset of $\partial\mathbb{D}$ such that for each $f, g \in \mathcal{H}$, $g \neq 0$ the meromorphic function f/g has nontangential limits a.e. on $\Delta(\mathcal{H})$. Furthermore, we will see that the question of whether or not $\Delta(\mathcal{H})$ has linear Lebesgue measure 0 is related to questions about the invariant subspace structure of M_ζ.
We further associate with $\mathcal{H}$ a second set $\Sigma(\mathcal{H}) \subseteq \partial\mathbb{D}$ which is defined in terms of the norm on $\mathcal{H}$ (or equivalently by use of the minimal co-isometric extension of M_ζ). For example, $\Sigma(\mathcal{H})$ has the property that $||\zeta^n f|| \to 0$ for all $f \in \mathcal{H}$ if and only if $\Sigma(\mathcal{H})$ has linear Lebesgue measure 0.
It turns out that $\Delta(\mathcal{H}) \subseteq \Sigma(\mathcal{H})$ a.e. and $\Delta(\mathcal{H}) \neq \Sigma(\mathcal{H})$ in general. We will present conditions that imply that $\Delta(\mathcal{H}) = \Sigma(\mathcal{H})$ a.e.. For example, one corollary to our results states that if dim $\mathcal{H}/\zeta\mathcal{H} = 1$ and if there is a $c > 0$ such that for all $f \in \mathcal{H}$ and all $\lambda \in \mathbb{D}$ we have $||\frac{\zeta-\lambda}{1-\bar{\lambda}\zeta} f|| \geq c||f||$, then $\Delta(\mathcal{H}) = \Sigma(\mathcal{H})$ a.e. and the following four conditions are equivalent:
(1) $||\zeta^n f|| \nrightarrow 0$ for some $f \in \mathcal{H}$,
(2) $||\zeta^n f|| \nrightarrow 0$ for all $f \in \mathcal{H}$, $f \neq 0$,
(3) $\Delta(\mathcal{H})$ has nonzero linear Lebesgue measure in $\partial\mathbb{D}$,
(4) every nonzero invariant subspace $\mathcal{M}$ of M_ζ has index 1, i.e. satisfies dim $\mathcal{M}/\zeta\mathcal{M} = 1$.