Raymond Mortini

Cours à l'Université du Luxembourg

Une deuxième vue sur l'analyse complexe
A second glimpse at complex analysis
Funktionentheorie: zum 2-ten

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'reading course' d'analyse.

Extraits du manuscrit du bouquin en préparation:

An introduction to extension problems, Bézout equations and stable ranks in classical function algebras
Accompanied by introductory chapters on point-set topology and function theory,

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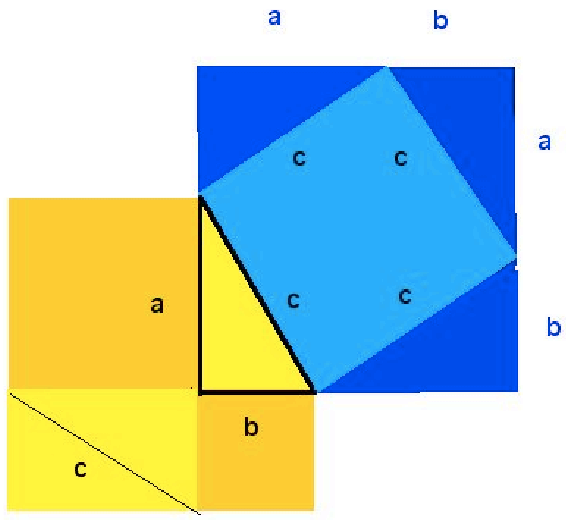
The Pythagorian school: the beginning of mathematics



blue area=yellow area $\Longrightarrow a^2 + b^2 = c^2$

Figure 1: $(a+b)^2 = a^2 + b^2 + 2ab = c^2 + 4\,\frac{ab}{2}$

Contents

Chapter 1

The Jordan curve Theorem, logarithms and fixed points

1.1 Logarithms of holomorphic functions

Definition 1.1. i) A *cycle* Γ is a finite union of closed, piecewise C^1-curves [1]. The *index* (or *winding number*) of a cycle with respect to the point $z \notin \Gamma$ is given by

$$n(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - z}\, d\zeta.$$

It coincides with the number of times the cycle surrounds the point z.

ii) A cycle Γ in a domain $G \subseteq \mathbb{C}$ is called *null-homologous* with respect to G if $n(\Gamma, a) = 0$ for every $a \in \mathbb{C} \setminus G$.

iii) A domain $\mathbb{D}$ in $\mathbb{C}$ is called a *Cauchy domain* if every cycle in D is null-homologous.

iv) A Jordan curve J is the homeomorphic image of the unit circle $\mathbb{T}$.

We note that $n(\Gamma, z)$ is constant on the components of $\mathbb{C} \setminus \Gamma$ and 0 on the unbounded component.

Theorem 1.2. *Let D be a domain in $\mathbb{C}$. The following assertions are equivalent:*

(1) D is a Cauchy domain; that is $n(\gamma, a) = 0$ for every $a \in \mathbb{C} \setminus D$ and every cycle γ in D. In other words, every cycle in D is null-homologous.

(2) Every $f \in H(D)$ admits a primitive.

(3) Every $f \in H(D)$ with $Z(f) = \emptyset$ admits a holomorphic logarithm in D; that is there is $F \in H(D)$ such that $f = e^F$.

(4) Every $f \in H(D)$ with $Z(f) = \emptyset$ admits a holomorphic square-root in D; that is there is $G \in H(D)$ such that $G^2 = f$.

[1] Where we made the usual convention of "identifying" the curve $\gamma = \phi([0,1])$ with the path ϕ itself.

Proof. (1) $\Longrightarrow$ (2): Fix $z_0 \in D$ and let γ_z be a path in D joining z_0 with z. Then the integral $F(z) := \int_{\gamma_z} f(\xi)d\xi$ is a well defined function in D, because by Cauchy's global integral theorem, $\int_\Gamma f(\xi)d\xi = 0$ for every closed curve Γ in D. It is now easy to verify that

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z).$$

Hence $F' = f$.

(2) $\Longrightarrow$ (3): Since $Z(f) = \emptyset$, the quotient f'/f belongs to $H(D)$. By hypothesis (2), f'/f admits a primitive; that is, there is $h \in H(D)$ with $h' = f'/f$. Consider the function $H := fe^{-h}$. Then

$$H' = f'e^{-h} - fe^{-h}h' \equiv 0 \text{ on } D.$$

Since D is connected, $H \equiv$ const. $:= c$ on D. Since $c \neq 0$, we may write c as $c = e^a$ for some $a \in \mathbb{C}$. Then

$$f = e^h e^a = e^{a+h}.$$

The function $F := a + h$ is now the desired logarithm of f.

(3) $\Longrightarrow$ (4): Let $f = e^F$ with $F \in H(D)$ and put $G = e^{F/2}$. Then $G \in H(D)$ and $G^2 = e^F = f$.

(4) $\Longrightarrow$ (1): Let us suppose, to the contrary, that there exists a cycle γ in D and $a \in \mathbb{C}\setminus D$ such that $N := n(\gamma, a) \neq 0$. Consider the function $f(z) = z - a$. Then $Z_D(f) = \emptyset$ and so, by hypothesis (4), we successively have:

(i) There is $f_1 \in H(D)$ with $Z_D(f_1) = \emptyset$ such that $f_1^2 = f$;

(ii) There is $f_2 \in H(D)$ with $Z_D(f_2) = \emptyset$ such that $f_2^2 = f_1$. Hence $f_2^{2^2} = f$.

(iii) $\cdots\cdots\cdots$

(iv) There is $f_n \in H(D)$ with $Z_D(f_n) = \emptyset$ such that $f_n^2 = f_{n-1}$. Hence $f_n^{2^n} = f$.

By taking logarithmic derivatives we deduce that

$$\frac{1}{z-a} = \frac{f'(z)}{f(z)} = \frac{2f_1f_1'}{f_1^2} = 2\frac{f_1'}{f_1} = 2\left(2\frac{f_2'}{f_2}\right) = \cdots = 2^n\frac{f_n'}{f_n}.$$

Hence

$$\begin{aligned} N &= \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz = \frac{1}{2\pi i}\int_\gamma 2^n\frac{f_n'(z)}{f_n(z)}dz \\ &= 2^n \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}dz}_{:=N_n} \\ &= 2^n N_n. \end{aligned}$$

Since $\frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}\,dz = n(f_n\circ\gamma, 0)$, $N_n \in \mathbb{Z}$ for every $n \in \mathbb{N}^*$. Thus $N_n \to 0$ and so $N_n = 0$ for every $n \geq n_0$. But, by assumption, $N \neq 0$. This is a contradiction. Thus we have shown that D is a Cauchy domain. $\square$

Proposition 1.3.

i) Let $f \in C(\mathbb{T}, \mathbb{C}^*)$ satisfy $f(-z) = -f(z)$ for all $z \in \mathbb{T}$. Then f has no square root in $C(\mathbb{T}, \mathbb{C})$.

ii) No odd function $f \in C(\mathbb{T}, \mathbb{C}^*)$ has a continuous logarithm in $C(\mathbb{T}, \mathbb{C})$,

Proof. i) Suppose that there is $g \in C(\mathbb{T}, \mathbb{C})$ such that $g^2 = f$. Since, by hypothesis, f has no zeros, g has no zeros. Hence $q(z) := g(z)/g(-z) \in C(\mathbb{T}, \mathbb{C}^*)$ from which we conclude that $q^2(z) = f(z)/f(-z) = -1$. Because $\mathbb{T}$ is connected, $q \equiv i$ or $q \equiv -i$. Therefore, since q is constant,

$$-1 = q^2(z) \stackrel{!}{=} q(z)q(-z) = \frac{g(z)}{g(-z)} \frac{g(-z)}{g(z)} = 1.$$

An obvious contradiction.

ii) This immediately follows from (i), since otherwise $e^L = f$ implies that $\left(e^{\frac{L}{2}}\right)^2 = e^L = f$; in other words f would have a square root, contradicting i). $\square$

1.2 The Jordan curve theorem

Lemma 1.4. *Let $K \subseteq \mathbb{R}^n$ be compact and C a component of $K^c := \mathbb{R}^n \setminus K$. Then C is open and $\partial C \subseteq \partial K \subseteq K$.*

Proof. It is easy to see that C is open. Now suppose, to the contrary, that there exists $x \in \partial C \setminus \partial K$. Since every neighborhood of x meets C and $K \cap C = \emptyset$, x cannot be an interior point of K. Thus $x \in K^c$ and so x belongs to some component $\tilde{C}$ of K^c. Let $x_n \to x$, $x_n \in C$. Because $\tilde{C}$ is open, $x_n \in \tilde{C}$ for almost all n. But components are either disjoint or coincide. Thus $\tilde{C} = C$. So $x \in C$. This contradicts the assumption that $x \in \partial C$ (note that, due to openness, $\partial C \cap C = \emptyset$). $\square$

Theorem 1.5 (The Jordan curve Theorem)**.** *Let $J = \phi(T)$ be a Jordan curve in $\mathbb{C}$. Then $\mathbb{C} \setminus J$ has exactly two components: the unbounded one, C_∞, and the bounded component C_1. Moreover $J = \partial C_\infty = \partial C_1$.*

Proof. First we show that J has at least one bounded component. Suppose, to the contrary, that $\mathbb{C} \setminus J$ is connected. Let $f : J \to \mathbb{T}$ be the inverse of ϕ. By Theorem 3.9, f has a continuous logarithm on J; say $f = e^h$ for some $h \in C(J)$.

Since f is the left-inverse of ϕ we get

$$\xi = f(\phi(\xi)) = e^{h(\phi(\xi))}, \quad |\xi| = 1.$$

Let H be a Tietze extension of $h \circ \phi : \mathbb{T} \to \mathbb{C}$ to $\mathbb{C}$. Then e^H is a zero-free extension of the identity on $\mathbb{T}$; a contradiction to Corollary 1.12 to Brouwer's fixed point theorem.

Next we show that there are at most two components. Suppose to the contrary that $\mathbb{C} \setminus J$ has two bounded components, C_1 and C_2. Let $a_j \in C_j$ and consider the function

$$f(z) = (z - a_1)^{s_2}/(z - a_2)^{s_1}, \quad (z \in J).$$

Here the $s_j \in \mathbb{Z}$ are chosen so that for $\xi \in \mathbb{T}$

$$\phi(\xi) - a_1 = \xi^{s_1} e^{h_1(\xi)} \text{ and } \phi(\xi) - a_2 = \xi^{s_2} e^{h_2(\xi)},$$

where $h_1, h_2 \in C(\mathbb{T}, \mathbb{C})$ (see Theorem 3.19). Hence, with $z = \phi(\xi) \in J$, $|\xi| = 1$,

$$f(z) = e^{s_2(h_1 \circ \phi^{-1})(z) - s_1(h_2 \circ \phi^{-1})(z)}.$$

Hence f has a continuous logarithm on J. By Theorem 3.16, $s_2 = s_1 = 0$. Therefore $\phi(\xi) - a_1 = e^{h_1(\xi)}$ and so $z - a_1 = e^{(h_1 \circ \phi^{-1})(z)}$ $(z \in J)$ is an exponential. In particular, applying Tietze to $h_1 \circ \phi^{-1}$, we see that $(z - a_1)|_J$ has a continuous zero-free extension to $\mathbb{C}$. This is a contradiction to Corollary 1.12. Hence, $\mathbb{C} \setminus J$ has exactly one bounded component, which we denote by C_1.

Now let C_∞ be the unbounded component of $\mathbb{C} \setminus J$. We conclude that $\mathbb{C} \setminus J = C_\infty \cup C_1$.

Next we show the assertion that $J = \partial C_\infty = \partial C_1$. By Lemma 1.4, for $j = 1$ or $j = \infty$, $\partial C_j \subseteq J$. Now

$$\mathbb{C} \setminus \partial C_j = \mathbb{C} \setminus (\overline{C}_j \setminus C_j) = (\mathbb{C} \setminus \overline{C}_j) \cup C_j.$$

Hence $\mathbb{C} \setminus \partial C_j$ is a disjoint union of open sets; so it is disconnected. Now, if ∂C_j would be a proper subset of J, then, by Lemma 1.6, $\mathbb{C} \setminus \partial C_j$ would be connected. A contradiction. Thus $\partial C_j = J$. □

The unbounded component C_∞ of $\mathbb{C} \setminus J$ is called the *exterior* domain and the bounded component C_1 the *interior* domain associated with the Jordan curve J. One also says that C_1 is the interior of J and C_∞ the exterior of J.

Lemma 1.6. *Let J be a Jordan curve in $\mathbb{C}$, If K is a proper subset of J, then $\mathbb{C} \setminus K$ is connected.*

Proof. We shall use Borsuk's Theorem 3.9 to conclude. So let $f : K \to \mathbb{C}^*$ be a continuous zero-free function on K. Let $S := \phi^{-1}(K)$. Since $\phi : \mathbb{T} \to J$ is a homeomorphism, S is a proper compact subset of $\mathbb{T}$. Hence $\mathbb{C} \setminus S$ is connected.

For $\xi \in \mathbb{T}$, let $h(\xi) = f(\phi(\xi))$. Then h is a zero-free continuous function on S. By Theorem 3.9, h has a continuous logarithm L on S. Thus, for $z \in K$,

$$f(z) = \exp(L(\phi^{-1}(z)),$$

and so f has a continuous logarithm on K. By Borsuk's Theorem again, $\mathbb{C} \setminus K$ is connected. □

1.3 Brouwer's fixed point theorem

Let $\mathbf{B}_n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : ||x|| := \sqrt{\sum_{j=1}^n |x_j|^2} \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$ and $S_{n-1} := \partial \mathbf{B}_n$ the unit sphere (boundary of $\mathbf{B}_n$).

Lemma 1.7. *There is no C^1-map $f : \mathbf{B}_n \to S_{n-1}$ such that $f(x) = x$ for all $x \in S_{n-1}$; in other words, the closed ball does not admit a (smooth) retract to its boundary* [2].

Proof. Let us suppose that there does exist such a map $f : \mathbf{B}_n \to S_{n-1}$. Consider the convex-combination f_t of the identity on $\mathbf{B}_n$ and f; that is, let $f_t(x) = (1-t)x + tf(x)$, $0 \le t \le 1$. Since $||f_t(x)|| \le 1$ we have that f_t is a self-map of $\mathbf{B}_n$ that fixes every point in S_{n-1}, too. Moreover, f_t writes as $f_t(x) = x + t(f(x) - x)$. With $g(x) := f(x) - x$, we see that g is a C^1-map of $\mathbf{B}_n$ and so, due to the convexity of $\mathbf{B}_n$, g satisfies a Lipschitz condition there. That is, there is a constant $C > 0$ such that for every $x, y \in \mathbf{B}_n$

$$||g(x) - g(y)|| \le C||x - y||.$$

We claim that there is $t_0 > 0$ such that for all $t \in [0, t_0]$, the function f_t is a bijection of $\mathbf{B}_n$ onto $\mathbf{B}_n$. First we show injectivity.

Suppose that $f_t(x_1) = f_t(x_2)$ for some points $x_1, x_2 \in \mathbf{B}_n$. Then $x_2 - x_1 = t(g(x_1) - g(x_2))$ and therefore

$$||x_2 - x_1|| = t||g(x_1) - g(x_2)|| \le Ct||x_2 - x_1||.$$

Hence, for $Ct < 1$, we get that $x_1 = x_2$. Thus f_t is injective on $\mathbf{B}_n$ whenever $0 \le t < 1/C$.

To show surjectivity, we look at the Jacobian matrix $J_{f_t}(x)$. Note that $J_{f_t}(x) = I_n + tJ_g(x)$, where I_n is the identity matrix in $M_n(\mathbb{R})$. Let

$$p(t, x) := \det J_{f_t}(x) = \det(I_n + tJ_g(x))$$

be the determinant of the Jacobian. Obviously, for each $x \in \mathbf{B}_n$, $p(t, x)$ is a polynomial in t. As g is C^1, we see that $p(t, x)$ is a uniformly continuous function on $[0, 1] \times \mathbf{B}_n$ with $p(0, x) = 1$. Thus there exists $t_0 \in]0, 1/C[$ such that $p(t, x) > 0$ for all $t \in [0, t_0]$ and $x \in \mathbf{B}_n$. By the inverse function theorem, $G_t := f_t((\mathbf{B}_n)^\circ)$ is an open set in $\mathbb{R}^n$ for these t. Note that $G_t \subseteq \mathbf{B}_n$, and that, due to continuity, $\overline{G}_t = f_t(\mathbf{B}_n)$. Assuming that $\overline{G}_t \ne \mathbf{B}_n$, the boundary ∂G_t of G_t intersects $\mathbf{B}_n$ at some point y_0 with $||y_0|| < 1$. Let $x_0 \in \mathbf{B}_n$ satisfy $f_t(x_0) = y_0$. Since G_t is open, $y_0 \notin G_t$ and so $x_0 \in \mathbf{B}_n \setminus (\mathbf{B}_n)^\circ = S_{n-1}$. Hence

$$y_0 = f_t(x_0) = x_0$$

has norm one; a contradiction. We conclude that f_t is a surjection of $\mathbf{B}_n$ onto itself.

Let us now consider the polynomial $F(t) = \int_{\mathbf{B}_n} p(t, x)dx$. Since for $t \in [0, t_0]$, $f_t : \mathbf{B}_n \to \mathbf{B}_n$ is a bijection, we obtain from the change of variable formula for multiple integrals that $F(t)$ is the volume σ_n of the image $f_t(\mathbf{B}_n)$. But $f_t(\mathbf{B}_n) = \mathbf{B}_n$. Hence $F(t)$ is constant for $t \in [0, t_0]$. Since $F(t)$ is a polynomial, $F(t)$ must be constant σ_n for each $t \in [0, 1]$. In particular, $F(1) = \sigma_n > 0$.

But on the other hand, $||f(x)|| = 1$ for each $x \in \mathbf{B}_n$ implies that $\sum_{j=1}^n \phi_j^2(x) \equiv 1$, where $f = (\phi_1, \dots, \phi_n)$. Hence, by taking the partial derivatives,

$$\sum_{j=1}^n \frac{\partial \phi_j}{\partial x_k} \phi_j \equiv 0.$$

[2] In Corollary 1.11 it will be shown that one can replace "smooth" by "continuous".

Thus each column of the Jacobian

$$J_f(x) = \begin{pmatrix} \frac{\partial \phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \phi_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \phi_n(x)}{\partial x_1} & \cdots & \frac{\partial \phi_n(x)}{\partial x_n} \end{pmatrix}$$

is orthogonal to the column vector $f(x)$ itself; hence the rank of $J_f(x)$ is less than $n-1$ and so its determinant $\det J_f(x)$ is zero for every $x \in \mathbf{B}_n$. But $f(x) = f_1(x)$ and so

$$F(1) = \int_{\mathbf{B}_n} \det J_{f_1}(x) dx = 0.$$

This contradicts the fact that $F(1) > 0$. □

Theorem 1.8 (Brouwer's fixed point theorem)**.** *Every continuous map $f : \mathbf{B}_n \to \mathbf{B}_n$ has a fixed point.*

Proof. By the Stone-Weierstrass theorem there exists a sequence (p_j) of polynomials $p_j \in \mathbb{R}[x_1, \ldots, x_n]$ converging uniformly to f on $\mathbf{B}_n$. Say

$$\max_{x \in \mathbf{B}_n} ||p_j(x) - f(x)|| \leq 1/j.$$

Now $q_j = (1 + 1/j)^{-1} p_j$ converges uniformly to f, too, and $||q_j(x)|| \leq 1$. Hence each q_j is a C^1 self-map of $\mathbf{B}_n$. We claim that q_j has a fixed point in $\mathbf{B}_n$. For if not, consider the map $Q_j : \mathbf{B}_n \to S_{n-1}$ that sends $x \in \mathbf{B}_n$ to the unique intersection point of the ray from $q_j(x)$ to x that meets S_{n-1}; that is $Q_j(x) = x + s(x)n(x)$ where $n(x) = (x - q_j(x))/||x - q_j(x)||$ and

$$s(x) = -x \cdot n(x) + \sqrt{1 - ||x||^2 + (x \cdot n(x))^2}.$$

Then Q_j is a continuous map from $\mathbf{B}_n$ to S_{n-1} that fixes each point in S_{n-1}. In order to apply Lemma 1.7, we need still to show that Q_j is a C^1-map. To see this, it suffices to show that $1 - ||x||^2 + \langle x, n(x) \rangle^2$ is never zero.

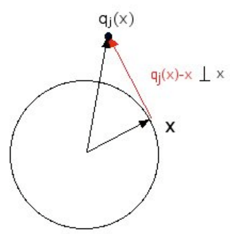


Figure 1.1: $q_j(x) - x$ orthogonal to x

This actually holds, since otherwise $1 + \langle x, n(x) \rangle^2 = ||x||^2 \leq 1$ implies $\langle x, n(x) \rangle = 0$ and $||x|| = 1$ for some $x \in \mathbf{B}_n$. Hence x is orthogonal to $x - q_j(x)$. By elementary geometry $||q_j|| > 1$, a contradiction (see figure 1.1).

This can also be seen as follows: Since $0 = \langle x - q_j(x), x\rangle = ||x||^2 - \langle q_j(x), x\rangle$, we see that

$$1 = ||x||^2 = \langle q_j(x), x\rangle = |\,\langle q_j(x), x\rangle\,| \leq ||x||\;||q_j(x)|| \leq 1.$$

Thus, by the equality case in the Cauchy-Schwarz inequality, $q_j(x) = \lambda x$ for some $\lambda > 0$. Consequently $|\lambda| = 1$ and so $q_j(x) = x$; a contradiction to the assumption that q_j does not admit a fixed point. Hence we have found a C^1-retract of $\mathbf{B}_n$ into S_{n-1}; a contradiction to Lemma 1.7. Thus, each of our maps q_j has a fixed point $x_j \in \mathbf{B}_n$. By passing to a subsequence, we may assume that x_j converges to some $\xi \in \mathbf{B}_n$. Due to the uniform convergence of (q_j) to f, we therefore have that $f(\xi) = \xi$, too. □

Corollary 1.9. *Every bounded continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point.*

Proof. Since f is bounded, there is a ball $B = \{x \in \mathbb{R}^n : ||x|| \leq r\}$, $r > 0$, such that $f(\mathbb{R}^n) \subseteq B$. Hence the restriction $f : B \to B$ is a continuous selfmap of B. Brouwer's fixed point Theorem 1.8 applied to the map

$$F : \begin{cases} \mathbf{B}_n \to \mathbf{B}_n \\ x \mapsto r^{-1} f(r\,x) \end{cases}$$

yields the desired fixed point of f in B. □

One of our major applications of Brouwer's fixed point theorem is the following result on the non-existence of invertible extensions of a certain vector-valued map in $\mathbb{R}^n$.

Theorem 1.10. *Let $K \subseteq \mathbb{R}^n$ be compact, C a bounded component of $\mathbb{R}^n \setminus K$ and $a \in C$. Then the n-tuple $x - a$, defined on K, is invertible, but does not admit a continuous extension to an invertible n-tuple on $K \cup C$.*

Proof. Let $f(x) = x - a$. Since $f \cdot \dfrac{f^t}{|f|^2} = 1$, we see that f is invertible. Next we use that, by Lemma 1.4, $\partial C \subseteq \partial K$. Let $L = K \cup C$. Suppose that $F_1 \in C(L, \mathbb{R}^n)$ is an invertible n-tuple extending f. Let

$$F_2(x) = \begin{cases} x - F_1(x) & \text{if } x \in \overline{C} \\ a & \text{if } x \in \mathbb{R}^n \setminus C. \end{cases}$$

Since

$$(\mathbb{R}^n \setminus C) \cap \overline{C} \subseteq \partial C \subseteq \partial K \subseteq K,$$

we see that F_2 is well-defined, because on this intersection both expressions are equal. Since F_2 is bounded, there is a closed ball $B \subseteq \mathbb{R}^n$ with $\overline{C} \subseteq B$ such that F_2 is a continuous self-map of B. By Brouwer's fixed point theorem (Theorem 1.8) there is a point $w \in B$ with $F_2(w) = w$. Because $a \in C$, the second case in the definition for $F_2(w)$ is not possible. Hence, $w \in \overline{C} \subseteq L$ and $w = F_2(w) = w - F_1(w)$. Thus $F_1(w)$ would be the zero vector, a contradiction to the invertibility of the n-tuple F_1 on L. We conclude that f cannot be extended to an invertible continuous n-tuple on L. □

Corollary 1.11. *Let $\mathbf{B}_n$ be the closed unit ball in $\mathbb{R}^n$. Then the identity map*

$$(x_1, \ldots, x_n) : \partial\mathbf{B}_n \to \partial\mathbf{B}_n$$

does not admit a zero-free [3]*, continuous extension to $\mathbf{B}_n$.*

Proof. Immediate from Theorem 1.10. □

Later, in Theorem 1.14, it will be shown that Corollary 1.11 is, indeed, equivalent to Brouwer's fixed-point theorem.

By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we immediately obtain the next Corollary.

Corollary 1.12. *Let $K \subseteq \mathbb{C}$ be compact, C a bounded component of $\mathbb{C} \setminus K$ and $a \in C$. Then the function $f(z) = z - a$ defined on K is zero-free on K, but does not admit a zero-free extension to $K \cup C$.*

Later on (see Lemma 3.15) we will prove the same result for integer powers of $z - a$.

Definition 1.13. A Hausdorff space X is said to be *contractible* to the point $x_0 \in X$, if there is a continuous map $H : X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$. Such a map is said to be a *homotopic contraction.*

Thus we have that X is contractible to the point x_0 if the identity map on X is homotopic to the constant map $x \mapsto x_0$.

Every convex set in $\mathbb{R}^n$ is contractible; in fact, if K is a convex set, then we use the map $H(x,t) = (1-t)x + t\,N$, $0 \le t \le 1$, where N is any fixed point of K and $x \in K$.

Theorem 1.14. *Let $E = (E, ||\cdot||)$ be a normed space, $B = \{x \in E : ||x|| \le 1\}$ the unit ball, and $S = \{x \in E : ||x|| = 1\}$ the unit sphere in E. Then the following assertions are equivalent:*

(1) Each continuous function $f : B \to B$ has a fixed point;

(2 The sphere S is no retract for B; that is there does not exist a continuous function $r : B \to S$ such that $r(x) = x$ for every $x \in S$;

(3) The sphere S is not contractible.

Proof. (1) $\Longrightarrow$ (2) Suppose that $r : B \to S$ is a retract. Then $f = -r$ is a selfmap of B; By (1) f admits a fixed point $x_0 \in B$. Since $f(B) \subseteq S$, we have that $x_0 \in S$. Hence

$$x_0 = f(x_0) = -r(x_0) = -x_0.$$

This is a contradiction, since $||x_0|| = 1$. Thus (2) holds.

[3]meaning that $(0, \ldots, 0)$ does not belong to the image

(2) $\Longrightarrow$ (3) Suppose that $H : [0,1] \times S \to S$ is a homotopy in S with $H(x,0) = x$ and $H(x,1) = x_0$ for every $x \in S$. Let

$$r(x) = \begin{cases} x_0 & \text{if } ||x|| \leq 1/2 \\ H(\frac{x}{||x||}, 2 - 2||x||) & \text{if } 1/2 \leq ||x|| \leq 1. \end{cases}$$

Then r is well defined (since for $||x|| = 1/2$ we have $r(x) = H(2x,1) = x_0$), hence r is a continuous map on B into S. Now for $x \in S$ we see that $r(x) = H(x, 2-2) = H(x,0) = x$. Thus r is a retract of B to S; a contradiction to (2).

(3) $\Longrightarrow$ (1)

Suppose that there exists a continuous selfmap f of B without any fixed points. Consider the function $H : [0,1] \times S \to S$ defined by

$$H(x,t) = \begin{cases} \frac{x - 2tf(x)}{||x - 2tf(x)||} & \text{if } 0 \leq t < 1/2 \\ \frac{2x - 2tx - f(2x-2tx)}{||2x-2tx-f(2x-2tx)||} & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Then H is well defined and continuous. (In fact, if $2x - 2tx - f(2x-2tx) = 0$, then $2x - 2tx$ would be a fixed point of f, and if $x - 2tf(x) = 0$, then $1 > 2t||f(x)|| = ||x|| = 1$. Twice a contradiction.) Moreover, $H(x,0) = x$ and $H(x,1) = x_0$, where $x_0 := -f(0)/|f(0)| \in S$. Thus H is a homotopy shrinking the sphere to a point. A contradiction to (3). $\square$

Chapter 2

The $\overline{\partial}$-calculus

In this chapter we present some advanced function theoretic tools that we need several times later on. The main feature is to derive explicit solutions to the $\overline{\partial}$-equation $\overline{\partial}v = f$ in $\mathbb{C}$ (also called inhomogeneous Cauchy-Riemann differential equation) and to derive in a simple and elegant way the Gauss-Green-Stokes formulas in the plane. We also include a purely computational section giving the explicit values of several Cauchy-type integrals $\int_{\mathbb{D}} f(\zeta)/(\zeta - z)\, d\sigma_2(\zeta)$. The set of all holomorphic functions on an open set Ω will be denoted by $H(\Omega)$.

2.1 The Wirtinger derivatives

Whereas in real analysis the partial derivatives $f_{x_j} := \partial f/\partial x_j$ play a central role, their counterparts in complex analysis are the so-called Wirtinger derivatives $\partial f = \partial f/\partial z$ and $\overline{\partial} f = \partial f/\partial \overline{z}$.

So suppose that $f : \Omega \to \mathbb{C}$ is $\mathbb{R}$-differentiable at $z_0 = x_0 + iy_0 \in \Omega$, $\Omega \subseteq \mathbb{C}$ open. Then, by writing $x - x_0 = \frac{1}{2}\big((z - z_0) + (\overline{z} - \overline{z}_0)\big)$ and $y - y_0 = \frac{1}{2i}\big((z - z_0) - (\overline{z} - \overline{z}_0)\big)$ we arrive at

$$\begin{aligned} f(z) &= f(z_0) + f_x(z_0)(x - x_0) + f_y(z_0)(y - y_0) + o(z - z_0) \\ &= f(z_0) + \frac{1}{2}\big(f_x(z_0) - if_y(z_0)\big)(z - z_0) + \frac{1}{2}\big(f_x(z_0) + if_y(z_0)\big)(\overline{z} - \overline{z}_0) + o(z - z_0). \end{aligned}$$

The Wirtinger derivatives are now defined by

$$f_z := \partial f/\partial z := \partial f := \frac{1}{2}\big(f_x - if_y\big)$$

$$f_{\overline{z}} := \partial f/\partial \overline{z} := \overline{\partial} f := \frac{1}{2}\big(f_x + if_y\big).$$

It is easy to see that the Cauchy-Riemann equations take the form $\overline{\partial} f = 0$. Also, if f is holomorphic at z_0, then $f'(z_0) = f_z(z_0) = f_x(z_0)$.

Here is a couple of useful formulas.

Proposition 2.1.

Let $f(z), g(z)$ and $h(w)$ be C^1-functions in $\mathbb{C}$. Then

(1) $(fg)_z = fg_z + f_z\, g$; $\quad (fg)_{\overline{z}} = fg_{\overline{z}} + f_{\overline{z}}\, g$;

(2) $\overline{(f_z)} = \overline{f}_{\overline{z}}$ *and* $\overline{(f_{\overline{z}})} = \overline{f}_z$*;*

(3) $(h \circ f)_z = (h_w \circ f)\, f_z + (h_{\overline{w}} \circ f)\, \overline{f}_z$*;* $\quad (h \circ f)_{\overline{z}} = (h_w \circ f)\, f_{\overline{z}} + (h_{\overline{w}} \circ f)\, \overline{f}_{\overline{z}}$*;*

(4) If $s(t)$ *is a* C^1*-path in* $\mathbb{C}$*, then*

$$(f \circ s)'(t) = f_z(s(t))\, s'(t) + f_{\overline{z}}(s(t))\, \overline{s'(t)}.$$

(5) If $\varphi(\theta, r) = f(re^{i\theta})$*,* $r > 0$*, then*

$$\varphi_r(\theta, r) = e^{i\theta} f_z + e^{-i\theta}\, f_{\overline{z}}$$
$$\varphi_\theta(\theta, r) = i\, re^{i\theta}\, f_z + re^{-i\theta}(-i)\, f_{\overline{z}};$$

(6) $\overline{\partial} f(z) = \dfrac{1}{2} e^{i\theta} \left(\dfrac{\partial}{\partial r} + \dfrac{i}{r} \dfrac{\partial}{\partial \theta} \right) \varphi(\theta, r);$

(7) If $f \in C^1(\mathbb{R}, \mathbb{R})$ *and* $h \in C^1(\mathbb{C}, \mathbb{R})$*, then*

$$(f \circ h)_z = (f' \circ h) h_z \text{ and } (f \circ h)_{\overline{z}} = (f' \circ h) h_{\overline{z}}.$$

(8) Let $u(z, w)$ *be a* C^1*-function in* $\mathbb{C}^2$*,* $h_1(t)$ *and* $h_2(t)$ *two* C^1*-paths in* $\mathbb{C}$*, and* $h(t) := u(h_1(t), h_2(t))$*. Then*

$$\begin{aligned} h'(t) &= u_z(h_1(t), h_2(t)) \cdot h_1'(t) + u_{\overline{z}}(h_1(t), h_2(t)) \cdot \overline{h_1'(t)} \\ &+ u_w(h_1(t), h_2(t)) \cdot h_2'(t) + u_{\overline{w}}(h_1(t), h_2(t)) \cdot \overline{h_2'(t)}. \end{aligned}$$

The proof is purely computational. Note that (5) is best done if one writes $\varphi(\theta, r) = f(r\cos\theta, r\sin\theta)$. Let us point out that the second exponential factor in the partial derivatives for φ is $e^{-i\theta}$ and not $e^{i\theta}$. Applying formula (6), we immediately see that the logarithm

$$\log_{D_j}(z) := \log|z| + i \arg z$$

is a holomorphic function on $D_1 := \mathbb{C} \setminus [0, \infty[$ or $D_2 = \mathbb{C} \setminus\,]-\infty, 0]$ for example.

Here are two formulas envolving the Laplace operator $\Delta u = u_{xx} + u_{yy}$ on $\mathbb{C}$:

Remark 2.2. Let u be a C^2-function in $\mathbb{C}$, and $f \in H(\mathbb{C})$. Then:

(1) $\Delta u = 4\, u_{z\overline{z}}$.

(2) $\Delta(u \circ f) = \big((\Delta u) \circ f\big)\, |f'|^2$.

Proof. (1) $u_{z\overline{z}} = \frac{1}{2}(u_x - iu_y)_{\overline{z}} = \frac{1}{4}\, (u_{xx} + iu_{xy}) - \frac{1}{4}\, i(u_{yx} + iu_{yy}) = \frac{1}{4}\, \Delta u$.
(2) Here we use Proposition 2.1 (3) and (2):

$$\begin{aligned} 4\Delta(u \circ f) &= (u \circ f)_{z\overline{z}} = \big((u_w \circ f) f_z\big)_{\overline{z}} \\ &= (u_{w\overline{w}} \circ f) \overline{f}_{\overline{z}}\, f_z \\ &= 4\big((\Delta u) \circ f\big)\, |f'|^2. \end{aligned}$$

□

2.2 Planar Cauchy-integrals and the $\overline{\partial}$-equation

One of our main tools will be the following Cauchy-type representation theorem for smooth functions with compact support. Planar Lebesgue measure is denoted by σ_2.

Theorem 2.3. *Let f be continuously $\mathbb{R}$-differentiable in $\mathbb{C}$ and suppose that f has compact support. Then*

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial} f(\zeta)}{\zeta - z} d\sigma_2(\zeta).$$

Proof. Fix $z \in \mathbb{C}$ and Let $\phi(r,\theta) = f(z + re^{i\theta})$. Then, by 2.1 (6), the integral above coincides with the limit, as $\varepsilon \to 0$, of

$$I(\varepsilon) := -\frac{1}{2\pi}\int_{\varepsilon}^{\infty}\int_{0}^{2\pi}\left(\frac{\partial\phi}{\partial r} + \frac{i}{r}\frac{\partial\phi}{\partial\theta}\right) d\theta\, dr. \tag{2.1}$$

Since ϕ and its partial derivatives are periodic in θ with period 2π, the integral of $\partial\phi/\partial\theta$ is 0. Hence

$$I(\varepsilon) = -\frac{1}{2\pi}\int_{0}^{2\pi}\int_{\varepsilon}^{\infty}\frac{\partial\phi}{\partial r} dr\, d\theta = \frac{1}{2\pi}\int_{0}^{2\pi}\phi(\varepsilon,\theta)\, d\theta.$$

Since $\phi(\varepsilon,\theta)$ tends uniformly (in θ) to $f(z)$ as $\varepsilon \to 0$, we see that $\lim_{\varepsilon\to 0} I(\varepsilon) = f(z)$. □

The existence of the integral

$$-\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial} f(\zeta)}{\zeta - z} d\sigma_2(\zeta)$$

with a singularity at z also follows from the fact that the convolution integral of a locally $L_1(d\sigma_2)$-function, here $1/\zeta$, with a continuous function, converges.

The following result is an analogue of the Cauchy-integral formula.

Proposition 2.4. *Let $\alpha \in C_c^\infty(\mathbb{C})$ and f continuous on $\mathbb{C}$. Suppose that f is holomorphic in Ω, where Ω is an open neighborhood of the support of α. Then*

$$\int_{\mathbb{C}} (\overline{\partial}\alpha)\, f\, d\sigma_2 = 0. \tag{2.2}$$

Proof. Let U be an open set such that $\operatorname{supp}\alpha \subseteq U \subseteq \overline{U} \subseteq \Omega$. Let $\phi \in C_c^\infty(\mathbb{C})$ satisfy $\phi = 1$ on U and $\operatorname{supp}\phi \subseteq \Omega$. Fix $a \in \mathbb{C}$ and define

$$F(z) = \begin{cases} \phi(z)f(z)(z-a) & \text{if } z \in \Omega \\ 0 & \text{if } z \notin \Omega. \end{cases}$$

Then $F \in C_c^\infty(\mathbb{C})$. Since F is holomorphic in $U \supseteq \operatorname{supp}(\alpha)$, we have $\overline{\partial}(\alpha F) = F\overline{\partial}\alpha$ on U. Thus, by Theorem 2.3,

$$\begin{aligned}
-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\alpha(\zeta)f(\zeta)\,d\sigma_2(\zeta) &= -\frac{1}{\pi}\int_U \frac{\overline{\partial}\alpha(\zeta)F(\zeta)}{\zeta-a}\,d\sigma_2(\zeta)\\
&= -\frac{1}{\pi}\int_U \frac{\overline{\partial}(\alpha F)(\zeta)}{\zeta-a}\,d\sigma_2(\zeta)\\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial}(\alpha F)(\zeta)}{\zeta-a}\,d\sigma_2(\zeta)\\
&= \alpha(a)F(a)=0.
\end{aligned}$$

□

A second proof of Proposition 2.4 will be given later 2.21 as an application of the complex version of Gauss' Theorem.

It is interesting to note that if we merely assume that f is holomorphic in a neighborhood of supp $\overline{\partial}\alpha$, then formula (2.2) does no longer hold, although the values of f outside supp $\overline{\partial}\alpha$ do not play any role, as what is demonstrated by the following integral equality:

$$\int_{\mathbb{C}} (\overline{\partial}\alpha)f\,d\sigma_2 = \int_{\text{supp }\overline{\partial}\alpha} (\overline{\partial}\alpha)f\,d\sigma_2.$$

Here is an example. Given $a\in\mathbb{C}$, choose $\alpha\in C_c^\infty(\mathbb{C})$ so that

$$R := (\text{supp }\alpha)^\circ \setminus \text{supp }\overline{\partial}\alpha \neq \emptyset,$$

$a\in R$ and $\alpha(a)=1$.

Let U be an open neighborhood of supp $\overline{\partial}\alpha$ and W an open set containing a with $\overline{U}\cap\overline{W}=\emptyset$. Let $g\in C_c^\infty(\mathbb{C})$ be chosen so that $g=0$ in a neighborhood of $\overline{W}$ and $g(z)=1$ if $z\in U$. Define f by

$$f(z)=\begin{cases}\dfrac{1}{z-a}\,g(z) & \text{if } z\in\mathbb{C}\setminus W\\ 0 & \text{if } z\in W.\end{cases} \tag{2.3}$$

Then $f\in C_c^\infty(\mathbb{C})$ and f is holomorphic in the neighborhood U of supp $\overline{\partial}\alpha$. By Theorem 2.3,

$$\begin{aligned}
-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\alpha(\zeta)\,f(\zeta)d\sigma_2 &= -\frac{1}{\pi}\int_{\text{supp }\overline{\partial}\alpha} \overline{\partial}\alpha(\zeta)\,f(\zeta)\,d\sigma_2\\
&= -\frac{1}{\pi}\int_{\text{supp }\overline{\partial}\alpha} \overline{\partial}\alpha(\zeta)\,\frac{1}{\zeta-a}\,d\sigma_2\\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial}\alpha(\zeta)}{\zeta-a}\,d\sigma_2\\
&= \alpha(a)\neq 0.
\end{aligned}$$

In view of Proposition 2.4 and the fact that $\int_\gamma f(z)dz = 0$ whenever f is the derivative of a holomorphic function and γ a closed C^1-path in Ω, it is tempting to conjecture that the following is true:

Desideratum 2.5. Let $\alpha \in C_c^\infty(\mathbb{C})$ and $f \in C(\mathbb{C})$. Suppose that in a neighborhood U of the support of $\overline{\partial}\alpha$, f is the derivative of a holomorphic function g. Then $\int_{\mathbb{C}}(\overline{\partial}\alpha)f\, d\sigma_2 = 0$.

Unfortunately, the assertion above is not true. Here is an example. Note that functions built on $1/(z-a)$ (as in the formula (2.3)) cannot be taken, since they do not have primitives on annuli surrounding a.

Let V be an open set containing $\operatorname{supp}\overline{\partial}\alpha$ with smooth boundary and such that $\overline{\partial}g = 0$ and $\partial g = f$ on $U \supseteq \overline{V} \supseteq V$. Then

$$\begin{aligned}
\int_{\mathbb{C}} (\overline{\partial}\alpha)\, f\, d\sigma_2 &= \int_V (\overline{\partial}\alpha)\, \partial g\, d\sigma_2 \\
&= \int_V \overline{\partial}(\alpha\partial g)\, d\sigma_2 - \frac{1}{4}\int_V \alpha\Delta g\, d\sigma_2 \\
&= \frac{1}{2i}\int_{\partial V} \alpha\partial g\, dz - 0.
\end{aligned}$$

Now let $\alpha(z) = z$ for $|z| < 2$ and $\alpha = 0$ for $|z| \geq 3$. Then

$$\operatorname{supp}\overline{\partial}\alpha \subseteq \{2 \leq |z| \leq 3\}.$$

Choose $g \in C(\mathbb{C})$ such that $g(z) = 1/z$ for $|z| \geq 1$ and $g(z) = \overline{z}$ for $|z| \leq 1$. Then g is holomorphic on $|z| > 1$, a neighborhood of $\operatorname{supp}\overline{\partial}\alpha$. As V we may take

$$V = \{3/2 < |z| < 7/2\}.$$

Then V is a neighborhood of $\operatorname{supp}\overline{\partial}\alpha$. But

$$\int_{\partial V} \alpha\partial g\, dz = 0 - \int_{\{|z|=3/2\}} z\frac{(-1)}{z^2}\, dz = 2\pi i \neq 0.$$

We are now ready to solve the $\overline{\partial}$-equation in the plane.

Theorem 2.6. *For $k = 1, 2, \ldots, \infty$, let $g \in C^k(\mathbb{C})$ and suppose that g has compact support. Then the Cauchy transform $G := C[g]$ of g, given by*

$$G(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} d\sigma_2(\zeta),$$

has the following properties:

1. *$G \in C^k(\mathbb{C})$ and $\lim_{z\to\infty} G(z) = 0$;*

2. *$\overline{\partial}G = g$.*

Proof. Let $\zeta = z + re^{i\theta}$. Then

$$G(z) = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} g(z+re^{i\theta})e^{-i\theta}dr\,d\theta.$$

This is a proper Riemann integral because g has compact support. Since $g \in C_c^1(\mathbb{C})$, the integral

$$J := -\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \overline{\partial}g(z+re^{i\theta})e^{-i\theta}dr\,d\theta$$

exists and, by interchanging the differential operators $\frac{d}{dx}$ and $\frac{d}{dy}$ with the integral operator, we see that $G \in C^k(\mathbb{C})$ and $\overline{\partial}G = J$. Since

$$J = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial}g(\zeta)}{\zeta - z}d\sigma_2(\zeta),$$

we may use Theorem 2.3 to conclude that $J = g$. Hence $\overline{\partial}G = g$. That $\lim_{z\to\infty} G(z) = 0$, follows immediately from the definition of G. □

Whereas the preceding results only used the Riemann integral, we must now deal with Lebesgue integration of continuous functions on compacta. We note that we can no longer use Riemann integrals (as above), since the compact sets K we are considering are not Jordan measurable; their boundaries will have positive Lebesgue measure in general.

Lemma 2.7. *Let $X \subseteq \mathbb{C}$ be a Borel set of finite planar measure. Then for all $z \in \mathbb{C}$*

$$\int_X \frac{1}{|\zeta - z|}d\sigma_2(\zeta) \le 2\sqrt{\pi\sigma_2(X)}.$$

In particular, $1/\zeta$ is in $L_1(K, \sigma_2)$ for each compact set $K \subseteq \mathbb{C}$.

Proof. Let $R = \sqrt{\sigma_2(X)/\pi}$, so $\sigma_2(X) = \pi R^2$. For fixed $z \in \mathbb{C}$, let D be the closed disk

$$D(z, R) = \{\zeta \in \mathbb{C} : |z - \zeta| \le R\}.$$

Because $\sigma_2(X) = \sigma_2(D)$, we conclude from

$$X \cup D = (X \setminus D) \cup D = (D \setminus X) \cup X$$

that $\sigma_2(X \setminus D) = \sigma_2(D \setminus X)$. Since $|\zeta - z| \le R$ for $\zeta \in D \setminus X$ and $|\zeta - z| \ge R$ for $\zeta \in X \setminus D$, it follows that

$$\int_{X\setminus D} \frac{1}{|\zeta - z|}\,d\sigma_2(\zeta) \le \int_{D\setminus X} \frac{1}{|\zeta - z|}\,d\sigma_2(\zeta).$$

Now $X = (X \cap D) \cup (X \setminus D)$. Hence

$$\int_X \frac{1}{|\zeta - z|}d\sigma_2(\zeta) = \int_{X\setminus D} \frac{1}{|\zeta - z|}d\sigma_2(\zeta) + \int_{X\cap D} \frac{1}{|\zeta - z|}d\sigma_2(\zeta)$$

$$\begin{aligned}
&\leq \int_{D\setminus X} \frac{1}{|\zeta - z|} d\sigma_2(\zeta) + \int_{D\cap X} \frac{1}{|\zeta - z|} d\sigma_2(\zeta) \\
&= \int_D \frac{1}{|\zeta - z|} d\sigma_2(\zeta) = \int_0^{2\pi} \int_0^R \frac{1}{r} r \, dr d\theta \\
&= 2\pi R = 2\sqrt{\pi\sigma_2(X)}.
\end{aligned}$$

□

As a special case we obtain that for all $z \in \mathbb{C}$

$$\boxed{I := \frac{1}{\pi} \int_{\{|\zeta|\leq R\}} \frac{1}{|\zeta - z|} \, d\sigma_2(\zeta) \leq 2R.} \tag{2.4}$$

A computation of the exact value of I is rather difficult (in form of infinite series) and involves Legendre polynomials. On the other hand it is very easy to determine the explicit value of

$$-\frac{1}{\pi} \int_{\{|\zeta|\leq R\}} \frac{1}{\zeta - z} \, d\sigma_2(\zeta)$$

(see below).

Theorem 2.8. *Let $K \subseteq \mathbb{C}$ be compact, $f \in C(K, \mathbb{C}) \cap C^k(K^\circ)$, $(k = 1, 2, \ldots, \infty)$, and for $z \in \mathbb{C}$ let*

$$v(z) = -\frac{1}{\pi} \int_K \frac{f(\zeta)}{\zeta - z} d\sigma_2(\zeta).$$

Then the following assertions hold:

(i) $v \in C(\widehat{\mathbb{C}}), v$ holomorphic in $\mathbb{C} \setminus K$, and $\lim_{z\to\infty} v(z) = 0$;

(ii) $v \in C^k(K^\circ)$;

(iii) $\overline{\partial} v = f$ in K°.

Proof. (i)is easy. To show (ii) and (iii), let $z_0 \in K^\circ$. Choose a closed disk $D(z_0, r)$ of radius r centered at z_0 such that $D(z_0, 2r) \subseteq K^\circ$. Let $\alpha \in C_c^\infty(\mathbb{C})$ satisfy $\alpha \equiv 1$ on $D(z_0, r)$ and $\alpha \equiv 0$ outside $D(z_0, 2r)$. Then for every $z \in D(z_0, r/2)$ we have

$$\begin{aligned}
-\pi \, v(z) &= \int_{D(z_0,r)} \frac{f(\zeta)}{\zeta - z} d\sigma_2(\zeta) + \int_{K\setminus D(z_0,r)} \frac{f(\zeta)}{\zeta - z} d\sigma_2(\zeta) \\
&= \int_{D(z_0,r)} \frac{(\alpha f)(\zeta)}{\zeta - z} d\sigma_2(\zeta) + \int_{K\setminus D(z_0,r)} \frac{f(\zeta)}{\zeta - z} d\sigma_2(\zeta) \\
&= \int_{\mathbb{C}} \frac{(\alpha f)(\zeta)}{\zeta - z} d\sigma_2(\zeta) - \int_{D(z_0,2r)\setminus D(z_0,r)} \frac{(\alpha f)(\zeta)}{\zeta - z} d\sigma_2(\zeta) \\
&\quad + \int_{K\setminus D(z_0,r)} \frac{f(\zeta)}{\zeta - z} d\sigma_2(\zeta)
\end{aligned}$$

Using Theorem 2.6 for the first integral and interchanging the differentiation operators ∂ and $\overline{\partial}$ with the integral operator in the remaining two integrals, allows us to conclude that $v \in C^k(K^\circ)$. Moreover, by Theorem 2.6

$$\overline{\partial}v(z_0) = \alpha f(z_0) + 0 + 0 = f(z_0). \qquad \square$$

Theorem 2.9. *Let Ω be a domain in $\mathbb{C}$ and let $k \in \{1, 2, \ldots, \infty\}$. If $v \in C^k(\Omega)$, then there exists $u \in C^k(\Omega)$ such that $\overline{\partial}u = v$.*

Proof. Let (K_n) be an exhaustion sequence of Ω with Ω-convex sets. For $j = 1, 2, \ldots,$ choose $\alpha_j \in C_c^\infty(\mathbb{C})$ so that $\operatorname{supp}\alpha_j \subseteq K_{j+1}^\circ$ and $\alpha_j \equiv 1$ on K_j and let $\alpha_0 \equiv 0$. Let

$$v_j(z) = \begin{cases} (\alpha_j(z) - \alpha_{j-1}(z))\, v(z) & \text{if } z \in K_{j+1} \\ 0 & \text{if } z \in \mathbb{C} \setminus K_{j+1}. \end{cases}$$

Then $v_j \in C^k(\mathbb{C})$ and $v_j \equiv 0$ on K_{j-1}, $j \geq 2$, and $\sum_{j=1}^N v_j = \alpha_N v$ on Ω. Hence, if $N \to \infty$, we see that on Ω, $\sum_{j=1}^\infty v_j = v$, the convergence being pointwise. Moreover, on K_N, we have

$$\sum_{j=1}^N v_j = v.$$

Since $v_j \in C_c^k(\mathbb{C})$, there is $u_j \in C^k(\mathbb{C})$ such that $\overline{\partial}u_j = v_j$ (see Theorem 2.6). In particular,

$$u_j \text{ is holomorphic on } K_{j-1}^\circ \text{ for } j \geq 2. \tag{2.5}$$

By Theorem ??, applied to the Ω-convex set K_{j-2}, there is $f_j \in H(\Omega)$ such that for $j = 3, 4, \ldots$

$$\sup_{K_{j-2}} |u_j - f_j| < 2^{-j}. \tag{2.6}$$

Now let

$$u = \sum_{n=1}^\infty (u_n - f_n).$$

We claim that u is the solution we are looking for.
In fact, using (2.6), we see that the series defining u converges locally uniformly on Ω. Moreover, by (2.5) and Weierstrass' convergence theorem, the remainder term

$$\sum_{n=N+1}^\infty (u_n - f_n)$$

is holomorphic on K_N°. Thus $u \in C^k(K_N^\circ)$ and, on K_N°,

$$\begin{aligned}
\overline{\partial} u &= \overline{\partial} \sum_{n=1}^{N} (u_n - f_n) + \overline{\partial} \sum_{n=N+1}^{\infty} (u_n - f_n) \\
&= \sum_{n=1}^{N} \overline{\partial} u_n + 0 \\
&= \sum_{n=1}^{N} v_n \\
&= v.
\end{aligned}$$

Since $\bigcup_{j=1}^{\infty} K_j^\circ = \Omega$, we obtain the assertion that $\overline{\partial} u = v$ on Ω and that $u \in C^k(\Omega)$. $\square$

2.3 Explicit values of some integrals

In this supplementary section we derive the explicit values of some Cauchy-type integrals. To deal with integrals of the form $\int_{\mathbb{D}} h(w,z) d\sigma_2(w)$, we use polar-coordinates (r, θ), $0 < r < 1$, $0 \le \theta < 2\pi$, and replace integration with respect to θ by complex integration on the unit circle; that is we put $d\theta = \dfrac{d\xi}{i\xi}$ where $|\xi| = 1$. As usual, the orientation of the circle in $\int_{|\xi|=1}$ is always to be taken counterclockwise (that is the positive orientation).

Proposition 2.10. *Let $f \in H^\infty$. Then*

$$\boxed{\frac{1}{\pi} \int_{\mathbb{D}} f(w)\, d\sigma_2(w) = f(0)} \tag{2.7}$$

and

$$\boxed{-\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(w)}{w - z}\, d\sigma_2(w) = \begin{cases} -f'(0) & \text{if } z = 0 \\ (|z|^2 - 1)\frac{f(z)-f(0)}{z} + f(0)\overline{z} & \text{if } 0 < |z| < 1 \\ \frac{f(0)}{z} & \text{if } |z| \ge 1. \end{cases}} \tag{2.8}$$

Remark 2.11. By the results in section 2.2 (in particular Theorem 2.6), the value $u(z)$ of the integral 2.8 is a function that is continuous on $\mathbb{C}$, holomorphic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and that satisfies the inhomogeneous Cauchy-Riemannn equation $\overline{\partial} u = f$ in $\mathbb{D}$. In particular, $u = \overline{z} f + h$, where $h \in H(\mathbb{D})$. It turns out that $h(z) = -\frac{f(z)-f(0)}{z}$ if $z \ne 0$ and $h(0) = -f'(0)$.

Proof. Let $J = \frac{1}{\pi} \int_{\mathbb{D}} f(w)\ d\sigma_2(w)$. Since f is bounded, the integral exists and we may use Fubini's theorem. Hence

$$J = 2 \int_{r=0}^{1} \Big[\frac{1}{2\pi i} \int_{|\xi|=1} f(r\xi) \frac{d\xi}{\xi} \Big]\, r dr = f(0) \int_{r=0}^{1} (2r) dr = f(0).$$

(This also follows from the fact that harmonic functions (here f) enjoy the planar mean-value property on disks. Moreover, instead of using Cauchy's integral formula, one could have used the power series representation $\sum_{n=0}^{\infty} a_n r^n e^{in\theta}$ of f, to derive the result.)

Let $u(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{f(w)}{w-z}\, d\sigma_2(w)$. By Theorem **??**, u converges absolutely and so we may apply Fubini's theorem. Since $\sigma_2(\{w \in \mathbb{C} : |w| = r\}) = 0$, we have $\int_{\mathbb{D}} = \int_{\mathbb{D}\setminus\{w\in\mathbb{C}:|w|=r\}}$. Henceforth, for fixed z, we may integrate only on curves $|\zeta| = r$ where $r \neq |z|$. [4] Thus

$$u(z) = -\frac{1}{2\pi i}\int\limits_{\substack{r=0\\ r\neq|z|}}^{1} \Big[\int\limits_{|\xi|=1} 2\,\frac{f(r\xi)}{r\xi - z}\,\frac{d\xi}{\xi}\Big]\, r dr = -\int\limits_{\substack{r=0\\ r\neq|z|}}^{1} \Big[\frac{1}{2\pi i}\int\limits_{|\xi|=1} 2\,\frac{f(r\xi)}{\xi - \frac{z}{r}}\,\frac{d\xi}{\xi}\Big]\, dr.$$

Let $0 < |z| < 1$. Then

$$u(z) = -\int\limits_{0<r<|z|} \Big[\frac{1}{2\pi i}\int\limits_{|\xi|=1} 2\,\frac{f(r\xi)}{(\xi - \frac{z}{r})\,\xi}\, d\xi\Big]\, dr - \int\limits_{|z|<r<1} \Big[\frac{1}{2\pi i}\int\limits_{|\xi|=1} 2\,\frac{f(r\xi)}{(\xi - \frac{z}{r})\,\xi}\, d\xi\Big]\, dr$$

If $|z/r| < 1$, then two singularities are surrounded: 0 and z/r; otherwise only 0 is sourrounded. By the residue theorem,

$$\begin{aligned} u(z) &= -2\int\limits_{0}^{|z|} \frac{f(0)}{(-\frac{z}{r})}\, dr - 2\int\limits_{|z|}^{1} \left(\frac{f(r\frac{z}{r})}{\frac{z}{r}} + \frac{f(0)}{(-\frac{z}{r})}\right)\, dr \\ &= \frac{f(0)}{z}|z|^2 - \frac{f(z)}{z}(1-|z|^2) + \frac{f(0)}{z}(1-|z|^2) \\ &= (|z|^2-1)\frac{f(z)-f(0)}{z} + f(0)\overline{z}. \end{aligned}$$

If $z = 0$, then

$$\begin{aligned} u(z) &= -2\int\limits_{r=0}^{1} \Big[\frac{1}{2\pi i}\int\limits_{|\xi|=1} \frac{f(r\xi)}{\xi^2} d\xi\Big]\, dr \\ &= -2\int\limits_{0}^{1} r f'(0)\, dr = -f'(0). \end{aligned}$$

Now let $|z| \geq 1$. Then, for $0 < r < 1$, $|z/r| > 1$; hence only the singularity 0 is surrounded. Thus

$$u(z) = -2\int\limits_{0}^{1} \frac{f(0)}{-\frac{z}{r}}\, dr = \frac{f(0)}{z}.$$

$\square$

[4] This is necessary in order to avoid singularities on the integration curve.

An analysis of the proof shows that the iterated integral

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\frac{f(re^{i\theta})}{re^{i\theta}-z}\,d\theta\,r dr$$

exists for every function $f \in H(\mathbb{D})$, f not necessary bounded, and that its value is deduced from (2.8).

As a particular case we obtain:

Example 2.12.

$$\boxed{-\frac{1}{\pi}\int_{\mathbb{D}}\frac{1}{w-z}\,d\sigma_2(w)=\begin{cases}\overline{z} & \text{if } |z|<1\\ \frac{1}{z} & \text{if } |z|\geq 1\end{cases}} \tag{2.9}$$

Note that the case $|z| \geq 1$ also follows directly from the case $|z| < 1$ without calculus; in fact, we know that $I(z)$ is continuous in $\mathbb{C}$ and holomorphic for $|z| > 1$. Now $1/z$ is the only holomorphic function outside the closed unit disk that has the boundary values $\overline{z}$.

2.4 The Cauchy, Gauss, Green, Stokes and Pompeiu formulas

In this section we give a very simple proof of Gauss' divergence theorem. We shall base our proof on the $\overline{\partial}$-calculus. For a better visualization, we exceptionally use here the double integral sign to denote planar integrals.

Lemma 2.13. *Let $\phi : [0, 1] \to \mathbb{C}$ be a C^1-path. Then its image, $\Gamma := \phi([0, 1])$, has two-dimensional Lebesgue measure zero.*

Proof. Since ϕ' is bounded on $[0, 1]$, we see that ϕ satisfies a Lipschitz condition on $[0, 1]$; that is $|\phi(s) - \phi(t)| \leq M|s - t|$ for every $s, t \in [0, 1]$. Fix $N \in \mathbb{N}$ and consider the decomposition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ of $[0, 1]$ into N intervals $I_j = [t_{j-1}, t_j]$ of length $1/N$. Then

$$\Gamma \subseteq \bigcup_{j=1}^{N}\phi(I_j).$$

Now for $z_j \in \phi(I_j)$ and $d_j := 2\,\mathrm{diam}\ \phi(I_j)$, we see that $\phi(I_j) \subseteq D(z_j, d_j)$. Hence

$$\sigma_2(\phi(I_j)) \leq \pi {d_j}^2.$$

But

$$d_j = 2\sup_{s,t\in I_j}|\phi(s)-\phi(t)| \leq 2M|I_j| \leq \frac{2M}{N}.$$

So we conclude that

$$\sigma_2(\Gamma) \leq \sum_{j=1}^{N}\sigma_2(\phi(I_j)) \leq N\pi\frac{4M^2}{N^2} = \frac{4M^2}{N} \to 0.$$

Thus $\sigma_2(\Gamma) = 0$. □

Theorem 2.14. *Let $\Omega \subseteq \mathbb{C}$ be an open set and suppose that Γ is a null-homologous cycle in Ω. Then the following formula holds for every $f \in C^1(\Omega)$:* [5]

$$\int_\Gamma f(z)\, dz = 2i \iint_\Omega \overline{\partial} f(\zeta)\, n(\Gamma,\zeta) d\sigma_2(\zeta).$$

Proof. Let $S = \Gamma \cup \{z \in \Omega \setminus \Gamma : n(\Gamma, z) \neq 0\}$. Then $S \subseteq \Omega$ and since $n(\Gamma, z)$ is locally constant on $\mathbb{C} \setminus \Gamma$, we see that S is compact. Choose $\psi \in C_c^\infty(\mathbb{C})$ such that $\psi \equiv 1$ in a neighborhood U of S with $S \subseteq U \subseteq \Omega$, and $\operatorname{supp} \psi \subseteq \Omega$. Let $g : \mathbb{C} \to \mathbb{C}$ be defined by

$$g(z) = \begin{cases} \psi(z) f(z) & \text{if } z \in \Omega \\ 0 & \text{if } z \notin \Omega. \end{cases}$$

Then $g \in C_c^1(\mathbb{C})$ and $\operatorname{supp} g \subseteq \Omega$. By Theorem 2.3

$$g(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\overline{\partial} g(\zeta)}{\zeta - z}\, d\sigma_2(\zeta).$$

Note that $g = f$ on Γ. Hence

$$I := \int_\Gamma f(z)\, dz = \int_\Gamma g(z)\, dz = -\frac{1}{\pi} \int_\Gamma \left(\iint_{\mathbb{C}} \frac{\overline{\partial} g(\zeta)}{\zeta - z}\, d\sigma_2(\zeta) \right) dz.$$

Since Γ has planar measure zero (Lemma 2.13), we conclude that

$$I = -\frac{1}{\pi} \int_\Gamma \left(\iint_{\mathbb{C}\setminus\Gamma} \frac{\overline{\partial} g(\zeta)}{\zeta - z}\, d\sigma_2(\zeta) \right) dz.$$

By Lemma 2.7 we see that

$$\frac{1}{\pi} \int_\Gamma \left(\iint_{\mathbb{C}} \left| \frac{\overline{\partial} g(\zeta)}{\zeta - z} \right| d\sigma_2(\zeta) \right) |dz| \leq 2\, L(\Gamma)\, ||\overline{\partial} g||_\infty \sqrt{\sigma_2(\operatorname{supp} g)/\pi}.$$

Hence we may use Fubini's Theorem to conclude that

$$I = 2i \iint_{\mathbb{C}\setminus\Gamma} \overline{\partial} g(\zeta) \left(\frac{1}{2\pi i} \int_\Gamma \frac{1}{z - \zeta}\, dz \right) d\sigma_2(\zeta) = 2i \iint_{\mathbb{C}\setminus\Gamma} \overline{\partial} g(\zeta)\, n(\Gamma,\zeta)\, d\sigma_2(\zeta).$$

By construction, $n(\Gamma, \zeta) = 0$ for $\zeta \notin S$. But on $U \supseteq S$, $f = g$. In particular, since U is open, $\overline{\partial} f = \overline{\partial} g$ on U. Hence, since $S \subseteq U \subseteq \Omega$,

$$I = 2i \iint_{S\setminus\Gamma} \overline{\partial} g(\zeta)\, n(\Gamma,\zeta)\, d\sigma_2(\zeta) = 2i \iint_{S\setminus\Gamma} \overline{\partial} f(\zeta)\, n(\Gamma,\zeta)\, d\sigma_2(\zeta).$$

Defining $n(\Gamma, z) = 0$ whenever $z \in \Gamma$, and using again that $\sigma_2(\Gamma) = 0$ we obtain

$$I = 2i \iint_\Omega \overline{\partial} f(\zeta)\, n(\Gamma,\zeta)\, d\sigma_2(\zeta).$$

□

[5] Here we use that $\sigma_2(\Gamma) = 0$; so that actually $\iint_\Omega = \iint_{\Omega\setminus\Gamma}$.

If $\Omega \subseteq \mathbb{C}$ is open, then we denote the set of all complex-valued functions f that are continuously $\mathbb{R}$-differentiable on Ω and for which f and the partial derivatives f_x and f_y are continuously extendable to $\overline{\Omega}$, by $C^1(\overline{\Omega})$.

Definition 2.15. i) Let $\Omega \subseteq \mathbb{C}$ be a bounded domain (=connected open set). We call Ω *admissible* if the boundary of Ω consists of finitely many closed, positively orientated, pairwise disjoint, piecewise-C^1 Jordan curves γ_j, $(j = 0, 1, \dots, n)$.

Recall that the boundary curve γ_j is positively orientated, if the domain lies to the left of the curve. Moreover, γ_0 denotes the outer boundary of the domain. It follows from the Jordan curve Theorem 1.5, that the cycle $\Gamma = \bigcup_{j=0}^{n} \gamma_j$ satisfies [6]

$$n(\Gamma, z) = 0 \text{ for every } z \in \mathbb{C} \setminus \Omega \text{ and } n(\Gamma, z) = 1 \text{ for every } z \in \Omega. \tag{2.10}$$

We note that instead of the piecewise smoothness, we could have stipulated the rectifiability of the curves.

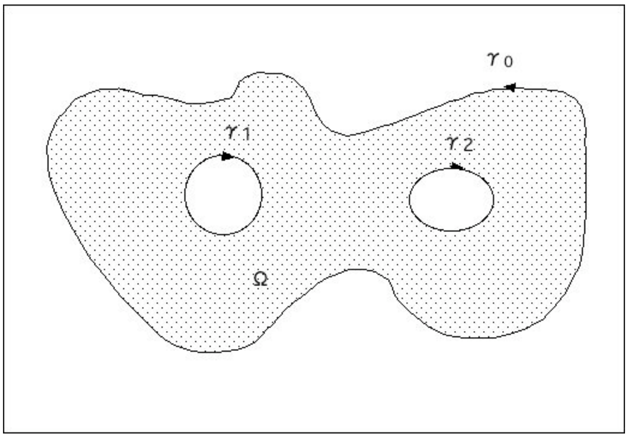


Figure 2.1: An admissible domain

Theorem 2.16 (Gauss' Theorem, complex version)**.** *Let Ω be an admissible domain. Suppose that f is continuously $\mathbb{R}$-differentiable in a neighborhood U of $\overline{\Omega}$. Then*

$$\int_{\partial\Omega} f(z)\, dz = 2i \iint_{\Omega} \overline{\partial} f(\zeta)\, d\sigma_2(\zeta).$$

Proof. This follows immediately from Theorem 2.14 by noticing that the cycle $\Gamma = \sum_{j=0}^{n} \gamma_j$ is null-homologuous in U and that $n(\Gamma, \zeta) = 1$ for every $\zeta \in \Omega$. $\square$

Corollary 2.17. *Let Ω be an admissible domain. Then the area of Ω (respectively $\overline{\Omega}$) is given by*

$$\sigma_2(\Omega) = \frac{1}{2i} \int_{\partial\Omega} \overline{z}\, dz.$$

[6] Readers who do not want to use, at this point, the Jordan curve Theorem 1.5, must additionally assume the conditions (2.10).

We are now able to deduce the classical Green-Riemann-Stokes formula and Gauss' divergence theorem:

Proposition 2.18 (Green-Riemann-Stokes)**.** *Let Ω be an admisible domain. Then, for every pair of real-valued functions $P, Q \in C^1(\overline{\Omega})$,*

$$\int_{\partial\Omega} Pdx + Qdy = \iint_{\Omega} (Q_x - P_y)dxdy.$$

Proof. For $z = x + iy$, identified with (x, y), let $f(z) = P(z) - iQ(z)$. Then

$$f\ dz = (P - iQ)(dx + idy) = Pdx + Qdy + i(Pdy - Qdx).$$

Moreover,

$$\overline{\partial} f = (1/2)(f_x + if_y) = (1/2)(P_x - iQ_x + i(P_y - iQ_y)) = (1/2)(P_x + Q_y + i(P_y - Q_x)).$$

Hence

$$\mathrm{Re} \int_{\partial\Omega} f(z)dz = \int_{\partial\Omega} Pdx + Qdy$$

and

$$\mathrm{Re} \left[2i \iint_{\Omega} \overline{\partial} f(\zeta) d\sigma_2(\zeta) \right] = \iint_{\Omega} (Q_x - P_y)dxdy.$$

□

Proposition 2.19 (Gauss)**.** *Let Ω be an admissible domain. Then, for every vector-valued function $f = (u, v) \in C^1(\overline{\Omega}, \mathbb{R}^2)$ and $\operatorname{div} f = u_x + v_y$,*

$$\iint_{\Omega} \operatorname{div} f\ dxdy = \int_{\partial\Omega} f \cdot \boldsymbol{n}\ ds,$$

where $\boldsymbol{n}$ is the outer normal to $\partial\Omega$ and ds is integration with respect to arc-length.

Proof. If $(x(t), y(t))$ is a parametrization of one of the boundary curves, then $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$ and the non-orientated normal $\boldsymbol{n}$ is given by

$$\boldsymbol{n} = \frac{(\dot{y}, -\dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

where the derivative with respect to t is denoted by the dot. By Proposition 2.18,

$$\iint_{\Omega} (u_x + v_y)\ dxdy = \int_{\partial\Omega} (-vdx + udy) = \int_{\partial\Omega} (u, v) \cdot \boldsymbol{n}\ ds.$$

□

Corollary 2.20 (Green)**.** *Let Ω be an admissible domain. Then, for any $h \in C^2(\overline{\Omega}, \mathbb{R})$ and $\Delta = h_{xx} + h_{yy}$,*

$$\iint_{\Omega} \Delta h\ dxdy = \int_{\partial\Omega} \frac{\partial h}{\partial \boldsymbol{n}}\ ds,$$

where $\dfrac{\partial h}{\partial \boldsymbol{n}} := \nabla h \cdot \boldsymbol{n}$ is the derivative along the outer normal $\boldsymbol{n}$.

Proof. Use Proposition 2.19 and the fact that $\Delta h = \operatorname{div}(\nabla h)$. □

Here are several corollaries to Gauss' Theorem 2.16. The first one coincides with Proposition 2.4. [7]

Corollary 2.21. *Let $\alpha \in C_c^\infty(\mathbb{C})$ and $f \in \mathbb{C}(\mathbb{C})$. Suppose that f is holomorphic in an neighborhood of the support of α. Then*

$$\int_{\mathbb{C}} (\overline{\partial}\alpha) f \, d\sigma_2 = 0.$$

In particular, $\iint_{\mathbb{C}} \overline{\partial}\alpha \, d\sigma_2 = \iint_{\mathbb{C}} \partial\alpha \, d\sigma_2 = 0$.

Proof. Let D be an open disk such that $\operatorname{supp}\alpha \subseteq D$. Then

$$\begin{aligned} I := \iint_{\mathbb{C}} (\overline{\partial}\alpha)\ f \, d\sigma_2 &= \iint_{\mathbb{C}} \overline{\partial}(\alpha f) \, d\sigma_2 - \iint_{\mathbb{C}} \alpha\, \overline{\partial} f \, d\sigma_2 \\ &= \iint_{D} \overline{\partial}(\alpha f) \, d\sigma_2 - \iint_{\operatorname{supp}\alpha} \alpha\, \overline{\partial} f \, d\sigma_2. \end{aligned}$$

Since $\overline{\partial} f = 0$ on $\operatorname{supp}\overline{\partial}\alpha$ and $\alpha = 0$ on ∂D we conclude from Theorem 2.16 that

$$I = \frac{1}{2i} \int_{\partial D} (\alpha f)(z) dz - 0 = 0.$$

The special case immediately follows when chosing $f \equiv 1$ and by noticing that $\partial\alpha = \overline{\overline{\partial}\overline{\alpha}}$ (Proposition 2.1). □

Corollary 2.22. *Let $f \in C^1(D(z_0, R))$. If the boundary of the disk $D_r := D(z_0, r)$ is surrounded counter-clockwise, then* [8]

$$\overline{\partial} f(z_0) = \lim_{r \to 0} \frac{1}{2\pi i\, r^2} \int_{\partial D_r} f(z) \, dz. \tag{2.11}$$

Proof. Since the disk is an admissible domain, we obtain from Theorem 2.16 that

$$\int_{\partial D_r} f(z) \, dz = 2i \iint_{D_r} \overline{\partial} f(\zeta) \, d\sigma_2(\zeta).$$

By the mean-value theorem for the planar integral of a continuous function u,

$$\frac{1}{\sigma_2(D_r)} \iint_{D_r} u(\zeta) \, d\sigma_2(\zeta) \to u(z_0) \tag{2.12}$$

as $r \to 0$. Applying this to $u = \overline{\partial} f$ yields the desired equality. □

[7] This second proof runs under the heading "integration by parts"

[8] This limit is called the "aerolar derivative" or "areal derivative" (by Pompeiu).

Remark 2.23. • A different proof can be given by using directly the representation

$$f(z) = f(z_0) + \partial f(z_0)(z - z_0) + \overline{\partial} f(z_0)(\overline{z} - \overline{z}_0) + o(z - z_0)$$

and integrating along the circles ∂D_r. To handle the little o-term, just note that if $o(z - z_0) = r(z)$, then

$$\int_{\partial D_r} r(z) dz = \int_{\partial D_r} \frac{r(z)}{z - z_0}(z - z_0)\, dz = o(1) \int_{\partial D_r} r|dz| = o(1) r^2.$$

• The limit (2.11) may exist without f being a C^1-function. For example if $f(z) = |z|$, then

$$\lim_{r \to 0} \frac{1}{2\pi i\, r^2} \int_{\partial D_r} f(\zeta)\, d\zeta = 0.$$

If we raise the denominator of the integrand $\overline{\partial} f(\zeta)/(\zeta - a)$ to certain integer powers, then the integral $\int_{\mathbb{C}} \overline{\partial} f(\zeta)/(\zeta - a)^n \, d\sigma_2(\zeta)$ is, in general, divergent for $a \in \operatorname{supp} \overline{\partial} f$. On the other hand, Theorem 2.16 will allow us to determine the explicit value whenever $a \notin \operatorname{supp} \overline{\partial} f$.

Corollary 2.24. *Let $f \in C^1(\mathbb{C})$ have compact support and let $n \in \mathbb{N}$. Then, for every $a \notin \operatorname{supp} \overline{\partial} f$* [9]

$$I := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{\partial} f(\zeta)}{(\zeta - a)^{n+1}}\, d\sigma_2(\zeta) = \frac{f^{(n)}(a)}{n!}.$$

Proof. Let $D := D(a, \varepsilon)$ be a small disk centered at a, so that f is holomorphic in D and let $R > 0$ be chosen so that $\overline{D} \cup \operatorname{supp} f \subseteq D(0, R)$. If $\Omega = D(0, R) \setminus D$, then, by Theorem 2.16,

$$\begin{aligned} I &= -\frac{1}{\pi} \int_{\Omega} \frac{\overline{\partial} f(\zeta)}{(\zeta - a)^{n+1}}\, d\sigma_2(\zeta) = -\frac{1}{\pi} \int_{\Omega} \overline{\partial} \left(\frac{f(\zeta)}{(\zeta - a)^{n+1}} \right) d\sigma_2(\zeta) \\ &= -\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - a)^{n+1}}\, dz = \frac{1}{2\pi i} \int_{|z - a| = \varepsilon} \frac{f(z)}{(z - a)^{n+1}}\, dz \\ &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

□

The methods above also allow us to give an analogue to Morera's theorem for disks.

Theorem 2.25 (Carleman)**.** *Let $f \in C(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is an open set. Suppose that for every disk D with $\overline{D} \subseteq \Omega$*

$$\int_{\partial D} f(z)\, dz = 0.$$

Then f is holomorphic in Ω.

[9] Note that f is holomorphic in a neighborhood of a.

Proof. We first prove the assertion under the condition that $f \in C^1(\Omega)$. By Corollary 2.22,

$$\overline{\partial} f(z_0) = \lim_{r\to 0} \frac{1}{2\pi i\, r^2} \int_{\partial D_r} f(z)\, dz = 0$$

for every $z_0 \in \Omega$. Hence f is holomorphic in Ω.

If f is merely continuous, we shall use an approximation argument. Fix $z_0 \in \Omega$ and consider a disk $D = D(z_0, 2r)$ with $\overline{D} \subseteq \Omega$. By Theorem **??**, let $\phi \in C_c^\infty(\mathbb{C})$ satisfy $0 \le \phi \le 1$, $\phi = 1$ on $D(z_0, r)$ and $\operatorname{supp}\phi \subseteq D$. Then the function F given by

$$F(z) = \begin{cases} \phi(z) f(z) & \text{if } z \in D \\ 0 & \text{if } z \notin D \end{cases}$$

is continuous in $\mathbb{C}$. Let $\psi \in C_c^\infty(\mathbb{C})$ be chosen so that $0 \le \psi \le 1$, $\operatorname{supp}\psi \subseteq \mathbb{D}$ and such that $\int_{\mathbb{C}} \psi(\zeta)\, d\sigma_2 = 1$ (for example we could take $\psi = \phi / \int_{\mathbb{C}} \phi(\zeta) d\sigma_2$ with $z_0 = 0$ and $r = 1/2$). For $\varepsilon > 0$, let $\psi_\varepsilon = \varepsilon^{-2}\psi(z/\varepsilon)$ be the associated C^∞-approximate identity. Consider the convolution

$$F_\varepsilon(z) = F * \psi_\varepsilon(z) := \iint_{\mathbb{C}} F(\zeta - z)\, \psi_\varepsilon(\zeta)\, d\sigma_2(\zeta).$$

Then, by Theorem **??**, $F_\varepsilon \in C_c^\infty(\mathbb{C})$ and by Theorem **??**,

$$\lim_{\varepsilon\to 0} \sup_{z\in\mathbb{C}} |F_\varepsilon(z) - F(z)| = 0.$$

We claim that F_ε is holomorphic in $D(z_0, r/2)$ whenever $\varepsilon < r/2$. Let $z_1 \in D(z_0, r/2)$. By Fubini's Theorem,

$$\begin{aligned} I := \int\limits_{|z-z_1|=1/n} F_\varepsilon(z) dz &= \int\limits_{|z-z_1|=1/n} \left(\iint\limits_{|\zeta|\le\varepsilon} f(z-\zeta)\psi_\varepsilon(\zeta)\, d\sigma_2(\zeta) \right) dz \\ &= \iint\limits_{|\zeta|\le\varepsilon} \Big(\int\limits_{|z-z_1|=1/n} f(z-\zeta)\, dz \Big) \psi_\varepsilon(\zeta)\, d\sigma_2(\zeta). \end{aligned}$$

Next we do a change of variables. Let $\xi = z - \zeta$. Then, when z moves on the circle $\partial D(z_1, 1/n)$, ξ moves on the circle $S := \partial D(z_1 - \zeta, 1/n)$. Hence

$$I = \iint\limits_{|\zeta|\le\varepsilon} \Big(\int\limits_{S} f(\xi) d\xi \Big) \psi_\varepsilon(\zeta)\, d\sigma_2(\zeta).$$

In order to apply the assumption of the Theorem, we need to show that for every ζ with $|\zeta| \le \varepsilon$, $\overline{D(z_1 - \zeta, 1/n)} \subseteq D(z_0, r)$ whenever n is large. To see this, let $w \in \overline{D(z_1 - \zeta, 1/n)}$. Then

$$|w - z_0| \le |w - (z_1 - \zeta)| + |\zeta| + |z_1 - z_0| < 1/n + \varepsilon + r/2 < r.$$

Hence the inner integral $\int_S f(\xi)d\xi = 0$. It follows that

$$I = \int\limits_{|z-z_1|=1/n} F_\varepsilon(z)\,dz = 0$$

for large n. Since F_ε is smooth, we conclude from Corollary 2.22 that $\overline{\partial}F_\varepsilon(z_1) = 0$. Thus $\overline{\partial}F_\varepsilon = 0$ on $D(z_0, r/2)$.
Since F_ε converges uniformly to F on $D(z_0, r)$ and since $f = F$ on $D(z_0, r)$, we conclude from Weierstrass' Theorem that f is holomorphic in $D(z_0, r/2)$. Since z_0 was arbitrarily chosen, $f \in H(\Omega)$. □

We conclude this section with the Cauchy-Pompeiu formula.

Theorem 2.26. *Let Ω be an admissible domain. Then the following formula holds for every $F \in C^1(\overline{\Omega})$ and $z \in \Omega$:*

$$F(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{F(\xi)}{\xi - z}\,d\xi - \frac{1}{\pi}\iint_\Omega \frac{\overline{\partial}F(\zeta)}{\zeta - z}\,d\sigma_2(\zeta),$$

Proof. First we note that both integrals exist Fix $z \in \Omega$ and consider a small disk $D(z,\varepsilon)$ centered at z such that $\overline{D}(z,\varepsilon) \subseteq \Omega$. The new domain $\Omega_\varepsilon := \Omega \setminus \overline{D}(z,\varepsilon)$ is admissible again. We orientate the circle $|\xi - z| = \varepsilon$ negatively. Hence, by Theorem 2.16,

$$\begin{aligned} I_2(\varepsilon) := \frac{1}{\pi}\iint_{\Omega_\varepsilon} \frac{\overline{\partial}F(\zeta)}{\zeta - z}\,d\sigma_2(\zeta) &= \frac{1}{\pi}\iint_{\Omega_\varepsilon} \overline{\partial}\left(\frac{F(\zeta)}{\zeta - z}\right)d\sigma_2(\zeta) \\ &= \frac{1}{2\pi i}\int\limits_{\partial\Omega_\varepsilon} \frac{F(\xi)}{\xi - z}\,d\xi \\ &= \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{F(\xi)}{\xi - z}\,d\xi - \frac{1}{2\pi i}\int\limits_{\partial D(z,\varepsilon)} \frac{F(\xi)}{\xi - z}\,d\xi. \end{aligned}$$

Since the integral

$$J := \frac{1}{\pi}\iint_\Omega \frac{\overline{\partial}F(\zeta)}{\zeta - z} d\sigma_2(\zeta)$$

converges absolutely, and $\lim_{\varepsilon\to 0} I_2(\varepsilon) = J$, it remains to show that

$$I_1(\varepsilon) := \int\limits_{\partial D(z,\varepsilon)} \frac{F(\xi)}{\xi - z}\,d\xi \to 2\pi i F(z) \text{ as } \varepsilon \to 0.$$

By using the parametrization $\xi(t) = z + \varepsilon\, e^{it}$, we obtain

$$I_1(\varepsilon) = \int\limits_0^{2\pi} \frac{F(\xi(t))}{\varepsilon\, e^{it}} \varepsilon e^{it} i\; dt = i\int\limits_0^{2\pi} F(z + \varepsilon\, e^{it})\,dt$$

$$\to i \int_0^{2\pi} F(z)\, dt = 2\pi i F(z).$$

□

As special cases we obtain Cauchy's integral formula (note that $\overline{\partial} F = 0$ in that case) and Theorem 2.3 whenever F has its support contained in Ω.

2.5 The Bézout equation in $H(U)$

Let R be a commutative unital ring. Then

$$U_n(R) := \{\boldsymbol{f} \in R^n : \boldsymbol{f} \cdot \boldsymbol{x} = \mathbf{1} \text{ for some } \boldsymbol{x} \in R^n\}$$

is the set of all invertible n-tuples.

Theorem 2.27. *Let R be a commutative unital ring. Suppose that $\boldsymbol{a} = (a_1, \dots, a_n)$ is an invertible n-tuple in R^n (that is $\boldsymbol{a} \in U_n(R)$), and let $\boldsymbol{x} = (x_1, \dots, x_n)$ satisfy $1 = \sum_{j=1}^n x_j a_j$; that is $\boldsymbol{x}\boldsymbol{a}^t = 1$. Then every other representation $1 = \sum_{j=1}^n y_j a_j$ of 1 can be deduced from the former by letting $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{a}H$, where H is an antisymmetric $(n \times n)-$matrix over R; that is $H = -H^t$, where H^t is the transpose of H.*

Proof. Suppose that $1 = \boldsymbol{x}\boldsymbol{a}^t$ and $1 = \boldsymbol{y}\boldsymbol{a}^t$. For $k = 1, \dots, n$, multiply the first equation by y_k and the second by x_k. Then

$$x_k - y_k = \sum_{j \neq k} a_j (y_j x_k - y_k x_j).$$

Thus $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{a}H$ for some antisymmetric matrix H. To prove the converse, let $1 = \boldsymbol{x}\boldsymbol{a}^t$. Note that

$$\boldsymbol{u}\boldsymbol{v}^t = \boldsymbol{v}\boldsymbol{u}^t \in R \tag{2.13}$$

for any $\boldsymbol{u}, \boldsymbol{v} \in R^n$. Since H is antisymmetric we have (due to the transitivity of matrix multiplication)

$$\begin{aligned} (\boldsymbol{a}H)\boldsymbol{a}^t &= \boldsymbol{a}(H\boldsymbol{a}^t) = \boldsymbol{a}(\boldsymbol{a}H^t)^t \\ &= \boldsymbol{a}(-\boldsymbol{a}H)^t \overset{(2.13)}{=} (-\boldsymbol{a}H)\boldsymbol{a}^t. \end{aligned}$$

Thus $(\boldsymbol{a}H)\boldsymbol{a}^t = 0$. Hence

$$\boldsymbol{y}\boldsymbol{a}^t = (\boldsymbol{x} + \boldsymbol{a}H)\boldsymbol{a}^t = \boldsymbol{x}\boldsymbol{a}^t + (\boldsymbol{a}H)\boldsymbol{a}^t = 1 + 0 = 1.$$

□

Proposition 2.28. *Let Ω be open in $\mathbb{C}$ and let $f_j \in H(\Omega)$ be different from the zero-function, $(j = 1, \dots, n)$.* [10] *Then the Bézout equation $\sum_{j=1}^n u_j f_j = 1$ admits a solution in $H(\Omega)$ if and only if the functions f_j do not have a common zero in Ω.*

[10] Note that f_j can be identically zero on some components of Ω

We would like to present an entirely different proof here, based on the Hörmander-Wolff method to solve a system of $\overline{\partial}$-equations. Consider $\boldsymbol{f} = (f_1, \dots, f_n)$ as a row matrix and let $\boldsymbol{f}^t$ be its transpose. Then $|\boldsymbol{f}|^2 = \sum_{j=1}^n |f_j|^2$; that is $|\boldsymbol{f}|^2 = \overline{\boldsymbol{f}}\boldsymbol{f}^t$.

Proof. Since the condition on the zeros of the functions f_j obviously holds whenever $\boldsymbol{u}\boldsymbol{f}^t = 1$, it remains to prove the converse. So assume that $\bigcap_{j=1}^n Z(f_j) = \emptyset$; that is $|\boldsymbol{f}| > 0$. Let $\boldsymbol{x} = \overline{\boldsymbol{f}}/|\boldsymbol{f}|^2$. Then all the coordinates of $\boldsymbol{x}$ belong to $C^\infty(\Omega)$ and $\boldsymbol{x}\boldsymbol{f}^t = 1$. By Theorem 2.27, any other solution $\boldsymbol{u} \in C^\infty(\Omega)$ to the Bézout equation $\boldsymbol{u}\boldsymbol{f}^t = 1$ is given by

$$\boldsymbol{u}^t = \boldsymbol{x}^t + H\boldsymbol{f}^t,$$

or equivalently

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{f}H,$$

where H is an $n \times n$ antisymmetric matrix over $C^\infty(\Omega)$; that is $H^t = -H$.

Let

$$F = \left(\left(\overline{\partial}\boldsymbol{x}^t \cdot \overline{\boldsymbol{f}} \right)^t - \overline{\partial}\boldsymbol{x}^t \cdot \overline{\boldsymbol{f}} \right) \frac{1}{|\boldsymbol{f}|^2}.$$

Since $\boldsymbol{x} \in C^\infty(\Omega)^n$, we see that F is an antisymmetric matrix over $C^\infty(\Omega)$. Thus, by Theorem 2.9 (applied to each component of Ω), the system $\overline{\partial}H = F$ admits a matrix solution H over $C^\infty(\Omega)$. Since F is antisymmetric, H can be chosen to be antisymmetric, too.

It is now easy to check that on Ω, $\overline{\partial}\boldsymbol{u} = \boldsymbol{0}$. In fact

$$\begin{aligned}
\overline{\partial}\boldsymbol{u} = \overline{\partial}\boldsymbol{x} - \boldsymbol{f} \cdot \overline{\partial}H &= \overline{\partial}\boldsymbol{x} - \boldsymbol{f} \cdot \left(\overline{\boldsymbol{f}}^t \cdot \overline{\partial}\boldsymbol{x} - \overline{\partial}\boldsymbol{x}^t \cdot \overline{\boldsymbol{f}} \right) \frac{1}{|\boldsymbol{f}|^2} \\
&= \frac{\left(\boldsymbol{f} \cdot \overline{\partial}\boldsymbol{x}^t\right) \overline{\boldsymbol{f}}}{|\boldsymbol{f}|^2} = \frac{\left(\overline{\partial}(\boldsymbol{f} \cdot \boldsymbol{x}^t)\right) \overline{\boldsymbol{f}}}{|\boldsymbol{f}|^2} = \frac{\left(\overline{\partial}(\boldsymbol{x} \cdot \boldsymbol{f}^t)^t\right) \overline{\boldsymbol{f}}}{|\boldsymbol{f}|^2} = \boldsymbol{0}. \qquad (2.14)
\end{aligned}$$

Thus $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{f}H$ is holomorphic in Ω. Hence $\boldsymbol{u}$ is a solution to the Bézout equation in $H(\Omega)$. ∎

Chapter 3

Continuous logarithms

3.1 Some Banach algebras

Lemma 3.1. *Let $A = (A, ||\cdot||)$ be a unital Banach algebra over $\mathbb{K}$, not necessarily commutative. We assume that $||\cdot||$ is submultiplicative, but $||\mathbf{1}||$ may be bigger than 1.*

a) Let $f \in A$ satisfy $||\mathbf{1} - f|| < 1$. Then $f \in A^{-1}$ and

$$f^{-1} = \sum_{n=0}^{\infty} (\mathbf{1} - f)^n$$

(Neumann series). Moreover,

$$||f^{-1}|| \leq \frac{1}{1 - ||\mathbf{1} - f||}.$$

b) Actually $f \in \exp A$.

Proof. Since $||\sum_{n=N}^{M} (\mathbf{1} - f)^n|| \leq \sum_{n=N}^{M} ||\mathbf{1} - f||^n \to 0$ as $N, M \to \infty$, the completeness of the norm implies that the Neumann series is convergent; that is there is $g \in A$ such that $g = \sum_{n=0}^{\infty} (\mathbf{1} - f)^n$. Since $||\mathbf{1} - f||^N \to 0$,

$$\begin{aligned} f \cdot \left(\sum_{n=0}^{N} (\mathbf{1} - f)^n \right) &= \big(\mathbf{1} - (\mathbf{1} - f)\big) \sum_{n=0}^{N} (\mathbf{1} - f)^n \\ &= \sum_{n=0}^{N} (\mathbf{1} - f)^n - \sum_{n=1}^{N+1} (\mathbf{1} - f)^n \\ &= \mathbf{1} - (\mathbf{1} - f)^{N+1} \to \mathbf{1}. \end{aligned}$$

Hence $fg = \mathbf{1}$. Similarily, $gf = \mathbf{1}$. The norm estimate of the inverse is clear (geometric series).

b) By Proposition 3.2, there is $L \in A$ such that $u = \mathbf{1} + (u - \mathbf{1}) = e^L$. $\square$

Proposition 3.2. *Let $A = (A, ||\cdot||)$ be a unital (not necessarily commutative) Banach algebra over $\mathbb{K}$, the norm $||\cdot||$ being submultiplicative.* [11]

[11] We do not assume that the norm of the identity element is one.

i) Suppose that $f \in A$ satisfies $||f|| < 1$. Then $\mathbf{1} + f$ admits a logarithm in A; that is there is $L \in A$ such that $\mathbf{1} + f = \exp L$. Moreover, L is given by

$$L = L_f := \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} f^j,$$

the series being unconditionnally/absolutely convergent. Finally, if $f_n \to 0$, then $L_{f_n} \to 0$.

ii) If $u \in A$ is nilpotent , that is $u^s = 0$ for some $s \in \mathbb{N}^*$, then $\mathbf{1} + u$ admits a logarithm $\tilde{L}$ in A, too. In that case $\tilde{L}$ is given by

$$\tilde{L} = \sum_{j=1}^{s-1} \frac{(-1)^{j-1}}{j} u^j.$$

iii) Assertion ii) holds in any unital algebra, not necessarily bearing a norm.

Proof. We first note that due to the submultiplicativity of the algebra norm, the series for L converges absolutely in the Banach space $(A, ||\cdot||)$. Now if $x \in \mathbb{R}$, $|x| < 1$, then the Taylor series of the logarithm is given by

$$L(x) := \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}.$$

By taking Cauchy-products, $L(x)^k = \sum_{n=0}^{\infty} a_{k,n} x^n$ for some uniquely determined coefficients $a_{k,n} \in \mathbb{R}$, $n \in \mathbb{N}$. Hence, by reordering the absolute converging series,

$$\begin{aligned} 1 + x = e^{\log(1+x)} &= \sum_{k=0}^{\infty} \frac{1}{k!} L(x)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=0}^{\infty} a_{k,n} x^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{1}{k!} a_{k,n} \right) x^n \end{aligned}$$

By the uniqueness of the coefficients, $\sum_{k=0}^{\infty} \frac{1}{k!} a_{k,n} = 0$ if $n \geq 2$ and $\sum_{k=0}^{\infty} \frac{1}{k!} a_{k,n} = 1$ if $n = 0$ or $n = 1$. Therefore, due to absolute convergence,

$$e^L = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{1}{k!} a_{k,n} \right) f^n = \mathbf{1} + f.$$

This formula remains valid if one replaces f by the nilpotent element u. Hence ii) holds.

If $f_n \to 0$, then $||f_n|| \leq \varepsilon \leq 1/2$ for all $n \geq n_0$. Hence, for these n,

$$||L_{f_n}|| \leq \sum_{j=1}^{\infty} ||f_n||^j \leq \frac{\varepsilon}{1-\varepsilon} \leq 2\varepsilon.$$

□

Lemma 3.3. *Let A be a unital Banach algebra over $\mathbb{K}$ and $f, g \in A$. The following equality holds:*

$$e^f = \lim_{n\to\infty} \left(\mathbf{1} + \frac{f}{n}\right)^n .$$

Moreover, if f and g commute, then $e^{f+g} = e^f e^g$.

Proof. Let $S_N = \sum_{n=0}^N f^n/n!$ and $a_N = (\mathbf{1} + f/N)^N$. Then

$$S_N - a_N = \sum_{n=0}^{N} \left(\frac{1}{n!} - \binom{N}{n}\right) f^n.$$

Hence, by noticing that $\frac{1}{n!} \leq \binom{N}{n}$,

$$||S_n - a_N|| \leq \sum_{n=0}^{N} \left(\binom{N}{n} - \frac{1}{n!}\right) ||f||^n = \left(1 + \frac{||f||}{N}\right)^N - \sum_{n=0}^{N} \frac{1}{n!}||f||^n \to 0.$$

The rest is a straightforward application of the Cauchy-product to represent $e^f e^g$. □

Theorem 3.4. *Let $A = (A, ||\cdot||)$ be a commutative unital Banach algebra over $\mathbb{K}$ and $G_A := A^{-1}$ the group of invertible elements. The following assertions hold:*

(1) G_A is an open subset of A;

(2) $\exp A$ is open-closed in G_A, path-connected, and coincides with the connected component in G_A of the identity element $\mathbf{1}$ in A;

Proof. (1) $G_A = A^{-1}$ is an open set in A. In fact, let $f \in A^{-1}$ and let $g \in A$ satisfy $||f - g|| < ||f^{-1}||^{-1}$. Then

$$||\mathbf{1} - f^{-1}g|| = ||f^{-1}(f - g)|| \leq ||f^{-1}|| \, ||f - g|| < 1.$$

Hence, by Lemma 3.1, $f^{-1}g \in A^{-1}$ and so, by (1), $g = f(f^{-1}g) \in A^{-1}$. We conclude that A^{-1} is open

(2) • Let $a \in A$. Consider the map $\Gamma : [0, 1] \to \exp A$, $t \mapsto e^{ta}$. Using that

$$||e^{sa} - e^{ta}|| \leq ||e^{sa}|| \, ||\mathbf{1} - e^{(t-s)a}|| \leq C \, e^{||sa||} \left|1 - e^{|t-s| \, ||a||}\right|,$$

where $C = \max\{1, ||\mathbf{1}||\}$, we see that Γ is a continuous path between the identity and e^a. Whence, $\exp A$ is path-connected.

• Next we show that $\exp A$ is an open set in A. Let $f = e^a \in \exp A$ and let $g \in A$ satisfy $||f - g|| < ||f^{-1}||^{-1}$. Then, as we know, $||1 - f^{-1}g|| < 1$ and so, by Lemma 3.1, $f^{-1}g \in \exp A$; say $f^{-1}g = e^b$ for some $b \in A$. Since A was assumed to be commutative, we may use Lemma 3.3 to conclude that

$$g = f(f^{-1}g) = e^a e^b = e^{a+b} \in \exp A.$$

• Next we show that $\exp A$ is (relatively) closed in $U_1(A)$. To this end, let (a_n) be a sequence in A and $u \in U_1(A)$ such that $||e^{a_n} - u|| \to 0$. Then

$$||e^{a_n}u^{-1} - \mathbf{1}|| = ||(e^{a_n} - u)u^{-1}|| \leq ||u^{-1}|| \, ||e^{a_n} - u|| \to 0.$$

By Proposition 3.2, $e^{a_n}u^{-1} \in \exp A$ for n sufficiently large. Since $\exp A$ is a subgroup of $U_1(A)$, we conclude that $u^{-1} \in \exp A$ and so $u \in \exp A$.

In summary, we have shown that $\exp A$ is the connected component of the identity element within $U_1(A)$. □

Corollary 3.5. *Let X be a compact, contractible Hausdorff space. Then every continuous, zero-free $\mathbb{C}$-valued function on X admits a continuous logarithm in $A = C(X, \mathbb{C})$, that is* $\exp A = G_A$.

Proof. Consider the following path Γ induced by the homotopic contraction H:

$$\Gamma : [0, 1] \to U_1(C(X, \mathbb{C})), \quad \Gamma(t) = f \circ H(\cdot, t).$$

This path connects f and $f(x_0)$. Hence both invertible tuples belong to the same component of $U_1(C(X, \mathbb{C}))$. Thus $f = e^g f(x_0) = e^p$. and note that $\mathbb{C}^* \cdot 1 \subseteq \exp C(X, \mathbb{C}) = G_{C(X,\mathbb{C})}$.

□

3.2 Zero-free extensions, logarithms and Eilenberg's Theorem

We introduce the following notation.

Notation 3.6. Let $K \subseteq \mathbb{C}$ be compact and let f, g be continuous and zero-free on K. Then we say that f is homotopic to g within $C(K, \mathbb{C}^*)$, denoted by $f \overset{K}{\sim} g$, if there is a continuous map $H : K \times [0, 1] \to \mathbb{C}^*$ such that $H(z, 0) = f(z)$ and $H(z, 1) = g(z)$ for $z \in K$.

Theorem 3.7 (Borsuk)**.** *Let $K \subseteq \mathbb{C}$ be compact and let $f \in C(K, \mathbb{C}^*)$ be a zero-free continuous function. Then the following assertions are equivalent:*

1. *f is homotopic in $C(K, \mathbb{C}^*)$ to a constant;*

2. *f has a continuous logarithm; that is there is $L \in C(K, \mathbb{C})$ such that $f = e^L$;*

3. *f has a zero-free continuous extension to $\mathbb{C}$.*

Proof. (1) $\Longrightarrow$ (2): Let $H : K \times [0, 1] \to \mathbb{C}^*$ be a homotopy with $H(z, 0) = f(z)$ and $H(z, 1) \equiv c$, $(z \in K)$. The path $\Gamma : [0, 1] \to U_1(C(K, \mathbb{C}))$, given by

$$\Gamma(t) = H(\cdot, t)$$

connects the invertible function f with the (invertible) constant function c. Thus f belongs to the principal component of $U_1(C(K, \mathbb{C}))$. Using Theorem 3.4, we conclude that f is an exponential.

(2) $\Longrightarrow$ (3): Let $f = e^L$, $L \in C(K, \mathbb{C})$. By Tietze's theorem applied to $\mathbb{R}^2 \sim \mathbb{C}$, we may extend L to a function L^* continuous in $\mathbb{C}$. Hence e^{L^*} is the desired zero-free extension of f.

(3) $\Longrightarrow$ (2): Let D be a closed disk centered at 0 such that $K \subseteq D^\circ$. If F is a zero-free continuous extension of $f : K \to \mathbb{C}^*$, then $F|_D$ is invertible in $C(D, \mathbb{C})$. But D is contractible (look at the homotopy $H(z, t) = tz, 0 \leq t \leq 1, z \in D$.) By Corollary 3.5, each zero-free $g \in C(D, \mathbb{C})$ has a logarithm; in particular there is $L \in C(K, \mathbb{C})$ such that $e^L = f$.

(2) $\Longrightarrow$ (1): If $e^L = f$, then we only have to consider the homotopy $H : K \times [0,1] \to \mathbb{C}^*$ given by
$$H(z,t) = e^{tL(z)}.$$
$\square$

Corollary 3.8. *Let $K \subseteq \mathbb{C}$ be compact and let $f, g \in C(K, \mathbb{C}^*)$ be two zero-free continuous functions. Then the following assertions are equivalent:*

1. *f and g are homotopic in $C(K, \mathbb{C}^*)$;*
2. *there is $h \in C(K, \mathbb{C})$ such that $f = ge^h$.*

In particular, if f and g are homotopic in $C(K, \mathbb{C}^*)$, then f has a continuous logarithm on K if and only if g has one.

Proof. This is an immediate consequence of Theorem 3.7 by using that $f \overset{K}{\sim} g$ if and only if $f/g \overset{K}{\sim} 1$. Note that if $H(z,t) : K \times [0,1] \to \mathbb{C}^*$ is a homotopy between f and g, then $\tilde{H}(z,t) := H(z,t)/H(z,1)$ is a homotopy between f/g and 1. $\square$

Theorem 3.9 (Borsuk)**.** *For a compact set $K \subseteq \mathbb{C}$, the following assertions are equivalent:*

1. *Every continuous, zero-free function $f \in C(K, \mathbb{C})$ has a continuous logarithm;*
2. *$\mathbb{C} \setminus K$ is connected.*

Proof. (1) $\Longrightarrow$ (2): If K has a hole G and $a \in G$, then Corollary 1.12 shows that the function $f(z) = z - a$ ($z \in K$), has no invertible continuous extension to $K \cup G$, and a fortiori no invertible extension to $\mathbb{C}$. Hence, by Theorem 3.7, f does not have a continuous logarithm.

(2) $\Longrightarrow$ (1): Special case of Theorem 3.19. $\square$

Remark 3.10. This result does no longer hold in higher dimensions: let $K = \{(x,y,t) \in \mathbb{R}^3 : t = 0,\ x^2 + y^2 = 1\}$ (in other words, K is the embedded unit circle). Then $\mathbb{R}^3 \setminus K$ is connected, but the function $f : K \to \mathbb{C}$ given by $f(x,y,t) = x + iy$ has no continuous logarithm on K (otherwise there would exist $L \in C(K, \mathbb{C})$ such that $e^L = f$. This in turn implies that the identity map $z : \mathbb{T} \to \mathbb{C}$ would have a continuous, zero-free extension to $\overline{\mathbb{D}}$, a contradiction to Corollary 1.12.)

Proposition 3.11. *Let $K \subseteq \mathbb{C}$ be compact, $f \in C(K, \mathbb{C})$, f holomorphic in K°. Suppose that there is $h \in C(K, \mathbb{C})$ such that $f = e^h$. Then h is holomorphic in K°.*

Proof. Let $a \in K^\circ$.

Case 1 f is not constant in a neighborhood of a. The holomorphy implies that there exists a disk $D(a, \varepsilon) \subseteq K^\circ$ such that $f(z) \neq f(a)$ for all $z \in D(a, \varepsilon) \setminus \{a\}$. Hence $h(z) \neq h(a)$ for all those z. Note that $\lim_{z\to a} h(z) = h(a)$. Hence, if $z \neq a$ is close to a, $e^{h(z)} \neq e^{h(a)}$. The differentiability of the exponential function now implies that

$$\begin{aligned}\frac{h(z)-h(a)}{z-a} &= \frac{f(z)-f(a)}{z-a}\left(\frac{e^{h(z)}-e^{h(a)}}{h(z)-h(a)}\right)^{-1} \\ &\to f'(a)e^{-h(a)} = \frac{f'(a)}{f(a)}.\end{aligned}$$

Hence h is $\mathbb{C}$-differentiable at a.

Case 2 If f is constant $f(a)$ in a neighborhood of a, then $f \equiv f(a)$ in the connected component C of K° containing a. Thus the continuous logarithm h of f is constant in C, too. In particular, h is holomorphic in K°. □

Lemma 3.12 (Eilenberg). *Let $K \subseteq \mathbb{C}$ be compact and $a, b \in \mathbb{C} \setminus K$. Let $f : K \to \mathbb{C}$ be defined by $f(z) = (z-a)/(z-b)$ $(z \in K)$. Then the following assertions are equivalent:*

(1) f has a continuous logarithm on K;

(2) a and b belong to the same component of $\mathbb{C} \setminus K$. [12]

Proof. (1) $\Longrightarrow$ (2): Suppose, to the contrary, that a and b belong to different components of $\mathbb{C} \setminus K$. Then at least one of them is a bounded component. Let us call this C. We may assume that $a \in C$. Now, if (1) is satisfied then, by Theorem 3.7, f admits a continuous zero-free extension, F, to $\mathbb{C}$. Hence, the function $A : K \cup C$ given by $A(z) = (z-b)F(z)$ $(z \in K \cup C)$, is a zero-free extension of $(z-b)f(z)|_K = (z-a)|_K$; a contradiction to Corollary 1.12

(2) $\Longrightarrow$ (1): Let $a, b \in C$, where C is a component of $\mathbb{C} \setminus K$. Let $\gamma : [0,1] \to C$ be a path in C joining a to b. Then $H : [0,1] \times K \to \mathbb{C}^*$, given by

$$H(t,z) = \frac{z-\gamma(t)}{z-b}$$

is a homotopy in $C(K,\mathbb{C}^*)$ between f and 1; in other words the path $\Gamma : [0,1] \to U_1(C(K,\mathbb{C}))$ given by $\Gamma(t) = H(t,\cdot)$ joins within $U_1(C(K,\mathbb{C}))$ the identity element 1 to f. By Theorem 3.4, f belongs to the principal component in $U_1(C(K,\mathbb{C}))$; that is, f is an exponential. □

As an application we now present Janiszewski's separation theorem.

Theorem 3.13 (Janiszewski). *For $j = 1, 2$, let K_j be two compact subsets of $\mathbb{C}$ and let Ω_j be a component of $\mathbb{C} \setminus K_j$. Suppose that $\{a, b\} \subseteq \Omega_1 \cap \Omega_2$ and that $K_1 \cap K_2$ is connected. Then a and b both lie in the same component of $\mathbb{C} \setminus (K_1 \cup K_2)$.*

Proof. Consider on $K_1 \cup K_2$ the function $f(z) = (z-a)/(z-b)$. By Lemma 3.12, there are $h_j \in C(K_j)$ such that $f(z) = e^{h_j(z)}$ whenever $z \in K_j$. Hence $e^{h_1(z)-h_2(z)} = 1$ whenever

[12] One also says that K does not separate a and b

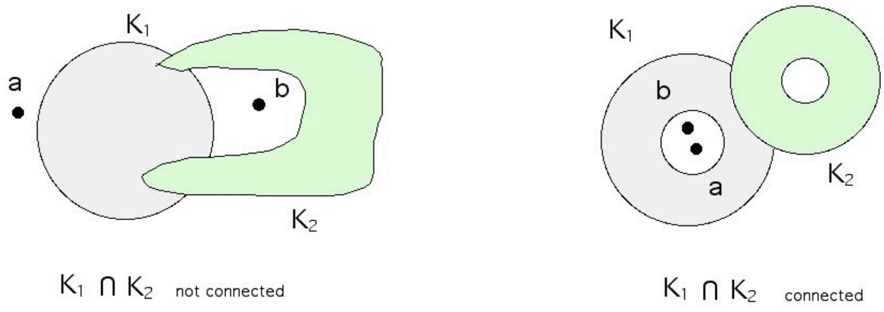


Figure 3.1: A separating and a non-separating union

$z \in K_1 \cap K_2$. Since $K_1 \cap K_2$ is connected, $h_1 - h_2$ is constant on $K_1 \cap K_2$, say $2\pi i m$, for some $m \in \mathbb{Z}$. Now define the function $L : K_1 \cup K_2 \to \mathbb{C}$ by

$$L(z) = \begin{cases} h_2(z) + 2\pi i m & \text{if } z \in K_1 \\ h_1(z) & \text{if } z \in K_2. \end{cases}$$

Then L is continuous on $K_2 \cup K_2$. Since $e^L = f$, Lemma 3.12 yields that a and b do not belong to distinct components of $\mathbb{C} \setminus (K_1 \cup K_2)$. □

Next we discuss what happens when we have expressions of the form

$$\prod_{j=1}^{n} (z - a_j)^{s_j}, \quad (s_j \in \mathbb{Z}).$$

Corollary 3.14. *Let a, b, p belong to the same component C of $\mathbb{C} \setminus K$ and let $m, n \in \mathbb{N}$. Then there exists $h \in C(K, \mathbb{C})$ such that for all $z \in K$*

$$f(z) := \frac{(z-a)^m}{(z-b)^n} = (z-p)^{m-n} e^{h(z)}.$$

Proof. By Lemma 3.12, there exists $h_1, h_2 \in C(K, \mathbb{C})$ such that $z - a = e^{h_1(z)}(z - p)$ and $z - b = e^{h_2(z)}(z - p)$. Hence

$$f(z) = e^{mh_1(z) - nh_2(z)} (z-p)^{m-n}.$$

□

Lemma 3.15. *Let $K \subseteq \mathbb{C}$ be compact and let C be a bounded component of $\mathbb{C} \setminus K$. If $a \in C$ and $s \in \mathbb{Z}$, $s \neq 0$, then $(z - a)^s|_K$ does not have a zero-free continuous extension to $K \cup C$.*

Proof. Let $D_r = D(a,r)$ be a closed disk of radius r centered at a such that $K \subseteq D_r^\circ$. Note that $C \subseteq D_r$, too. Suppose that there is $F \in C(K \cup C)$, F zero-free, with $F(z) = (z-a)^s$ for $z \in K$. Let

$$F_1(z) = \begin{cases} F(z) & \text{if } z \in K \cup C \\ (z-a)^s & \text{if } z \in D_r \setminus (K \cup C). \end{cases}$$

Then F_1 is zero-free on D_r. Since on the boundary of $K \cup C$ the functions F and $(z-a)^s$ coincide, F_1 is continuous on D_r. By Theorem 3.9, F_1 has a continuous logarithm on D_r; say $F_1 = e^L$. Now $f(z) := e^{(1/s)L(z)}$ is continuous on D_r and $f^s = F_1$. In particular, $f^s(z) = (z-a)^s$ for $|z-a| = r$. Now $z \mapsto f(z)/(z-a)$ is continuous on $|z-a| = r$ and $\big(f(z)/(z-a)\big)^s = 1$ for each z with $|z-a| = r$. Because the circle is connected, $f(z)/(z-a) \equiv e^{2k\pi i/s}$ for some $k \in \mathbb{Z}$, independent of $z \in \partial D_r$. Note that f is zero-free on D_r. Hence $e^{-2k\pi i/s} f$ is a zero-free extension of $(z-a)|_{\partial D_r}$ to D_r. This contradicts Corollary 1.12. □

Theorem 3.16. *Let $C_1, \dots, C_n$ be distinct bounded components of the complement $\mathbb{C} \setminus K$ of the compact set K and let $a_j \in C_j$. Suppose that for some $s_j \in \mathbb{Z}$, $(j = 1, \dots, n)$, the function*

$$f(z) = \prod_{j=1}^{n} (z-a_j)^{s_j}, \quad (z \in K)$$

has a continuous logarithm on K. Then $s_1 = \dots = s_n = 0$.

Proof. In view of Lemma 3.15 and Theorem 3.7, we may assume that $n \geq 2$. Let $f = e^h$, where $h \in C(K, \mathbb{C})$. Then

$$(z-a_1)^{s_1} = e^{h(z)} \prod_{j=2}^{n} (z-a_j)^{-s_j} =: R(z).$$

Because $a_j \notin C_1$, $(j = 2, \dots, n)$, we see that the function $R(z)$ (and hence $(z-a_1)^{s_1}$) has a zero-free continuous extension to $K \cup C_1$. By Lemma 3.15, $s_1 = 0$. By the same reasoning, $s_j = 0$ for the remaining j. □

It is important to note that if the point a belongs to the unbounded component of $\mathbb{C} \setminus K$, then $z - a$ does have a continuous logarithm on K, as the following Proposition shows:

Proposition 3.17. *Let $K \subseteq \mathbb{C}$ be compact and suppose that $f : K \to \mathbb{C}^*$ is continuous. If 0 belongs to the unbounded component C_0 of $\mathbb{C} \setminus f(K)$, then f has a continuous logarithm on K. This holds in particular for the function $f(z) = z - a$, when a belongs to the unbounded component of $\mathbb{C} \setminus K$.*

Proof. Choose $r > \max_K |f|$ and let $\gamma : [0,1] \to C_0$ be a path joining within C_0 the point 0 to r. Then the homotopy $H : [0,2] \times K \to \mathbb{C}^*$ given by

$$H(t,z) = \begin{cases} f(z) - \gamma(t) & \text{if } (t,z) \in [0,1] \times K \\ (2-t)f(z) - r & \text{if } (t,z) \in [1,2] \times K \end{cases}$$

shows that f is homotopic in $C(K, \mathbb{C}^*)$ to the constant $-r$. Hence, by Theorem 3.7, f has a continuous logarithm on K. □

Our proof of Eilenberg's theorem relies on the "primitive" version 3.18 of Runge's theorem and on the $\overline{\partial}$-calculus.

Theorem 3.18 (Runge)**.** *Let $K \subseteq \mathbb{C}$ be compact. Then every function holomorphic in a neighborhood of K can be uniformly approximated on K by rational functions vanishing at infinity and with (simple) poles outside K.*

Proof. Let f be holomorphic in the open set U with $K \subseteq U$. Choose bounded open sets V and W such that

$$K \subseteq V \subseteq \overline{V} \subseteq W \subseteq \overline{W} \subseteq U.$$

Let $\phi \in C_c^\infty(\mathbb{C})$ satisfy

$$\phi = 1 \text{ on } \overline{V},\ \phi = 0 \text{ on } \mathbb{C} \setminus W$$

and $0 \leq \phi \leq 1$. Then $F := \phi f \in C_c^\infty(\mathbb{C})$ and $F = f$ on V. By Theorem 2.3, for every $z \in \mathbb{C}$,

$$F(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{\partial} F(\xi)}{\xi - z} d\sigma_2(\xi).$$

Since $\overline{\partial} F = 0$ in $V \cup Q^c$, where Q is a closed dyadic square with $\overline{W} \subseteq Q^\circ$, we obtain

$$F(z) = -\frac{1}{\pi} \int_{Q \setminus V} \frac{\overline{\partial} F(\xi)}{\xi - z} d\sigma_2(\xi). \tag{3.1}$$

The Riemann sums associated with this integral are rational functions (in z) and uniformly approximate F (hence f) on K. Here are the details of the proof.

Let $d := \frac{1}{4}\mathrm{dist}\,(K, Q \setminus V)$. Now the function $h : K \times (Q \setminus V) \to \mathbb{C}$, given by

$$h(z, \xi) = -\frac{1}{\pi} \frac{\overline{\partial} F(\xi)}{\xi - z}$$

is uniformly continuous. In particular, for every $\varepsilon > 0$ there is $\delta \in]0, d[$ such that

$$|h(z, \xi) - h(z, \eta)| < \varepsilon$$

for any $z \in K$ and $\xi, \eta \in Q \setminus V$ with $|\xi - \eta| < \delta$. If we choose a covering of $Q \setminus V$ with adjacent (closed) dyadic squares $Q_j \subseteq Q$, $j = 1, \ldots, N$, of fixed diameter κ less than δ, [13] then for $\xi_j \in Q_j \setminus V$

$$\left| F(z) - \sum_{j=1}^{N} h(z, \xi_j) \sigma_2(Q_j) \right| \leq \varepsilon \sum_{j=1}^{N} \sigma_2(Q_j) \leq \varepsilon \sigma_2(Q).$$

Thus F has been uniformly approximated by rational functions of the form

$$\sum_{j=1}^{N} \frac{\mu_j}{\xi_j - z},$$

where the poles ξ_j belong to $\mathbb{C} \setminus K$. □

[13] For example, take a dyadic lattice of the plane with squares of diameter κ and delete all those squares whose interior does not meet $Q \setminus V$.

Theorem 3.19 (Eilenberg)**.** *Let $K \subseteq \mathbb{C}$ be compact and for each bounded component C of $\mathbb{C}\setminus K$, let $a_C \in C$. Suppose that $f : K \to \mathbb{C}^*$ is continuous. Then there exist finitely many bounded components C_j of $\mathbb{C}\setminus K$, integers $s_j \in \mathbb{Z}$ $(j = 1, \dots, n)$, and $L \in C(K, \mathbb{C})$ such that for all $z \in K$*

$$f(z) = \prod_{j=1}^{n} (z - a_{C_j})^{s_j}\, e^{L(z)}.$$

Proof. Since f has no zeros on the compact set K, the continuity of f implies that $\min_K |f| > 0$. We may also assume that f is continuous and zero-free in a neighborhood of K. By Weierstrass' approximation theorem **??**, let F be a smooth function (say a polynomial in the real variables x and y) with $||F - f||_K < \min_K |f|$. Then F is zero free in a neighborhood of K. Moreover, $||\frac{F}{f} - 1||_K < 1$. By Proposition 3.2, there exists $g \in C(K, \mathbb{C})$ such hat $F/f = e^g$. Hence $f = Fe^{-g}$. Next we wish to write F as $e^u G$, where G is holomorphic in a neighborhood of K and $u \in C^1(U)$. Here we use the $\overline{\partial}$-calculus. In fact, let u be a smooth solution to the $\overline{\partial}$-equation $\overline{\partial} u = \frac{\overline{\partial} F}{F}$ in a neighborhood U of K (see Theorem 2.8). Then $\overline{\partial}(e^{-u}F) = 0$ and so $G := e^{-u}F$ is holomorphic on U. Moreover,

$$f = Fe^{-g} = (e^u G)e^{-g} = Ge^{u-g},$$

and G is zero-free in U. Now we use Runge's approximation theorem 3.18 to get a rational function r with poles outside K such that $||r - G||_K < \min_K |G|$. Then $||(r/G) - 1||_K < 1$ and so, by Proposition 3.2, $G = re^v$ for some $v \in C(K, \mathbb{C})$. Summing up, we have obtained on K the representation

$$f = re^{v+u-g},$$

r continuous and zero-free on K. Next we have to deal with the zeros of r inside the components of $\mathbb{C} \setminus K$. Since r is a rational function, there are of course only finitely many components C_j, $j = 0, 1, \dots, N$ that contain zeros and/or poles. Here C_0 is the unbounded component. Let s_j be the number of zeros (multiplicities included) minus the number of poles in C_j. Choose a point $a_{C_0} \in C_0$. Using Corollary 3.14 several times, we finally obtain a continuous function $w \in C(K, \mathbb{C})$ such that $r = \prod_{j=0}^{N} (z - a_{C_j})^{s_j} e^w$. By Proposition 3.17, $z - a_{C_0}$ has a continuous logarithm on K. Thus $r = \prod_{j=1}^{N} (z - a_{C_j})^{s_j} e^h$ for some $h \in C(K, \mathbb{C})$. To sum up, we have shown that

$$f = \prod_{j=1}^{N} (z - a_{C_j})^{s_j} e^{h+v+u-g}.$$

□

Corollary 3.20. *Let $S \subseteq \mathbb{C}$ be compact. If a and b belong to distinct components of $\mathbb{C} \setminus S$, then the polynomials $z - a$ and $z - b$ are not homotopic in $C(S, \mathbb{C}^*)$.*

Proof. Suppose to the contrary that $f(z) := z - a$ and $g(z) := z - b$ are homotopic in $C(S, \mathbb{C}^*)$. Then, by Corollary 3.8, there is $h \in C(S, \mathbb{C})$ such that $f = ge^h$. Hence $(z - a)/(z - b)$ has a continuous logarithm on S. This contradicts Theorem 3.16 if both numbers a and b belong to (distinct) bounded components of $\mathbb{C} \setminus S$. If one of them, say a, belongs to the unbounded component, we already have $f(z) = z - a = e^{\ell(z)}$ for some $\ell \in C(S, \mathbb{C})$ (by Proposition 3.17. But then we also would have have $g = e^{\ell - h}$. In other words, $z - b$ has a logarithm, say $z - b = e^{L(z)}, z \in S$. Let Ω denote that bounded component of $\mathbb{C} \setminus S$ the point b belongs to. Note that $\partial\Omega \subseteq S$. Then $(z - b)|_{\partial\Omega}$ has a zero-free continuous extension to $\overline{\Omega}$; a contradiction to Corollary 1.12. $\square$

Theorem 3.21. *Let K_1 and K_2 be two compact sets in $\mathbb{C}$ and let $h : K_1 \to K_2$ be a homeomorphism. Then K_1 and K_2 have the same number of holes.*

Proof. i) If K_1 has no holes then $\mathbb{C} \setminus K_1$ is connected and so, by Borsuk's theorem 3.9, any $f \in C(K_1, \mathbb{C}^*)$ has a logarithm; say $f = e^g$. Now, if $F \in C(K_2, \mathbb{C}^*)$, then $f := F \circ h \in C(K_1, \mathbb{C}^*)$. Hence $F = f \circ h^{-1} = e^{g \circ h^{-1}}$. Since F was arbitrary, we conclude, again from Borsuk's theorem 3.9, that $\mathbb{C} \setminus K_2$ is connected. In other words, K_2 has no holes.

ii) Assume that K_1 has finitely many holes, say $U_1, \dots, U_n$, and let $V_1, \dots, V_m$ be holes of K_2. By i), $m \geq 1$. In view of achieving a contradiction, we may suppose that $m \geq n + 1$. Let $a_k \in U_k$ and $b_j \in V_j$ be fixed and consider for $w \in K_2$ the functions

$$f_j(w) = w - b_j, \quad (j = 1, \dots, m).$$

For $z \in K_1$, let $F_j(z) = f_j(h(z))$, $j = 1, \dots, m$. Then $F_j \in C(K_1, \mathbb{C}^*)$. By Eilenberg's theorem 3.19, there are $L_j \in C(K_1, \mathbb{C})$ and $s_{k,j} \in \mathbb{Z}$ such that for $z \in K_1$

$$F_j(z) = \prod_{k=1}^{n} (z - a_k)^{s_{k,j}}\, e^{L_j(z)}.$$

Consider the "horizontal" (n, m)-matrix

$$M = \begin{pmatrix} s_{1,1} & \dots & \dots & s_{1,m} \\ \vdots & & & \vdots \\ s_{n,1} & \dots & \dots & s_{n,m} \end{pmatrix}.$$

The rank of M over $\mathbb{Q}$ is at most n. Since $m > n$, the m columns are linear dependent over $\mathbb{Q}$. Thus there exist $s_j \in \mathbb{Z}$, $(j = 1, \dots, m)$, not all of them zero, such that

$$\sum_{j=1}^{m} s_{k,j} s_j = 0 \text{ for all } k = 1, \dots, n.$$

Hence

$$\begin{aligned} \prod_{j=1}^{m} F_j(z)^{s_j} &= \left[\prod_{j=1}^{m} \prod_{k=1}^{n} (z - a_k)^{s_{k,j} s_j}\right] \exp\Big(\overbrace{\sum_{j=1}^{m} s_j L_j(z)}^{:=L(z)}\Big) \\ &= \left[\prod_{k=1}^{n} (z - a_k)^{\sum_{j=1}^{m} s_{k,j} s_j}\right] \exp L(z) \\ &= \exp L(z). \end{aligned}$$

On the other hand, $F(z) := \prod_{j=1}^{m} F_j(z)^{s_j} = \prod_{j=1}^{m}(h(z) - b_j)^{s_j}$. Hence, with $z = h^{-1}(w)$,

$$\prod_{j=1}^{m}(w - b_j)^{s_j} = F(h^{-1}(w)) = \exp L(h^{-1}(w)),\ w \in K_2.$$

By Theorem 3.16, $s_j = 0$ for every j. This is a contradiction to the choice of s_j. Hence $m \le n$. Interchanging the role of m and n yields that $m = n$.

iii) If K_1 has infinitely many holes, then K_2 must have infinitely many holes, too (otherwise we apply the previous case where the roles of K_1 and K_2 are interchanged). $\square$

Proposition 3.22. *Let f be a zero-free continuous function in $\mathbb{C}$. Then f has a continuous logarithm; that is, there exists a continuous function g with $e^g = f$.*

Proof. First approach. By Borsuk's theorem 3.9, f admits a continuous logarithm on every closed square. Now we cover $\mathbb{C}$ with unit squares $[n, n+1] \times [m, m+1]$, $n, m \in \mathbb{Z}$ and enumerate those in the following way: one starts with the square $Q_1 := [0, 1]^2$, denoted by 1, and spirals around:

37	36	35	34	33	32	31
$\downarrow$	17	16	15	14	13	30
	18	5	4	3	12	29
	19	6	1	2	11	28
	20	7	8	9	10	27
$\downarrow$	21	22	23	24	25	26
	$\rightarrow$				$\rightarrow$	

Let $h_1 := g_1 = \log f$ on Q_1; choose $g_2 = \log f$ on Q_2 so that $g_2 = h_1$ on the common boundary and put

$$h_2 = \begin{cases} g_1 & \text{on } Q_1 \\ g_2 & \text{on } Q_2 \end{cases}.$$

Then $h_2 \in C(Q_1 \cup Q_2)$ and $e^{h_2} = f$. Inductively, choose $g_n = \log f$ so that g_n coincides with h_{n-1} on the intersection I of the boundary of Q_n with the previous boundaries. Note that I is a union of at most two adjacent segments, hence a connected set. The function $g = g_n$ on Q_n, $(n = 1, 2, 3, \dots)$, is then well-defined and satisfies $e^g = f$ in $\mathbb{C}$.

Second approach Let $g_n = \log f$ on the disk $D_n = \{|z| \le n\}$, where the branch is chosen so that $g_n = g_{n-1}$ on D_{n-1}. We claim that the function g, defined by $\lim_n g(z)$, is a continuous logarithm of f. In fact, let $z_0 \in \mathbb{C}$ be fixed, and consider the smallest disk D_{n_0} containing z_0 in its interior. Then $g_n(z) = g_{n_0}(z)$ for every $z \in D_{n_0}$ and $n \ge n_0$. Thus $g(z) = g_{n_0}(z)$ for all $z \in D_{n_0}$ and so $e^g = f$ on D_{n_0}. $\square$

Theorem 3.23. *Let G be a simply connected domain in $\mathbb{C}$. Then every zero free continuous function on G admits a continuous logarithm.*

Proof. According to Theorem **??**, let (K_n) be an exhaustion sequence of Ω-convex connected compacta. Since Ω is simply connected, each K_n actually is polynomial convex (Proposition **??**). By Borsuk's Theorem 3.9, there is $g_n \in C(K_n, \mathbb{C})$ with $e^{g_n} = f$ on K_n and where the branch is chosen so that $g_n = g_{n-1}$ on K_{n-1} whenever $n \geq 2$. As above, we see that the function g, defined by $\lim_n g(z)$, is a continuous logarithm of f on G. □

We conclude this Section with Brouwer's fixed point theorem for the closed unit disk.

Theorem 3.24 (Brouwer)**.** *Let $\boldsymbol{D}$ be the closed unit disk and $f \in C(\boldsymbol{D}, \boldsymbol{D})$. Then f admits a fixed point; that is there is $a \in \boldsymbol{D}$ with $f(a) = a$.*

Proof. Suppose that f does not admit a fixed point. Then $g(z) := z - f(z)$ is a zero-free continuous function on $\boldsymbol{D}$. Note that $\boldsymbol{D}$ is contractible. Hence, due to Corollary 3.5, $g = e^G$ for some $G \in C(\boldsymbol{D}, \mathbb{C})$. Now consider the function $h : \boldsymbol{D} \to \boldsymbol{D}$ given by $h(z) = 1 - \overline{z} f(z)$. Then h is zero free on $\boldsymbol{D}$, because otherwise $\overline{b} f(b) = 1$ for some $b \in \boldsymbol{D}$. Since on $\mathbb{D}$, $|bf(b)| < 1$, we deduce that $|b| = 1$. Thus $f(b) = b$; a contradiction to the assumption. Again, by Corollary 3.5, there is $H \in C(\boldsymbol{D}, \mathbb{C})$ with $h = e^H$. But on $\mathbb{T} = \partial \boldsymbol{D}$, we have $zh(z) = z - f(z) = g(z)$. Thus $ze^{H(z)} = e^{G(z)}$ and so

$$z = e^{G(z) - H(z)} \text{ with } |z| = 1.$$

This contradicts Proposition 1.3, for example. We conclude that f has a fixed point in $\boldsymbol{D}$. □

Chapter 4

Conformal maps

4.1 Conformal maps and the Riemann mapping theorem

Here we will present the most important theorem in geometric function theory dealing with conformal maps: namely the Riemann mapping theorem. We will use our original Definition 4.1 of simple-connectedness.

Definition 4.1.

i) A *domain* in $\mathbb{C}$ is an open connected set [14].

ii) A domain $G \subseteq \mathbb{C}$ is said to be *simply connected* if $\mathbb{C} \setminus G$ has no bounded components.

Recall from Definition 1.1 that a Cauchy domain is a domain in $\mathbb{C}$ for which every cycle is null-homologous.

Theorem 4.2. *Let D_1 and D_2 be two domains in $\mathbb{C}$ and suppose that $f : D_1 \to D_2$ is a conformal map (that is a holomorphic bijection). If D_1 is a Cauchy domain, then $D_2 = f(D_1)$ is a Cauchy domain.*

Proof. Let $\Gamma = w([0,1]) \subseteq D_2$ be a closed, piecewise C^1-curve. We need to show that $n(\Gamma, a) = 0$ for every $a \in \mathbb{C} \setminus D_2$. To this end, let $\gamma = f^{-1} \circ \Gamma$. Then γ is a closed piecewise C^1-curve in D_1 parametrized by $z(t) = f^{-1}(w(t))$, or equivalently, $w(t) = f(z(t)), 0 \leq t \leq 1$. By assumption, γ is null-homologous in D_1. Hence,

$$\begin{aligned} I := n(\Gamma, a) &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{w-a}\, dw \\ &= \frac{1}{2\pi i} \int_0^1 \frac{f'(z(t)) \cdot \dot{z}(t)}{f(z(t)) - a}\, dt \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-a}\, dz. \end{aligned}$$

Since $f'/(f-a) \in H(D_1)$, the Cauchy theorem for null-homologous cycles implies that $I = 0$. Hence Γ is null-homologous in D_2 and so D_2 is a Cauchy domain, too. □

[14] If D is a *nonvoid* domain, then D is sometimes called a *region* (see [?]).

Theorem 4.3 (Riemann mapping theorem)**.** *Let G be a simply connected domain, $z_0 \in G$ and $G \neq \mathbb{C}$. Then there is a unique conformal map $f : G \to \mathbb{D}$ (called the Riemann map) with $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. Uniqueness

Let f and g be two conformal maps of G onto $\mathbb{D}$ with $f(z_0) = g(z_0) = 0$ and $f'(z_0) > 0$, $g'(z_0) > 0$. Then $h := g \circ f^{-1}$ is a conformal map of $\mathbb{D}$ onto $\mathbb{D}$ satisfying $h(0) = 0$. Hence, by Schwarz's Lemma, $h(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi[$. But

$$e^{i\theta} = h'(0) = g'(f^{-1}(0))(f^{-1})'(0) = g'(z_0)\frac{1}{f'(z_0)} > 0.$$

Consequently $\theta = 0$. Hence $h(z) = z$ and so $f = g$.

Existence

Step 1 Reduction of the problem to a Cauchy domain $G := D_3$ with $D_3 \subseteq \mathbb{D}$ and $z_0 := 0 \in D_3$.

Let $a \in \mathbb{C} \setminus G$. Then $f(z) = z - a$ is zero-free in G. Now, due to Example **??**, G is a Cauchy domain. Hence, by Theorem 1.2, there is $g \in H(G)$ with $g^2(z) = z - a$ and $Z(g) = \emptyset$. Moreover, g is injective. Let $D_1 := g(G)$. Then D_1 is a Cauchy domain, too (Appendix 4.2).

Claim 1 D_1 is antisymetric; that is, if $w_0 \in G_1$, then $-w_0 \notin G_1$.

In fact, supposing the contrary, there are $z_1, z_2 \in G$ with $g(z_1) = w_0$ and $g(z_2) = -w_0$. Hence

$$z_1 - a = g^2(z_1) = g^2(z_2) = z_2 - a,$$

from which we deduce that $z_1 = z_2$ and so $w_0 = 0$; a contradiction to $Z(g) = \emptyset$.

Claim 2 D_1 has exterior points; this means that $\mathbb{C} \setminus \overline{D_1}$ has nonvoid interior.

To see this, let $w_0 \in D_1$ and let $r > 0$ be such that $D(w_0, r) \subseteq D_1$ (note that D_1 is open). Then $D(-w_0, r) \cap D_1 = \emptyset$ by the first claim.

Now consider the Möbius transform $S(z) = \dfrac{r}{z + w_0}$. This function maps the extended exterior of $D(-w_0, r)$ to $\mathbb{D}$; that is

$$S(\widehat{\mathbb{C}} \setminus D(-w_0, r)) = \mathbb{D} \text{ with } S(\infty) = 0.$$

Since $D_1 \subseteq \mathbb{C} \setminus D(-w_0, r)$, we see that $D_2 := S(D_1) \subseteq \mathbb{D}$. Finally, using a conformal selfmap T of $\mathbb{D}$ with $T \circ S \circ g(z_0) = 0$; for example $T(\xi) = (S(g(z_0)) - \xi)/(1 - \overline{S(g(z_0))}\xi)$, we obtain a conformal map $F := T \circ S \circ g$ of G onto a Cauchy domain $D_3 := F(G) \subseteq \mathbb{D}$ such that $F(z_0) = 0$.

Step 2 "Left-composing F to get a surjection ".

Consider the family

$$\mathcal{F} = \{f \in H(D_3) : f(D_3) \subseteq \mathbb{D}, f(0) = 0, f \text{ injective}\} \cup \{0\}.$$

Since $\mathcal{F}$ is uniformly bounded (by 1), we may apply Montel's normality criterion **??**, to deduce that $\mathcal{F}$ is a normal family. But $\mathcal{F}$ is also closed in the metric space $(H(D_3), d)$ (given in

Observation **??**), hence (sequentially) compact. In fact, if (f_n) is a sequence in $\mathcal{F}\setminus\{0\}$ converging locally uniformly to some $f \in H(D_3)$, then, by Hurwitz's Theorem **??**, f either is injective or constantly 0.

Since the map $L : \begin{cases} (H(D_3), d) & \to \mathbb{C} \\ f & \mapsto f'(0) \end{cases}$ is continuous, the compactness of $\mathcal{F}$ yields that $L(\mathcal{F})$ is a compact set in $\mathbb{C}$. Thus, there is $f_0 \in \mathcal{F}$ (called an extremal function) such that

$$|f_0'(0)| = \sup\{|f'(0)| : f \in \mathcal{F}\}.$$

Claim 3 f_0 is a conformal map of D_3 onto $\mathbb{D}$.

We first note that $|f_0'(0)| \geq 1$, because the identity map $f(z) = z$, $(z \in D_3 \subseteq \mathbb{D})$, satisfies $f'(0) = 1$ and belongs to $\mathcal{F}$. Thus $f_0 \not\equiv 0$ and so f_0 is injective.

To prove the surjectivity of f_0, we suppose that this is not the case. Then there is $b \in \mathbb{D} \setminus f_0(D_3)$. Let $S_b(\xi) = \frac{b-\xi}{1-\bar{b}\xi}$. Then the function $S_b \circ f_0 : D_3 \to \mathbb{D}$ is zero-free. Since D_3 is a Cauchy domain, we may apply Theorem 1.2 again to deduce that there is $q \in H(D_3)$ with $q^2 = S_b \circ f_0$. Note that $q(D_3) \subseteq \mathbb{D}$. The injectivity of $S_b \circ f_0$ implies that q is also injective. Let $c := q(0)$. Then the function $h := S_c \circ q \in H(D_3)$, $h(D_3) \subseteq \mathbb{D}$, and $h(0) = 0$. But h is also injective. Hence $h \in \mathcal{F}$. But $|h'(0)| > |f_0'(0)|$. In fact, if $\psi(w) = w^2$, then

$$f_0 = \underbrace{S_b^{-1} \circ \psi \circ S_c^{-1}}_{:=\varphi} \circ h,$$

and so

$$|f_0'(0)| = |\varphi'(0)|\,|h'(0)|. \tag{4.1}$$

Since $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\varphi(0) = 0$, we deduce from the Schwarz Lemma that $|\varphi'(0)| < 1$ (note that φ is not a rotation). Hence $|f_0'(0)| < |h'(0)|$. This is a contradiction though to the extremality of f_0. We conclude that f_0 must be surjective.

Finally, let $f_1 := f_0 \circ F$ and

$$f := \frac{|f_1'(0)|}{f_1'(0)} f_1.$$

Note that $f_1'(0) \neq 0$ because f_1 is a conformal map. Then f is the desired conformal map of G onto $\mathbb{D}$ with $f'(0) > 0$ and $f(0) = 0$. □

Theorem 4.4. *If D_1 and D_2 are two simply connected domains in $\mathbb{C}$ with $D_j \neq \mathbb{C}$, then there is a conformal map of D_1 onto D_2 (we say that D_1 is conformally equivalent to D_2).*

Proof. Let $f_j : \mathbb{D}_j \to \mathbb{D}$ be the Riemann maps associated with D_j. Then $f_2^{-1} \circ f_1$ is a conformal map of D_1 onto D_2. □

For the following observe that $\mathbb{D}$ is homeomorphic with $\mathbb{C}$ (just consider the map $h : \mathbb{C} \to \mathbb{D}$ given by $h(z) = z/(1+|z|)$ and its inverse $h^{-1}(w) = w/(1-|w|)$). Of course $\mathbb{C}$ is not conformally equivalent with $\mathbb{D}$ in view of Liouville's Theorem.

Theorem 4.5. *Let G be a domain in $\mathbb{C}$ with $G \neq \mathbb{C}$. The following assertions are equivalent:*

(1) G is conformally equivalent with $\mathbb{D}$.

(2) G is homeomorphic to $\mathbb{D}$.

(3) G is simply connected.

Proof. (1) $\Longrightarrow$ (2) This is obvious.

(2) $\Longrightarrow$ (3) By example **??**, G is simply connected (in the sense of Definition 4.1) if and only if it is a psc-space. But by Corollary **??**, being a psc-space is invariant under homeomorphisms. Hence G is simply connected as the homeomorphic image of $\mathbb{D}$.

(3) $\Longrightarrow$ (1) This is the Riemann mapping Theorem, Appendix 4.3. □

Theorem 4.6. Here is a conformal map of the unit disk onto the cusp domain

$$G = \{z \in \mathbb{C} : |z + 1/2| < 1/2\} \setminus \left(\left\{\left|z - \tfrac{i}{2}\right| \le \tfrac{1}{2}\right\} \cup \left\{\left|z + \tfrac{i}{2}\right| \le \tfrac{1}{2}\right\}\right).$$

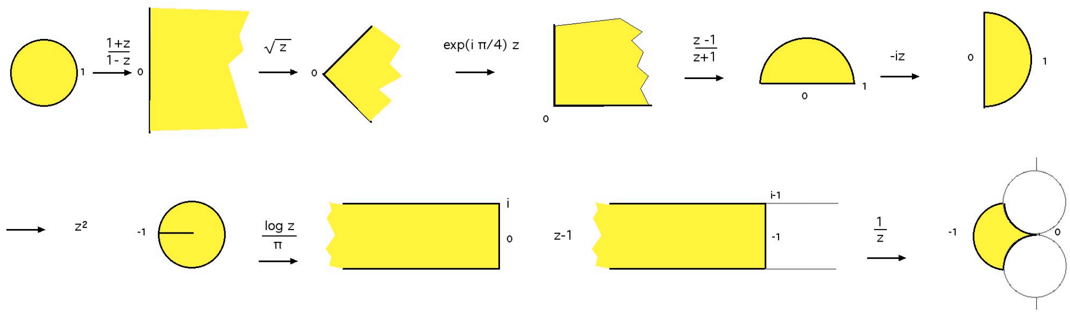


Figure 4.1: From the disk to a circular triangle

And here is a conformal map of the lunar crescent G to the unit disk

$$G = \mathbb{D} \setminus \{z \in \mathbb{C} : |z - 1/2| \le 1/2\}$$

$$f(z) = i \tan \frac{\pi}{2}\left(\frac{1+z}{1-z} - \frac{1}{2}\right).$$

4.2 Extensions of conformal maps

Theorem 4.7. *Let $f : \mathbb{D} \to G$ be a conformal map of the unit disk $\mathbb{D}$ onto the bounded domain $G \subseteq \mathbb{C}$. Then the following assertions are equivalent:*

(1) f has a continuous extension to $\overline{\mathbb{D}}$;

(2) ∂G is a curve; that is $\partial G = \{\phi(\xi) : \xi \in \partial\mathbb{D}\}$ with continuous ϕ;

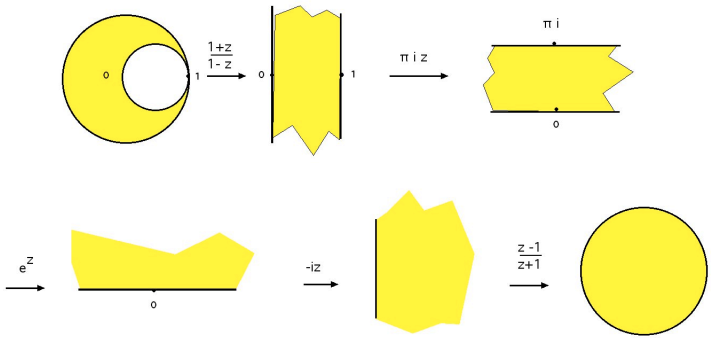


Figure 4.2: From the lunar crescent to the disk

(3) ∂G *is locally connected;*

(4) $\mathbb{C} \setminus G$ *is locally connected.*

Theorem 4.8 (Carathéodory)**.** *Let* $f : \mathbb{D} \to G$ *be a conformal map of the unit disk* $\mathbb{D}$ *onto the bounded domain* $G \subseteq \mathbb{C}$*. Then the following assertions are equivalent:*

(1) f *has a continuous injective extension to* $\overline{\mathbb{D}}$*;*

(2) ∂G *is a Jordan curve;*

(3) ∂G *is locally connected and has no cut-points.*