# Real symmetric extensions of invertible tuples of multivariable continuous functions

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Saarbrücken, Januar, 2010

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$F := (f, g) \in C(A) \times C(A)$ invertible, $A \subseteq \mathbb{R}$ compact;
$\exists$? invertible extension to $\mathbb{R}$?

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**Lemma**

*Let $K, L$ be two compact subsets in $\mathbb{R}^n$ with $K \subseteq L$. Then a continuous map $f : K \to \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ admits a continuous extension $F : L \to \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ if and only if $f/|f| : K \to S^{n-1}$ admits a continuous extension $F^* : L \to S^{n-1}$.*

## Theorem

*Let $K, L$ be two compact subsets in $\mathbb{R}^n$ with $K \subseteq L$. Then the following assertions are equivalent:*

1. *Every continuous map $f : K \to S^{n-1}$ admits a continuous extension $F : L \to S^{n-1}$,*
2. *Every invertible n-tuple of real-valued continuous functions on $K$ admits an extension to an invertible n-tuple of real-valued continuous functions on $L$,*
3. *Every component of $\mathbb{R}^n \setminus K$ contains a component of $\mathbb{R}^n \setminus L$.*

◂ symmetric case

## Corollary

*Let $K \subseteq \mathbb{C}^n$ be compact and suppose that $K$ has no holes. Then every invertible n-tuple $f = (f_1, \ldots, f_n)$ of real-valued (respectively complex-valued) continuous functions can be extended to an invertible n-tuple $F = (F_1, \ldots, F_n)$ of complex-valued continuous functions on $\mathbb{C}^n$.*

## Proposition

*Let $K, L$ be two compact subsets in $\mathbb{R}^n$ with $K \subseteq L$. Then every invertible n-tuple $f = (f_1, \ldots, f_n)$ of complex-valued continuous functions on $K$ admits an extension to an invertible n-tuple of complex-valued continuous functions on $L$.*

## Proposition

*Let $K, L$ be two compact subsets in $\mathbb{R}^n$ with $K \subseteq L$. Suppose that $m \geq n+1$. Then every invertible m-tuple $f = (f_1, \ldots, f_m)$ of real-valued continuous functions on $K$ admits an extension to an invertible m-tuple of real-valued continuous functions on L.*

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Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then $\overline{z} = (\overline{z}_1, \dots, \overline{z}_n)$, where $\overline{z}_j$ denotes the complex conjugate of $z_j \in \mathbb{C}$.

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A set $K$ in $\mathbb{C}^n$ is said to be *real symmetric* if $\overline{z} \in K$ whenever $z \in K$. If $X \subseteq \mathbb{C}^n$, then $X^* = \{z \in \mathbb{C}^n : \overline{z} \in X\}$. The set $\mathbb{R}^n$ of real tuples will be viewed as a subset of $\mathbb{C}^n$.

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A complex-valued function $f$ defined on a real symmetric set $K$ in $\mathbb{C}^n$ is said to be *real symmetric*, if $\overline{f(\overline{z})} = f(z)$ for any $z \in K$.

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If $K \subseteq \mathbb{C}^n$ is real symmetric and compact, then every function $f \in C(K)_{\rm sym}$ has a real symmetric extension to $\mathbb{C}^n$. Indeed, if $\phi$ is any continuous Tietze extension to $\mathbb{C}^n$, just put

$$F(z) = \big(\phi(z) + \overline{\phi(\overline{z})}\big)/2,$$

the symmetrization of $\phi$. Then $F$ is real symmetric.

## Theorem

*Let $K, L$ be two highly symmetric, compact subsets of $\mathbb{C}^n$ with $K \subseteq L$. Then the following assertions are equivalent:*

1. *Every map $f \in C_{\rm sym}(K, S^{2n-1})$ can be extended to a map $F \in C_{\rm sym}(L, S^{2n-1})$,*
2. *Every invertible n-tuple of functions in $C(K)_{\rm sym}$ can be extended to an invertible n-tuple of functions in $C(L)_{\rm sym}$,*
3. *Every component of $\mathbb{C}^n \setminus K$ contains a component of $\mathbb{C}^n \setminus L$* **and** *every component of $\mathbb{R}^n \setminus K$ contains a component of $\mathbb{R}^n \setminus L$.*

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To show this, let $\check{K} = \partial B \cup \partial B^*$. The real symmetric invertible $n$-tuple

$$f(z) = \begin{cases} z - a & \text{if } z \in \partial B, \\ z - \overline{a} & \text{if } z \in \partial B^* \end{cases}$$

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$$g(z) = \begin{cases} \hat{f}(z) & \text{on } \mathbb{C}^n \setminus (B \cup B^*), \\ z - a & \text{on } B, \\ z - \overline{a} & \text{on } B^*. \end{cases}$$

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Now

$$d(g,\Omega,0)=\sum_{\xi:g(\xi)=0}\operatorname{sgn}\det J_g(\xi),$$

where $J_g$ is the Jacobian of the map $g$ (note that $g$ is holomorphic around its zeros; that is in the balls $B$ and $B^*$). Thus $d(g,\Omega,0)=1$ if $a$ is in $\Omega$ and $\overline{a}\notin\Omega$, and $d(g,\Omega,0)=2$ if both $a$ and $\overline{a}$ are in $\Omega$. In any case $d(g,\Omega,0)>0$ and hence $d(F,\Omega,0)>0$. That implies that $F$ has a zero in $\Omega$; this is a contradiction because $F$ is invertible in $L\supseteq\Omega$. Hence $\Omega$ cannot be contained in $L$. Thus we have shown that each component $\Omega$ of $\mathbb{C}^n\setminus K$ contains a component of $\mathbb{C}^n\setminus L$.

# Covering dimension

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## Definition

Let $X$ be a topological space, $\mathscr{U}$ a system of open sets, $p \in X$. The order, $\operatorname{ord}_p \mathscr{U}$, of $\mathscr{U}$ at $p$ is the number of members of $\mathscr{U}$ that contain $p$. The order of $\mathscr{U}$ is given by $\operatorname{ord} U = \sup\{\operatorname{ord}_p \mathscr{U} : p \in X\}$. Let $n \in \mathbb{N}$. $X$ is said to have covering dimension ( or Čech-Lebesgue dimension) $\leq n$, $\dim X \leq n$, if for any finite open covering $\mathscr{U}$ of $X$ there exists an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that $\operatorname{ord} \mathscr{V} \leq n + 1$.

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If $A = C(X, \mathbb{K})$, (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), or $A = C(K)_{\rm sym}$, then we denote the set of all invertible $n$-tuples in $A$ by $U_n(A)$. Note that

$$U_n(A) = \{(f_1, \dots, f_n) \in A^n \mid \exists g = (g_1, \dots, g_n) \in A^n : \sum_{j=1}^{n} f_j g_j = 1\}.$$

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## Theorem

*Let $X$ be a normal space. Then the following assertions are equivalent:*

1. $0 \leq \dim X \leq n$;
2. *For every closed subspace $A$ of $X$ any continuous mapping $f : A \to S^n$ admits a continuous extension $F : X \to S^n$;*
3. *For every closed subspace $A$ of $X$ any invertible $(n+1)$-tuple $f \in U_{n+1}(C(A, \mathbb{R}))$ admits an extension to an invertible $(n+1)$-tuple $F \in U_{n+1}(C(X, \mathbb{R}))$.*

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The smallest integer $n$ for which every element in $U_{n+1}(A)$ is reducible is called the *Bass stable rank* of $A$ and is denoted by $\mathrm{bsr}(A)$. If no such integer exists, then $\mathrm{bsr}(A) = \infty$.

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A related concept is that of the *topological stable rank*, $\mathrm{tsr}(A)$, of $A$. This is the smallest integer $n$ such that $U_n(A)$ is dense in $A^n$. If no such $n$ exists, then $\mathrm{tsr}(A) = \infty$. It is well known that $\mathrm{bsr}(A) \leq \mathrm{tsr}(A)$.

**Theorem**

*The $(n+1)$-tuple $(f_1, \ldots, f_n, g)$ in $C(X)$ is reducible if and only if $(f_1, \ldots, f_n)$ admits an invertible extension from $Z(g)$ to $X$.*

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**Theorem (Vaserstein, Rieffel)**

*Let $X$ be a compact Hausdorff space. Then*

$$\operatorname{tsr}(C(X, \mathbb{C})) = \operatorname{bsr}(C(X, \mathbb{C})) = \left[\frac{\dim X}{2}\right] + 1$$

$$\operatorname{tsr}(C(X, \mathbb{R})) = \operatorname{bsr}(C(X, \mathbb{R})) = \dim X + 1.$$

## Theorem

*Let $X$ be a real symmetric, compact set in $\mathbb{C}^n$. Then*

$$\operatorname{bsr} C(X)_{\text{sym}} = \operatorname{tsr} C(X)_{\text{sym}} = \max\left\{\left[\frac{\dim X}{2}\right], \dim\,(X \cap \mathbb{R}^n)\right\} + 1.$$