On several different notions of stable ranks for algebras of holomorphic functions

Raymond Mortini
Université Paul Verlaine - Metz

Saarbrücken, August, 2009

Invertible tuples

Invertible tuples

A commutative unital ring, unit 1.
$f \in A$ invertible if $\exists g \in A : fg = 1$

Invertible tuples

A commutative unital ring, unit 1.
$f \in A$ invertible if $\exists g \in A : fg = 1$
$f = (f_1, \dots, f_n) \in A^n$ invertible if $\exists g \in A^n : fg = 1$, where
$fg = \sum_{j=1}^n f_j g_j$;

Invertible tuples

A commutative unital ring, unit 1.
$f \in A$ invertible if $\exists g \in A : fg = 1$
$f = (f_1, \dots, f_n) \in A^n$ invertible if $\exists g \in A^n : fg = 1$, where
$fg = \sum_{j=1}^n f_j g_j$;
$U_n(A)$ set of invertible n-tuples

Invertible tuples

A commutative unital ring, unit 1.
$f \in A$ invertible if $\exists g \in A : fg = 1$
$f = (f_1, \dots, f_n) \in A^n$ invertible if $\exists g \in A^n : fg = 1$, where $fg = \sum_{j=1}^n f_j g_j$;
$U_n(A)$ set of invertible n-tuples

Problem Characterize the invertible tuples

Invertible tuples

A commutative unital ring, unit 1.
$f \in A$ invertible if $\exists g \in A : fg = 1$
$f = (f_1, \dots, f_n) \in A^n$ invertible if $\exists g \in A^n : fg = 1$, where $fg = \sum_{j=1}^n f_j g_j$;
$U_n(A)$ set of invertible n-tuples

Problem Characterize the invertible tuples
a) $A = \mathbb{Z}: \quad (p_1, \dots, p_n) \in U_n(\mathbb{Z}) \longleftrightarrow \gcd(p_1, \dots, p_n) = 1$

Invertible tuples

A commutative unital ring, unit 1.
$f \in A$ invertible if $\exists g \in A : fg = 1$
$f = (f_1, \dots, f_n) \in A^n$ invertible if $\exists g \in A^n : fg = 1$, where
$fg = \sum_{j=1}^n f_j g_j$;
$U_n(A)$ set of invertible n-tuples

Problem Characterize the invertible tuples
a) $A = \mathbb{Z} : \quad (p_1, \dots, p_n) \in U_n(\mathbb{Z}) \longleftrightarrow \gcd(p_1, \dots, p_n) = 1$
b) $A = H(\Omega) : (f_1, \dots, f_n) \in U_n(A) \longleftrightarrow \bigcap_{j=1}^n Z_\Omega(f_j) = \emptyset$
(Wedderburn)

c) A commutative unital complex Banach algebra:

c) A commutative unital complex Banach algebra:

$$(f_1, \dots, f_n) \in U_n(A) \longleftrightarrow \bigcap_{j=1}^{n} Z(\widehat{f_j}) = \emptyset,$$

where $\widehat{f}$ is the Gelfand transform given by

$$\widehat{f} : m \mapsto m(f), m \in M(A)$$

and $M(A)$ the spectrum of A (=space of multiplicative linear functionals $\neq 0$ endowed with the weak-$*$-topology $\sigma(A^*, A)_{|M(A)}$).)

c) A commutative unital complex Banach algebra:

$$(f_1, \dots, f_n) \in U_n(A) \longleftrightarrow \bigcap_{j=1}^{n} Z(\widehat{f_j}) = \emptyset,$$

where $\widehat{f}$ is the Gelfand transform given by

$$\widehat{f} : m \mapsto m(f), m \in M(A)$$

and $M(A)$ the spectrum of A (=space of multiplicative linear functionals $\neq 0$ endowed with the weak-$*$-topology $\sigma(A^*, A)_{|M(A)}$).)

ex.: $A = C(X)$, X compact Hausdorff space, $M(A) = X$, via $x \in X \sim \Phi_x : f \mapsto f(x)$ point functional.

c1) $A = A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}), f$ holomorphic in $\mathbb{D}\}$:

$$(f_1, \ldots, f_n) \in U_n(A) \longleftrightarrow \bigcap_{j=1}^{n} Z(f_j) = \emptyset,$$

where $Z(f) = \{z \in \overline{\mathbb{D}} : f(z) = 0\}$, or equivalently:

$$(f_1, \ldots, f_n) \in U_n(A) \longleftrightarrow \sum_{j=1}^{n} |f_j| \geq \delta > 0.$$

c1) $A = A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}), f$ holomorphic in $\mathbb{D}\}$:

$$(f_1, \dots, f_n) \in U_n(A) \longleftrightarrow \bigcap_{j=1}^{n} Z(f_j) = \emptyset,$$

where $Z(f) = \{z \in \overline{\mathbb{D}} : f(z) = 0\}$, or equivalently:

$$(f_1, \dots, f_n) \in U_n(A) \longleftrightarrow \sum_{j=1}^{n} |f_j| \geq \delta > 0.$$

c2) <u>Corona-Theorem (Carleson)</u> $A = H^\infty(\mathbb{D})$:

$$(f_1, \dots, f_n) \in U_n(A) \longleftrightarrow \delta := \inf_{z \in \mathbb{D}} \sum_{j=1}^{n} |f_j(z)| > 0,$$

or in topological terms: **D** is dense in $M(H^\infty(\mathbb{D}))$, where $\mathbf{D} = \{\phi_a : a \in \mathbb{D}\}$ set of evaluation functionals $f \mapsto \phi_a(f) = f(a)$.

d1) $A = A(\mathbb{B}_N)$, ball,
$\mathbb{B}_N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \sum_{j=1}^N |z_j|^2 < 1\}$,
$M(A(\mathbb{B}_N)) = \overline{\mathbb{B}_N}$.

d1) $A = A(\mathbb{B}_N)$, ball,
$\mathbb{B}_N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \sum_{j=1}^N |z_j|^2 < 1\}$,
$M(A(\mathbb{B}_N)) = \overline{\mathbb{B}_N}$.

d2) $A = A(\mathbb{D}^N)$, polydisk,
$M(A(\mathbb{D}^N)) = \overline{\mathbb{D}^N}$.

d1) $A = A(\mathbb{B}_N)$, ball,
$\mathbb{B}_N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : \sum_{j=1}^N |z_j|^2 < 1\}$,
$M(A(\mathbb{B}_N)) = \overline{\mathbb{B}_N}$.

d2) $A = A(\mathbb{D}^N)$, polydisk,
$M(A(\mathbb{D}^N)) = \overline{\mathbb{D}^N}$.

two open Problems:
1) Is $\mathbb{B}_N$ dense in $M(H^\infty(\mathbb{B}_N))$?

d1) $A = A(\mathbb{B}_N)$, ball,
$\mathbb{B}_N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : \sum_{j=1}^N |z_j|^2 < 1\}$,
$M(A(\mathbb{B}_N)) = \overline{\mathbb{B}_N}$.

d2) $A = A(\mathbb{D}^N)$, polydisk,
$M(A(\mathbb{D}^N)) = \overline{\mathbb{D}^N}$.

two open Problems:
1) Is $\mathbb{B}_N$ dense in $M(H^\infty(\mathbb{B}_N))$?
2) Is $\mathbb{D}^N$ dense in $M(H^\infty(\mathbb{D}^N))$?

Topological stable rank

Topological stable rank

Question: For which BA A each n-tuple $(f_1, \dots, f_n) \in A^n$ can be approximated by invertible n-tuples? (property app_n.)

Topological stable rank

Question: For which BA A each n-tuple $(f_1, \dots, f_n) \in A^n$ can be approximated by invertible n-tuples? (property app_n.)

Def (M. Rieffel, 1983) The topological stable rank tsr(A) of A is the least integer n with property app_n. If no such integer exists, then $\mathrm{tsr}(A) = \infty$.

Topological stable rank

Question: For which BA A each n-tuple $(f_1, \dots, f_n) \in A^n$ can be approximated by invertible n-tuples? (property app_n.)

Def (M. Rieffel, 1983) The topological stable rank tsr(A) of A is the least integer n with property app_n. If no such integer exists, then $\mathrm{tsr}(A) = \infty$.

Examples

• $\mathrm{tsr}(C_{\mathbb{C}}([0, 1]))=1$
(Weierstrass; move real zeros of the approximating polynomial a little bit to the upper half plane).

Topological stable rank

Question: For which BA A each n-tuple $(f_1, \dots, f_n) \in A^n$ can be approximated by invertible n-tuples? (property app_n.)

Def (M. Rieffel, 1983) The topological stable rank tsr(A) of A is the least integer n with property app_n. If no such integer exists, then $\mathrm{tsr}(A) = \infty$.

Examples

• tsr($C_{\mathbb{C}}([0,1])$)=1
(Weierstrass; move real zeros of the approximating polynomial a little bit to the upper half plane).

• tsr($C_{\mathbb{R}}([0,1])$)=2

Proof: $f = x - 1/2$ cannot be uniformly approximated by invertibles, since $||f - u|| < 1/4$ implies $u(0) < 0$ and $u(1) > 0$. Now, by Weierstrass $||(f, g) - (p, q)|| < \varepsilon$; move the real zeros of q in common with those of p a little to the left so that at the end p and q have no real zeros in common.

In general (Rieffel)

- $\mathrm{tsr}(C_{\mathbb{R}}(X)) = \dim X + 1$ and $\mathrm{tsr}(C_{\mathbb{C}}(X)) = \lfloor \frac{\dim X}{2} \rfloor + 1$.

In general (Rieffel)

- $\mathrm{tsr}(C_{\mathbb{R}}(X)) = \mathrm{dim}X + 1$ and $\mathrm{tsr}(C_{\mathbb{C}}(X)) = \lfloor \frac{\mathrm{dim}X}{2} \rfloor + 1$.
- $\mathrm{tsr}(\mathrm{A}(\mathbb{D})) = 2$ (Rieffel)
- $\mathrm{tsr}(\mathrm{H}^{\infty}(\mathbb{D})) = 2$ (Suarez).

In general (Rieffel)

- $\mathrm{tsr}(C_{\mathbb{R}}(X)) = \mathrm{dim} X + 1$ and $\mathrm{tsr}(C_{\mathbb{C}}(X)) = \lfloor \frac{\mathrm{dim} X}{2} \rfloor + 1$.
- $\mathrm{tsr}(\mathrm{A}(\mathbb{D})) = 2$ (Rieffel)
- $\mathrm{tsr}(\mathrm{H}^{\infty}(\mathbb{D})) = 2$ (Suarez).

Unknown whether $\mathbb{D}$ can be replaced by $\mathbb{B}_n$ or $\mathbb{D}^n$ ($n > 1$).

In general (Rieffel)

- $\mathrm{tsr}(C_{\mathbb{R}}(X)) = \mathrm{dim} X + 1$ and $\mathrm{tsr}(C_{\mathbb{C}}(X)) = \lfloor \frac{\mathrm{dim} X}{2} \rfloor + 1$.
- $\mathrm{tsr}(\mathrm{A}(\mathbb{D})) = 2$ (Rieffel)
- $\mathrm{tsr}(\mathrm{H}^\infty(\mathbb{D})) = 2$ (Suarez).

Unknown whether $\mathbb{D}$ can be replaced by $\mathbb{B}_n$ or $\mathbb{D}^n$ $(n > 1)$.

Let $H^\infty_{\mathbb{R}}(\mathbb{D}) = \{f \in H^\infty : f(z) = \overline{f(\overline{z})}\}$. Then $\mathrm{tsr}(\mathrm{H}^\infty_{\mathbb{R}}(\mathbb{D})) = 2$ (Mortini-Wick).

Bass stable rank

Let $(f, g) \in U_2(A)$; then $\exists (x, y) \in A^2 : xf + yg = 1$.
Question: When x itself can be chosen to be invertible? Or in other words, when $\exists y \in A$ such that $x + yg \in U_1(A)$?

Bass stable rank

Let $(f,g) \in U_2(A)$; then $\exists (x,y) \in A^2 : xf + yg = 1$.
Question: When x itself can be chosen to be invertible? Or in other words, when $\exists y \in A$ such that $x + yg \in U_1(A)$?

More generally, let $(\underline{f}, g) := (f_1, \ldots, f_n, g) \in U_{n+1}(A)$. When does there exist $\underline{x} = (x_1, \ldots, x_n) \in A^n$ such that $(f_1 + x_1 g, \ldots, f_n + x_n g) \in U_n(A)$?

Bass stable rank

Let $(f, g) \in U_2(A)$; then $\exists (x, y) \in A^2 : xf + yg = 1$.
Question: When x itself can be chosen to be invertible? Or in other words, when $\exists y \in A$ such that $x + yg \in U_1(A)$?

More generally, let $(\underline{f}, g) := (f_1, \dots, f_n, g) \in U_{n+1}(A)$. When does there exist $\underline{x} = (x_1, \dots, x_n) \in A^n$ such that $(f_1 + x_1 g, \dots, f_n + x_n g) \in U_n(A)$?
In that case we call f reducible.

Bass stable rank

Let $(f, g) \in U_2(A)$; then $\exists (x, y) \in A^2 : xf + yg = 1$.
Question: When x itself can be chosen to be invertible? Or in other words, when $\exists y \in A$ such that $x + yg \in U_1(A)$?

More generally, let $(\underline{f}, g) := (f_1, \dots, f_n, g) \in U_{n+1}(A)$. When does there exist $\underline{x} = (x_1, \dots, x_n) \in A^n$ such that
$(f_1 + x_1 g, \dots, f_n + x_n g) \in U_n(A)$?
In that case we call f reducible.

Lemma If each $(f_1 \dots, f_n, g) \in U_{n+1}(A)$ is reducible, then each $(f_1, \dots, f_{n+1}, g) \in U_{n+2}(A)$ is reducible.

Def (Bass) Let A be a commutative unital ring. The least integer n for which every $(\underline{f}, g) \in U_{n+1}(A)$ is reducible, is called the Bass stable rank $\mathrm{bsr}(A)$ of A. If no such integer exists, then $\mathrm{bsr}(A) = \infty$.

Def (Bass) Let A be a commutative unital ring. The least integer n for which every $(\underline{f}, g) \in U_{n+1}(A)$ is reducible, is called the Bass stable rank $\mathrm{bsr}(A)$ of A. If no such integer exists, then $\mathrm{bsr}(A) = \infty$.

Proposition (Rieffel) Let A be BA. Then $\mathrm{bsr}(A) \leq \mathrm{tsr}(A)$.

Proof Let $(f_1, \ldots, f_n, h) \in U_{n+1}(A)$.

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$.

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $||u_j - x_j||_A < \varepsilon$.

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $||u_j - x_j||_A < \varepsilon$. Also $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because $(u_1, \dots, u_n)$ is invertible.

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $||u_j - x_j||_A < \varepsilon$. Also $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because $(u_1, \dots, u_n)$ is invertible. Hence

$$\sum_{j=1}^n u_j(f_j + h_j h) = \sum_{j=1}^n u_j f_j + xh$$

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $||u_j - x_j||_A < \varepsilon$. Also $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because $(u_1, \dots, u_n)$ is invertible. Hence

$$\sum_{j=1}^n u_j(f_j + h_j h) = \sum_{j=1}^n u_j f_j + xh$$

$$= (\sum_{j=1}^n x_j f_j + xh) + \sum_{j=1}^n (u_j - x_j) f_j = 1 + u,$$

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $||u_j - x_j||_A < \varepsilon$. Also $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because $(u_1, \dots, u_n)$ is invertible. Hence

$$\sum_{j=1}^n u_j(f_j + h_j h) = \sum_{j=1}^n u_j f_j + xh$$

$$= \Big(\sum_{j=1}^n x_j f_j + xh\Big) + \sum_{j=1}^n (u_j - x_j) f_j = 1 + u,$$

where we have defined $u := \sum_{j=1}^n (u_j - x_j) f_j$. Moreover, we have $||u||_A \leq \varepsilon \sum_{j=1}^n ||f_j||_A$.

Proof Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $||u_j - x_j||_A < \varepsilon$. Also $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because $(u_1, \dots, u_n)$ is invertible. Hence

$$\sum_{j=1}^n u_j(f_j + h_j h) = \sum_{j=1}^n u_j f_j + xh$$

$$= (\sum_{j=1}^n x_j f_j + xh) + \sum_{j=1}^n (u_j - x_j) f_j = 1 + u,$$

where we have defined $u := \sum_{j=1}^n (u_j - x_j) f_j$. Moreover, we have $||u||_A \leq \varepsilon \sum_{j=1}^n ||f_j||_A$. Hence for $\varepsilon > 0$ small enough, $1 + u$ is invertible in A, and so $(f_1 + h_1 h, \dots, f_n + h_n h) \in U_n(A)$.

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr}C(X) = \mathrm{tsr}C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr}A(\mathbb{D}) = 1$

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr} A(\mathbb{D}) = 1 \neq \mathrm{tsr} A(\mathbb{D}) = 2$,

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr}C(X) = \mathrm{tsr}C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr}A(\mathbb{D}) = 1 \neq \mathrm{tsr}A(\mathbb{D}) = 2$,
- (Corach, Suarez (1997); Mikkola, Sasane (2009))

$\mathrm{bsr}A(X) = \lfloor \frac{n}{2} \rfloor + 1 \neq \mathrm{tsr}A(X) = n + 1$, where $X = \mathbb{D}^n$ or $X = \mathbb{B}_n$

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr} A(\mathbb{D}) = 1 \neq \mathrm{tsr} A(\mathbb{D}) = 2$,
- (Corach, Suarez (1997); Mikkola, Sasane (2009))

$\mathrm{bsr} A(X) = \lfloor \frac{n}{2} \rfloor + 1 \neq \mathrm{tsr} A(X) = n + 1$, where $X = \mathbb{D}^n$ or $X = \mathbb{B}_n$

- (Treil; 1992) $\mathrm{bsr} H^\infty(\mathbb{D}) = 1$

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr} A(\mathbb{D}) = 1 \neq \mathrm{tsr} A(\mathbb{D}) = 2$,
- (Corach, Suarez (1997); Mikkola, Sasane (2009))

$\mathrm{bsr} A(X) = \lfloor \frac{n}{2} \rfloor + 1 \neq \mathrm{tsr} A(X) = n + 1$, where $X = \mathbb{D}^n$ or $X = \mathbb{B}_n$

- (Treil; 1992) $\mathrm{bsr} H^\infty(\mathbb{D}) = 1 \neq \mathrm{tsr} H^\infty(\mathbb{D}) = 2$,

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr} A(\mathbb{D}) = 1 \neq \mathrm{tsr} A(\mathbb{D}) = 2$,
- (Corach, Suarez (1997); Mikkola, Sasane (2009))

$\mathrm{bsr} A(X) = \lfloor \frac{n}{2} \rfloor + 1 \neq \mathrm{tsr} A(X) = n + 1$, where $X = \mathbb{D}^n$ or $X = \mathbb{B}_n$

- (Treil; 1992) $\mathrm{bsr} H^\infty(\mathbb{D}) = 1 \neq \mathrm{tsr} H^\infty(\mathbb{D}) = 2$,
- (Rupp, Sasane; >2008) $\mathrm{bsr} A_{\mathbb{R}}(\mathbb{D}) = \mathrm{tsr} A_{\mathbb{R}}(\mathbb{D}) = 2$,

Theorems:

- (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,
- (Jones, Marshall, Wolff; 1986) $\mathrm{bsr} A(\mathbb{D}) = 1 \neq \mathrm{tsr} A(\mathbb{D}) = 2$,
- (Corach, Suarez (1997); Mikkola, Sasane (2009))

$\mathrm{bsr} A(X) = \lfloor \frac{n}{2} \rfloor + 1 \neq \mathrm{tsr} A(X) = n + 1$, where $X = \mathbb{D}^n$ or $X = \mathbb{B}_n$

- (Treil; 1992) $\mathrm{bsr} H^\infty(\mathbb{D}) = 1 \neq \mathrm{tsr} H^\infty(\mathbb{D}) = 2$,
- (Rupp, Sasane; >2008) $\mathrm{bsr} A_{\mathbb{R}}(\mathbb{D}) = \mathrm{tsr} A_{\mathbb{R}}(\mathbb{D}) = 2$,
- (Mortini, Wick; >2008) $\mathrm{bsr} H^\infty_{\mathbb{R}}(\mathbb{D}) = \mathrm{tsr} H_{\mathbb{R}}(\mathbb{D}) = 2$,

Theorems:

• (Vaserstein 1971, Rieffel 1983) $\mathrm{bsr} C(X) = \mathrm{tsr} C(X)$,

• (Jones, Marshall, Wolff; 1986) $\mathrm{bsr} A(\mathbb{D}) = 1 \neq \mathrm{tsr} A(\mathbb{D}) = 2$,

• (Corach, Suarez (1997); Mikkola, Sasane (2009))
$\mathrm{bsr} A(X) = \lfloor \frac{n}{2} \rfloor + 1 \neq \mathrm{tsr} A(X) = n + 1$, where $X = \mathbb{D}^n$ or $X = \mathbb{B}_n$

• (Treil; 1992) $\mathrm{bsr} H^\infty(\mathbb{D}) = 1 \neq \mathrm{tsr} H^\infty(\mathbb{D}) = 2$,

• (Rupp, Sasane; >2008) $\mathrm{bsr} A_{\mathbb{R}}(\mathbb{D}) = \mathrm{tsr} A_{\mathbb{R}}(\mathbb{D}) = 2$,

• (Mortini, Wick; >2008) $\mathrm{bsr} H^\infty_{\mathbb{R}}(\mathbb{D}) = \mathrm{tsr} H_{\mathbb{R}}(\mathbb{D}) = 2$,

• (Suarez; 1996) $\mathrm{bsr}(AP) = \infty$, where AP is the uniform algebra of almost periodic functions on $\mathbb{R}$; AP is generated by $g(t) = \sum_{j=1}^n c_j e^{i\lambda_j t}$, $c_k \in \mathbb{C}, \lambda_j \in \mathbb{R}$.

Proof that $\mathrm{bsr} H^\infty_{\mathbb{R}}(\mathbb{D}) > 1$:

Proof that $\mathrm{bsr} H^\infty_{\mathbb{R}}(\mathbb{D}) > 1$:

$f(z) = z, g(z) = 1 - z^2$; suppose
$u(z) := z + h(1 - z^2) \in H^\infty_{\mathbb{R}}(\mathbb{D}))^{-1}$.

Proof that $\mathrm{bsr} H^\infty_{\mathbb{R}}(\mathbb{D}) > 1$:

$f(z) = z, g(z) = 1 - z^2$; suppose $u(z) := z + h(1 - z^2) \in H^\infty_{\mathbb{R}}(\mathbb{D}))^{-1}$. Then on $]-1, 1[$: $\lim_{x \to -1} u(x) = -1$ and $\lim_{x \to 1} u(x) = 1$. Hence $\exists x_0 \in]-1, 1[$ such that $u(x_0) = 0$. $\lightning$

Proof that $\mathrm{bsr} H^\infty_{\mathbb{R}}(\mathbb{D}) > 1$:

$f(z) = z, g(z) = 1 - z^2$; suppose $u(z) := z + h(1 - z^2) \in H^\infty_{\mathbb{R}}(\mathbb{D}))^{-1}$.Then on $]-1, 1[$: $\lim_{x \to -1} u(x) = -1$ and $\lim_{x \to 1} u(x) = 1$. Hence $\exists x_0 \in]-1, 1[$ such that $u(x_0) = 0$. ↯

For the proof that $\mathrm{bsr} A(\mathbb{D}) = 1$, we need the following Lemma of Corach/Suarez.

Lemma Let A be a commutative unital Banach algebra. Then, for $g \in A$, the set

$$R_n(g) = \{(f_1, \ldots, f_n) \in A^n : (f_1, \ldots, f_n, g) \text{ is reducible } \}$$

is open-closed inside

$$I_n(g) = \{(f_1, \ldots, f_n) \in A^n : (f_1, \ldots, f_n, g) \in U_{n+1}(A)\}.$$

In particular, for $n = 1$, if $\phi : [0, 1] \to I(g)$ is a continuous curve and $(\phi(0), g)$ is reducible, then $(\phi(1), g)$ is reducible.

(Proof that $\mathrm{bsr} A(\mathbb{D}) = 1$)

(Proof that $\mathrm{bsr} A(\mathbb{D}) = 1$)
Let $(f, g) \in U_2(A(\mathbb{D}))$. Approximating f by a polynomial p, it is sufficient to show reducibility of (p, g), where p and g have no common zeros in $\overline{\mathbb{D}}$.

(Proof that $\mathrm{bsr} A(\mathbb{D}) = 1$)
Let $(f, g) \in U_2(A(\mathbb{D}))$. Approximating f by a polynomial p, it is sufficient to show reducibility of (p, g), where p and g have no common zeros in $\overline{\mathbb{D}}$. Now $p(z) = \alpha \prod_j (z - a_j)$. Since the reducibility of (p_1, g) and (p_2, g)implies that of $(p_1 p_2, g)$, we need only consider $(z - a_j, g)$ with $a_j \in \overline{\mathbb{D}}$.

(Proof that $\mathrm{bsr} A(\mathbb{D}) = 1$)
Let $(f, g) \in U_2(A(\mathbb{D}))$. Approximating f by a polynomial p, it is sufficient to show reducibility of (p, g), where p and g have no common zeros in $\overline{\mathbb{D}}$. Now $p(z) = \alpha \prod_j (z - a_j)$. Since the reducibility of (p_1, g) and (p_2, g)implies that of $(p_1 p_2, g)$, we need only consider $(z - a_j, g)$ with $a_j \in \overline{\mathbb{D}}$.Let $\phi : [0, 1] \to I(g)$ be a continuous curve with $\phi(0) = a_j$ and $\phi(1) = 2$.

(Proof that $\mathrm{bsr} A(\mathbb{D}) = 1$)
Let $(f, g) \in U_2(A(\mathbb{D}))$. Approximating f by a polynomial p, it is sufficient to show reducibility of (p, g), where p and g have no common zeros in $\overline{\mathbb{D}}$. Now $p(z) = \alpha \prod_j (z - a_j)$. Since the reducibility of (p_1, g) and (p_2, g)implies that of $(p_1 p_2, g)$, we need only consider $(z - a_j, g)$ with $a_j \in \overline{\mathbb{D}}$.Let $\phi : [0, 1] \to I(g)$ be a continuous curve with $\phi(0) = a_j$ and $\phi(1) = 2$. Since $z - \phi(1)$ is invertible, the pair $(z - \phi(1), g)$ is reducible. Thus we get from the Lemma the reducibility of $(z - a_j, g)$.

Further notions of stable ranks

Let **B** be the class of all commutative unital Banach algebras over a field $\mathbb{K}$.

Further notions of stable ranks

Let **B** be the class of all commutative unital Banach algebras over a field $\mathbb{K}$. We will always assume that algebra homomorphisms, f, between members of **B** are continuous and satisfy $f(1_A) = 1_B$.

Further notions of stable ranks

Let **B** be the class of all commutative unital Banach algebras over a field $\mathbb{K}$. We will always assume that algebra homomorphisms, f, between members of **B** are continuous and satisfy $f(1_A) = 1_B$. Also, if $f : A \to B$ is an algebra homomorphism, then $\underline{f}$ will denote the associated map given by $\underline{f} : (a_1, \dots, a_n) \mapsto (f(a_1), \dots, f(a_n))$ from A^n to B^n.

Def The surjective stable rank $\mathrm{ssr}(A)$ of $A \in \mathbf{B}$ is the smallest integer n such that for every $B \in \mathbf{B}$ and every *surjective* algebra homomorphism $f : A \to B$ the induced map of $U_n(A) \to U_n(B)$ is surjective, too. Again, if there is no such n, then we write $\mathrm{ssr}(A) = \infty$.

Def The surjective stable rank $\mathrm{ssr}(A)$ of $A \in \mathbf{B}$ is the smallest integer n such that for every $B \in \mathbf{B}$ and every *surjective* algebra homomorphism $f : A \to B$ the induced map of $U_n(A) \to U_n(B)$ is surjective, too. Again, if there is no such n, then we write $\mathrm{ssr}(A) = \infty$.

Def The dense stable rank $\mathrm{dsr}(A)$ of $A \in \mathbf{B}$ is the smallest integer n such that for every $B \in \mathbf{B}$ and every algebra homomorphism $f : A \to B$ with *dense image* the induced map $U_n(A) \to U_n(B)$ has dense image. If there is no such n, we write $\mathrm{dsr}(A) = \infty$.

Theorem (Corach, Larotonda; Mortini-Wick) Let A be a commutative unital Banach algebra. The following assertions are equivalent:

1. $\text{bsr}(A) \leq n$;
2. $\underline{\pi}(U_n(A)) = U_n(A/I)$ for every closed ideal I in A;
3. $\underline{\pi}(U_n(A))$ is dense in $U_n(A/I)$ for every closed ideal I in A.

Here $\pi : A \to A/I$ is the canonical quotient mapping and $\underline{\pi}$ the associated map on A^n.

Theorem (Corach, Larotonda; Mortini-Wick)
If A is a commutative unital Banach algebra, then

$$\mathrm{bsr}(A) = \mathrm{ssr}(A) \leq \mathrm{dsr}(A) \leq \mathrm{tsr}(A).$$

The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem above.

The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem above. Indeed, let $n = \mathrm{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\mathrm{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$.

The assertion that $\text{bsr}(A) = \text{ssr}(A)$ follows from Theorem above. Indeed, let $n = \text{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\text{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem above, $m := \text{bsr}(A) \leq n$.

The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem above. Indeed, let $n = \mathrm{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\mathrm{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem above, $m := \mathrm{bsr}(A) \leq n$.To show that $n \leq m$, let $f : A \to B$ be a surjective homomorphism.

The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem above. Indeed, let $n = \mathrm{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\mathrm{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem above, $m := \mathrm{bsr}(A) \leq n$.To show that $n \leq m$, let $f : A \to B$ be a surjective homomorphism. Then the canonical injection $\check{f} : \tilde{A} = A/\mathrm{Ker} f \mapsto B$ is an algebra isomorphism and so $U_m(\tilde{A})$ is mapped onto $U_m(B)$ by $\underline{\check{f}}$.

The assertion that $\text{bsr}(A) = \text{ssr}(A)$ follows from Theorem above. Indeed, let $n = \text{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\text{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem above, $m := \text{bsr}(A) \leq n$.To show that $n \leq m$, let $f : A \to B$ be a surjective homomorphism. Then the canonical injection $\check{f} : \tilde{A} = A/\text{Ker} f \mapsto B$ is an algebra isomorphism and so $U_m(\tilde{A})$ is mapped onto $U_m(B)$ by $\underline{\check{f}}$.Since $m = \text{bsr}(A)$, by Theorem above, $\underline{\pi}(U_m(A)) = U_m(A/\text{Ker} f)$.

The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem above. Indeed, let $n = \mathrm{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\mathrm{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem above, $m := \mathrm{bsr}(A) \leq n$.To show that $n \leq m$, let $f : A \to B$ be a surjective homomorphism. Then the canonical injection $\check{f} : \tilde{A} = A/\mathrm{Ker} f \mapsto B$ is an algebra isomorphism and so $U_m(\tilde{A})$ is mapped onto $U_m(B)$ by $\underline{\check{f}}$.Since $m = \mathrm{bsr}(A)$, by Theorem above, $\underline{\pi}(U_m(A)) = U_m(A/\mathrm{Ker} f)$.Thus $\underline{f}(U_m(A)) = U_m(B)$. This means that $\mathrm{ssr}(A) \leq m$.

The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem above. Indeed, let $n = \mathrm{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\mathrm{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem above, $m := \mathrm{bsr}(A) \leq n$.To show that $n \leq m$, let $f : A \to B$ be a surjective homomorphism. Then the canonical injection $\check{f} : \tilde{A} = A/\mathrm{Ker} f \mapsto B$ is an algebra isomorphism and so $U_m(\tilde{A})$ is mapped onto $U_m(B)$ by $\underline{\check{f}}$.Since $m = \mathrm{bsr}(A)$, by Theorem above, $\underline{\pi}(U_m(A)) = U_m(A/\mathrm{Ker} f)$.Thus $\underline{f}(U_m(A)) = U_m(B)$. This means that $\mathrm{ssr}(A) \leq m$.All together we have shown that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$.

The absolute stable rank

Def (Magurn,van der Kallen, Vaserstein (1988))
Let A be a uniform algebra, $M(A)$ its spectrum and
$Z(f) = \{x \in M(A) : f(x) = 0\}$. Let $(f_1, \dots, f_n, g) \in A^{n+1}$.

The absolute stable rank

Def (Magurn,van der Kallen, Vaserstein (1988))
Let A be a uniform algebra, $M(A)$ its spectrum and $Z(f)=\{x\in M(A): f(x)=0\}$. Let $(f_1,\dots,f_n,g)\in A^{n+1}$.
Consider the property (Z):
$\exists (x_1,\dots,x_n)\in A^n$ such that $\bigcap_{j=1}^n Z(f_j+x_jg)\subseteq Z(g)$.

The absolute stable rank

Def (Magurn,van der Kallen, Vaserstein (1988))
Let A be a uniform algebra, $M(A)$ its spectrum and $Z(f) = \{x \in M(A) : f(x) = 0\}$. Let $(f_1, \dots, f_n, g) \in A^{n+1}$.
Consider the property (Z):
$\exists (x_1, \dots, x_n) \in A^n$ such that $\bigcap_{j=1}^n Z(f_j + x_j g) \subseteq Z(g)$.
Equivalently, the ideal generated by $\{f_j + x_j g : j = 1, \dots, n\}$ is either unproper or is contained only in maximal ideals that contain g.

The absolute stable rank

Def (Magurn,van der Kallen, Vaserstein (1988))
Let A be a uniform algebra, $M(A)$ its spectrum and
$Z(f) = \{x \in M(A) : f(x) = 0\}$. Let $(f_1, \dots, f_n, g) \in A^{n+1}$.
Consider the property (Z):
$\exists (x_1, \dots, x_n) \in A^n$ such that $\bigcap_{j=1}^n Z(f_j + x_j g) \subseteq Z(g)$.
Equivalently, the ideal generated by $\{f_j + x_j g : j = 1, \dots, n\}$ is either unproper or is contained only in maximal ideals that contain g.
The absolute stable rank $\mathrm{abs}(A)$ is the least integer n for which each (n+1)-tuple in A has property Z.

Theorem (Swan, Vaserstein (1986))
$\mathrm{abs}C(X) = \mathrm{bsr}C(X) = \mathrm{tsr}C(X)$.

Theorem (Swan, Vaserstein (1986))
$\mathrm{abs}C(X) = \mathrm{bsr}C(X) = \mathrm{tsr}C(X)$.

Observation Mortini
$\mathrm{abs}A(\mathbb{D}) \geq 2$ and $\mathrm{abs}H^\infty(\mathbb{D}) \geq 2$

Theorem (Swan, Vaserstein (1986))
$\mathrm{abs}C(X) = \mathrm{bsr}C(X) = \mathrm{tsr}C(X)$.

Observation Mortini
$\mathrm{abs}A(\mathbb{D}) \geq 2$ and $\mathrm{abs}H^\infty(\mathbb{D}) \geq 2$

Proof
b infinite interpolating Blaschke product with positive zeros; then
$(b(1-z), (1-z)^2) \in A(\mathbb{D})^2$ has not property (Z) since
$b(1-z) + h(1-z)^2$ has always infinite many zeros in $\mathbb{D}$.

Theorem (Swan, Vaserstein (1986))
$\mathrm{abs}C(X) = \mathrm{bsr}C(X) = \mathrm{tsr}C(X)$.

Observation Mortini
$\mathrm{abs}A(\mathbb{D}) \geq 2$ and $\mathrm{abs}H^\infty(\mathbb{D}) \geq 2$

Proof
b infinite interpolating Blaschke product with positive zeros; then
$(b(1-z), (1-z)^2) \in A(\mathbb{D})^2$ has not property (Z) since
$b(1-z) + h(1-z)^2$ has always infinite many zeros in $\mathbb{D}$.

Open Problem: determine $\mathrm{abs}A(\mathbb{D})$ and $\mathrm{abs}H^\infty(\mathbb{D})$.

Matricial stable ranks

Matricial stable ranks

Theorem (Vaserstein)
Let R be a commutative unital ring, and let A be the ring of matrices $R^{n\times n}$ with entries from R. Then

$$\operatorname{bsr} A = \left\lceil \frac{\operatorname{bsr} R - 1}{n} \right\rceil + 1,$$

Theorem (Mortini, Rupp, Sasane; 2009)
Let R be a normed commutative ring with identity having topological stable rank at most 2. Let $n \geq 2$ and $m \geq 1$. If $N \in R^{n \times n}$ and $D \in R^{m \times n}$, then given $\epsilon > 0$, there exist $\widetilde{N} \in R^{n \times n}$, $\widetilde{D} \in R^{m \times n}$, $X \in R^{n \times n}$ and $Y \in R^{n \times m}$ such that

$$\|D - \widetilde{D}\|_{op} + \|N - \widetilde{N}\|_{op} < \epsilon,$$

and

$$X\widetilde{N} + Y\widetilde{D} = I_n.$$

Moreover, if the Bass stable rank of R is 1, then X can be chosen to be invertible.