Two distinguished real Banach algebras

Raymond Mortini
Université Paul Verlaine - Metz

Saarbrücken, October 27, 2008

The real disk-algebra

Let $C_s(\mathbb{T})$ be the set of all (complex-valued) continuous functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ that are symmetric; this means that $\overline{f(e^{-it})} = f(e^{it})$.

The real disk-algebra

Let $C_s(\mathbb{T})$ be the set of all (complex-valued) continuous functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ that are symmetric; this means that $\overline{f(e^{-it})} = f(e^{it})$.Associated with $C_s(\mathbb{T})$ is the subset, A_s, of those functions in $C_s(\mathbb{T})$ that admit a holomorphic extension to the interior $\mathbb{D}$ of the unit disk.

The real disk-algebra

Let $C_s(\mathbb{T})$ be the set of all (complex-valued) continuous functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ that are symmetric; this means that $\overline{f(e^{-it})} = f(e^{it})$.Associated with $C_s(\mathbb{T})$ is the subset, A_s, of those functions in $C_s(\mathbb{T})$ that admit a holomorphic extension to the interior $\mathbb{D}$ of the unit disk.
The holomorphic extension to the disk is given by the Poisson-integral

$$P[f](z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta.$$

Note that for $z = re^{it}$ this coincides with the convolution

$$f * P_r(t) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) P_r(t - \theta) d\theta$$

of f with the Poisson-kernel $P_r(t) := \frac{1-r^2}{1+r^2-2r\cos t}$. Using Fourier representation, we see that the Fourier series of an element $f \in A_s$ formally writes as $\mathcal{F}[f] = \sum_{n=0}^{\infty} \hat{f}_n e^{int}$, where the Fourier coefficients, given as usual by the formula $\hat{f}_n := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt$, are *real* numbers.

of f with the Poisson-kernel $P_r(t) := \frac{1-r^2}{1+r^2-2r\cos t}$. Using Fourier representation, we see that the Fourier series of an element $f \in A_s$ formally writes as $\mathcal{F}[f] = \sum_{n=0}^{\infty} \hat{f}_n e^{int}$, where the Fourier coefficients, given as usual by the formula $\hat{f}_n := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt$, are *real* numbers.
It is standard knowledge that the Taylor-MacLaurin series for $P[f]$ writes as $P[f](z) = \sum_{n=0}^{\infty} \hat{f}_n z^n$.

When A_s is endowed with the supremum-topology, then A_s is isomorphically isometric to the real Banach algebra, $A_{\mathbb{R}}(\mathbb{D})$, of all holomorphic functions on the disk that are real on the interval $]-1,1[$ and that admit a continuous extension to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$.

When A_s is endowed with the supremum-topology, then A_s is isomorphically isometric to the real Banach algebra, $A_{\mathbb{R}}(\mathbb{D})$, of all holomorphic functions on the disk that are real on the interval $]-1,1[$ and that admit a continuous extension to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$.
Let $A(\mathbb{D})$ be the disk-algebra, that is the algebra of all functions continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. This is the most prominent example of a complex Banach algebra. Its relation with $A_{\mathbb{R}}(\mathbb{D})$ is given by the following:

$$A_{\mathbb{R}}(\mathbb{D}) = \{f \in A(\mathbb{D}) : \overline{f(\overline{z})} = f(z)\}.$$

By Gelfand-theory, the maximal ideals in a complex commutative unital Banach algebra are exactly the kernels of the $\mathbb{C}$-valued multiplicative linear functionals. Since the analytic polynomials are dense in $A(\mathbb{D})$ it follows from that theory that the $\mathbb{C}$-valued characters of $A(\mathbb{D})$ are exactly the point evaluations $\Phi_a : f \mapsto f(a)$ for some $a \in \overline{\mathbb{D}}$; hence each maximal ideal has the form $\{f \in A(\mathbb{D}) : f(a) = 0\}$.

Lemma

The set of rational functions of the form

$$\frac{\sum_{n=0}^{M} a_n z^n}{z^N}, \quad |z| = 1,$$

where $M, N \in \mathbb{N}$ and where a_n are real, is uniformly dense in $C_s(\mathbb{T})$.

$$M_a := \{f \in A_{\mathbb{R}}(\mathbb{D}) : f(a) = 0\}.$$

M_a is an ideal. Is M_a maximal?

Lemma

Let I be an ideal in $A_{\mathbb{R}}(\mathbb{D})$. Suppose that for some $a \in \mathbb{D}$ the elements $f, g \in I$ satisfy $f(a) = 0$ and $g(a) \neq 0$. Then the following assertions hold:

1. *If $-1 < a < 1$, then $\frac{f}{z-a} \in I$;*
2. *If $a \in \mathbb{D} \setminus [-1, 1]$, then $\frac{f}{(z-a)(z-\overline{a})} \in I$.*

Proof $f(a) = 0 \longrightarrow f(\overline{a}) = 0$.
$f(z)/(z-a) \in A_{\mathbb{R}}(\mathbb{D})$ whenever $a \in]-1, 1[$ and
$\frac{f}{(z-a)(z-\overline{a})} \in A_{\mathbb{R}}(\mathbb{D})$ whenever $a \in \mathbb{D} \setminus [-1, 1]$.

Proof $f(a)=0 \longrightarrow f(\overline{a})=0$.
$f(z)/(z-a) \in A_{\mathbb{R}}(\mathbb{D})$ whenever $a \in]-1,1[$ and
$\frac{f}{(z-a)(z-\overline{a})} \in A_{\mathbb{R}}(\mathbb{D})$ whenever $a \in \mathbb{D} \setminus [-1,1]$. Similarly,

$$\frac{g(z)-g(a)}{z-a} \in A_{\mathbb{R}}(\mathbb{D}), \quad a \in]-1,1[,$$

Proof $f(a)=0 \longrightarrow f(\overline{a})=0$.
$f(z)/(z-a) \in A_{\mathbb{R}}(\mathbb{D})$ whenever $a \in]-1,1[$ and
$\frac{f}{(z-a)(z-\overline{a})} \in A_{\mathbb{R}}(\mathbb{D})$ whenever $a \in \mathbb{D} \setminus [-1,1]$. Similarly,

$$\frac{g(z)-g(a)}{z-a} \in A_{\mathbb{R}}(\mathbb{D}), \quad a \in \,]-1,1[,$$

and

$$\frac{(g(z)-g(a))(g(z)-g(\overline{a}))}{(z-a)(z-\overline{a})} \in A_{\mathbb{R}}(\mathbb{D}), \quad a \in \mathbb{D} \setminus [-1,1].$$

Note also that $g(\overline{a})=\overline{g(a)}$.

Assertion (1) now follows from

$$\frac{f(z)}{z-a} = -\frac{1}{g(a)}\left(\frac{g(z)-g(a)}{z-a}\,f(z) - \frac{f(z)}{z-a}\,g(z)\right), \qquad (1)$$

Assertion (1) now follows from

$$\frac{f(z)}{z-a} = -\frac{1}{g(a)}\left(\frac{g(z)-g(a)}{z-a}\, f(z) - \frac{f(z)}{z-a}\, g(z)\right), \qquad (1)$$

and (2) follows from

$$\frac{f(z)}{(z-a)(z-\overline{a})} =$$

$$\frac{1}{|g(a)|^2}\left(\frac{(g(z)-g(a))(g(z)-g(\overline{a}))}{(z-a)(z-\overline{a})}\, f(z) - \frac{f(z)\Big(g(z)-(g(a)+g(\overline{a})\Big)}{(z-a)(z-\overline{a})}\, g(z)\right)$$

Theorem

An ideal M in $A_{\mathbb{R}}(\mathbb{D})$ is maximal if and only if $M = M_a$ for some $a \in \overline{\mathbb{D}}$.

Theorem

An ideal M in $A_{\mathbb{R}}(\mathbb{D})$ is maximal if and only if $M = M_a$ for some $a \in \overline{\mathbb{D}}$.

Proof

We first show that the ideals M_a are maximal.

Theorem

An ideal M in $A_{\mathbb{R}}(\mathbb{D})$ is maximal if and only if $M = M_a$ for some $a \in \overline{\mathbb{D}}$.

Proof

We first show that the ideals M_a are maximal. So suppose that $f \in A_{\mathbb{R}}(\mathbb{D})$ does not vanish at a. Then $\big(f - f(a)\big)\,\big(f - \overline{f(a)}\big) \in M(a)$ and

Theorem

An ideal M in $A_{\mathbb{R}}(\mathbb{D})$ is maximal if and only if $M = M_a$ for some $a \in \overline{\mathbb{D}}$.

Proof

We first show that the ideals M_a are maximal. So suppose that $f \in A_{\mathbb{R}}(\mathbb{D})$ does not vanish at a. Then $\left(f - f(a)\right)\left(f - \overline{f(a)}\right) \in M(a)$ and

$$1 = \frac{\left(f - f(a)\right)\left(f - \overline{f(a)}\right)}{|f(a)|^2} - f\,\frac{f - (f(a) + \overline{f(a)})}{|f(a)|^2}.$$

Theorem

An ideal M in $A_{\mathbb{R}}(\mathbb{D})$ is maximal if and only if $M = M_a$ for some $a \in \overline{\mathbb{D}}$.

Proof

We first show that the ideals M_a are maximal. So suppose that $f \in A_{\mathbb{R}}(\mathbb{D})$ does not vanish at a. Then $\left(f - f(a)\right)\left(f - \overline{f(a)}\right) \in M(a)$ and

$$1 = \frac{\left(f - f(a)\right)\left(f - \overline{f(a)}\right)}{|f(a)|^2} - f\,\frac{f - (f(a) + \overline{f(a)})}{|f(a)|^2}.$$

Hence the ideal, $[M_a, f]$, generated by M_a and f is the whole algebra and so M_a is maximal.

Next we show that every maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ has this form. First we note that a function $f \in A_{\mathbb{R}}(\mathbb{D})$ is invertible in $A_{\mathbb{R}}(\mathbb{D})$ if and only if f does not vanish on $\overline{\mathbb{D}}$.

Next we show that every maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ has this form. First we note that a function $f \in A_{\mathbb{R}}(\mathbb{D})$ is invertible in $A_{\mathbb{R}}(\mathbb{D})$ if and only if f does not vanish on $\overline{\mathbb{D}}$. Now let M be a maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ and suppose that M is not contained in any ideal of the form M_a.

Next we show that every maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ has this form. First we note that a function $f \in A_{\mathbb{R}}(\mathbb{D})$ is invertible in $A_{\mathbb{R}}(\mathbb{D})$ if and only if f does not vanish on $\overline{\mathbb{D}}$. Now let M be a maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ and suppose that M is not contained in any ideal of the form M_a. Then for every $a \in \overline{\mathbb{D}}$ there exists $f_a \in M$ such that $f_a(a) \neq 0$.

Next we show that every maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ has this form. First we note that a function $f \in A_{\mathbb{R}}(\mathbb{D})$ is invertible in $A_{\mathbb{R}}(\mathbb{D})$ if and only if f does not vanish on $\overline{\mathbb{D}}$. Now let M be a maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ and suppose that M is not contained in any ideal of the form M_a. Then for every $a \in \overline{\mathbb{D}}$ there exists $f_a \in M$ such that $f_a(a) \neq 0$. By a compactness argument, this shows that there are finitely many functions $f_j \in M$ such that

$$\delta := \min_{z \in \overline{\mathbb{D}}} \sum_{j=1}^{n} |f_j(z)|^2 > 0. \tag{2}$$

Next we show that every maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ has this form. First we note that a function $f \in A_{\mathbb{R}}(\mathbb{D})$ is invertible in $A_{\mathbb{R}}(\mathbb{D})$ if and only if f does not vanish on $\overline{\mathbb{D}}$. Now let M be a maximal ideal in $A_{\mathbb{R}}(\mathbb{D})$ and suppose that M is not contained in any ideal of the form M_a. Then for every $a \in \overline{\mathbb{D}}$ there exists $f_a \in M$ such that $f_a(a) \neq 0$. By a compactness argument, this shows that there are finitely many functions $f_j \in M$ such that

$$\delta := \min_{z \in \overline{\mathbb{D}}} \sum_{j=1}^{n} |f_j(z)|^2 > 0. \tag{2}$$

We are going to show that $1 = \sum_{j=1}^{n} g_j f_j$ for some $g_j \in A_{\mathbb{R}}(\mathbb{D})$, contradicting the fact that M is a proper ideal.

Let

$$q_k = \frac{\overline{f_k}}{\sum_{j=1}^{n} |f_j|^2}, k = 1, \ldots, n.$$

Let

$$q_k = \frac{\overline{f_k}}{\sum_{j=1}^{n} |f_j|^2}, k = 1, \ldots, n.$$

Then $\sum_{k=1}^{n} q_k f_k = 1$. This equality holds on $\overline{\mathbb{D}}$.

Let

$$q_k = \frac{\overline{f_k}}{\sum_{j=1}^{n} |f_j|^2}, k = 1, \ldots, n.$$

Then $\sum_{k=1}^{n} q_k f_k = 1$. This equality holds on $\overline{\mathbb{D}}$.In order to get a solution $(g_1, \ldots, g_n) \in (A_{\mathbb{R}}(\mathbb{D}))^n$ of our Bezout equation $\sum_{k=1}^{n} g_k f_k$, we switch to the boundary $\mathbb{T}$ of $\mathbb{D}$.

Let

$$q_k = \frac{\overline{f_k}}{\sum_{j=1}^{n} |f_j|^2}, k = 1, \dots, n.$$

Then $\sum_{k=1}^{n} q_k f_k = 1$. This equality holds on $\overline{\mathbb{D}}$.In order to get a solution $(g_1, \dots, g_n) \in (A_{\mathbb{R}}(\mathbb{D}))^n$ of our Bezout equation $\sum_{k=1}^{n} g_k f_k$, we switch to the boundary $\mathbb{T}$ of $\mathbb{D}$.Since $q_k \in C_s(\mathbb{T})$, there exist by Lemma 1 rational functions $r_k(z) = \frac{\sum_{j=0}^{M} a_j z^j}{z^N}$, $a_j \in \mathbb{R}$, such that on $\mathbb{T}$

$$||q_k - r_k||_\infty < \frac{1}{2} \frac{1}{\sum_{j=1}^{n} ||f_j||_\infty}.$$

Let

$$q_k = \frac{\overline{f_k}}{\sum_{j=1}^n |f_j|^2}, k = 1, \ldots, n.$$

Then $\sum_{k=1}^n q_k f_k = 1$. This equality holds on $\overline{\mathbb{D}}$.In order to get a solution $(g_1, \ldots, g_n) \in (A_{\mathbb{R}}(\mathbb{D}))^n$ of our Bezout equation $\sum_{k=1}^n g_k f_k$, we switch to the boundary $\mathbb{T}$ of $\mathbb{D}$.Since $q_k \in C_s(\mathbb{T})$, there exist by Lemma 1 rational functions $r_k(z) = \frac{\sum_{j=0}^M a_j z^j}{z^N}$, $a_j \in \mathbb{R}$, such that on $\mathbb{T}$

$$||q_k - r_k||_\infty < \frac{1}{2} \frac{1}{\sum_{j=1}^n ||f_j||_\infty}.$$

Note that N and M can be chosen to be independent of k (just by adding, if necessary, 0 coefficients).

Thus, on $\mathbb{T}$,

$$\left|\sum_{k=1}^{n} z^N r_k f_k\right| = \left|\sum_{k=1}^{n} r_k f_k\right| \geq$$

Thus, on $\mathbb{T}$,

$$|\sum_{k=1}^{n} z^N r_k f_k| = |\sum_{k=1}^{n} r_k f_k| \geq$$

$$|\sum_{k=1}^{n} q_k f_k| - \sum_{k=1}^{n} ||q_k - r_k||_\infty \, ||f_k||_\infty \geq 1 - 1/2 = 1/2.$$

Thus, on $\mathbb{T}$,

$$|\sum_{k=1}^{n} z^N r_k f_k| = |\sum_{k=1}^{n} r_k f_k| \geq$$

$$|\sum_{k=1}^{n} q_k f_k| - \sum_{k=1}^{n} ||q_k - r_k||_\infty \, ||f_k||_\infty \geq 1 - 1/2 = 1/2.$$

Hence the function $q := \sum_{k=1}^{n} (z^N r_k) f_k$ has no zeros on $\mathbb{T}$.

Thus, on $\mathbb{T}$,

$$|\sum_{k=1}^{n} z^N r_k f_k| = |\sum_{k=1}^{n} r_k f_k| \geq$$

$$|\sum_{k=1}^{n} q_k f_k| - \sum_{k=1}^{n} ||q_k - r_k||_\infty \, ||f_k||_\infty \geq 1 - 1/2 = 1/2.$$

Hence the function $q := \sum_{k=1}^{n}(z^N r_k) f_k$ has no zeros on $\mathbb{T}$.Moreover, since $z^N r_k$ is a polynomial with real coefficients, it belongs to $A_{\mathbb{R}}(\mathbb{D})$. Thus $q \in A_{\mathbb{R}}(\mathbb{D})$.

Thus, on $\mathbb{T}$,

$$|\sum_{k=1}^{n} z^N r_k f_k| = |\sum_{k=1}^{n} r_k f_k| \geq$$

$$|\sum_{k=1}^{n} q_k f_k| - \sum_{k=1}^{n} ||q_k - r_k||_\infty \, ||f_k||_\infty \geq 1 - 1/2 = 1/2.$$

Hence the function $q := \sum_{k=1}^{n}(z^N r_k) f_k$ has no zeros on $\mathbb{T}$.Moreover, since $z^N r_k$ is a polynomial with real coefficients, it belongs to $A_{\mathbb{R}}(\mathbb{D})$. Thus $q \in A_{\mathbb{R}}(\mathbb{D})$. Moreover, $q \in I(f_1, \dots, f_n)$, the ideal generated by the f_j in $A_{\mathbb{R}}(\mathbb{D})$.

Thus, on $\mathbb{T}$,

$$|\sum_{k=1}^{n} z^N r_k f_k| = |\sum_{k=1}^{n} r_k f_k| \geq$$

$$|\sum_{k=1}^{n} q_k f_k| - \sum_{k=1}^{n} ||q_k - r_k||_\infty \, ||f_k||_\infty \geq 1 - 1/2 = 1/2.$$

Hence the function $q := \sum_{k=1}^{n}(z^N r_k)f_k$ has no zeros on $\mathbb{T}$.Moreover, since $z^N r_k$ is a polynomial with real coefficients, it belongs to $A_{\mathbb{R}}(\mathbb{D})$. Thus $q \in A_{\mathbb{R}}(\mathbb{D})$. Moreover, $q \in I(f_1, \dots, f_n)$, the ideal generated by the f_j in $A_{\mathbb{R}}(\mathbb{D})$.By analyticity, q has only finitely many zeros in $\mathbb{D}$. The symmetry of the functions in $A_{\mathbb{R}}(\mathbb{D})$ implies that these zeros are symmetric with respect to the real axis.

Using Lemma 2 we shall now divide out these zeros (taking pairs $(\xi, \overline{\xi})$ whenever the zero ξ is not real) without leaving the ideal $I(f_1, \dots, f_n)$.

Using Lemma 2 we shall now divide out these zeros (taking pairs $(\xi, \overline{\xi})$ whenever the zero ξ is not real) without leaving the ideal $I(f_1, \dots, f_n)$.(Note that for each $a \in \overline{\mathbb{D}}$ there is $g \in \{f_1, \dots, f_n\}$ such that $g(a) \neq 0$.)

Using Lemma 2 we shall now divide out these zeros (taking pairs $(\xi, \overline{\xi})$ whenever the zero ξ is not real) without leaving the ideal $I(f_1, \dots, f_n)$.(Note that for each $a \in \overline{\mathbb{D}}$ there is $g \in \{f_1, \dots, f_n\}$ such that $g(a) \neq 0$.)At the end, after finitely many steps, we obtain a function $f \in I(f_1, \dots, f_n)$ that dose not have any zeros in $\overline{\mathbb{D}}$.

Using Lemma 2 we shall now divide out these zeros (taking pairs $(\xi, \overline{\xi})$ whenever the zero ξ is not real) without leaving the ideal $I(f_1, \dots, f_n)$.(Note that for each $a \in \overline{\mathbb{D}}$ there is $g \in \{f_1, \dots, f_n\}$ such that $g(a) \neq 0$.)At the end, after finitely many steps, we obtain a function $f \in I(f_1, \dots, f_n)$ that dose not have any zeros in $\overline{\mathbb{D}}$. Hence f is invertible in $A_{\mathbb{R}}(\mathbb{D})$ and so $1 = f(f^{-1}) \in I(f_1, \dots, f_n) \subseteq M$.

Using Lemma 2 we shall now divide out these zeros (taking pairs $(\xi, \overline{\xi})$ whenever the zero ξ is not real) without leaving the ideal $I(f_1, \dots, f_n)$.(Note that for each $a \in \overline{\mathbb{D}}$ there is $g \in \{f_1, \dots, f_n\}$ such that $g(a) \neq 0$.)At the end, after finitely many steps, we obtain a function $f \in I(f_1, \dots, f_n)$ that dose not have any zeros in $\overline{\mathbb{D}}$. Hence f is invertible in $A_{\mathbb{R}}(\mathbb{D})$ and so $1 = f(f^{-1}) \in I(f_1, \dots, f_n) \subseteq M$. This contradiction shows that M actually *is* contained in some M_a. Since M is maximal, $M = M_a$.

Whereas in complex (commutative unital) Banach algebras maximal ideals always have co-dimension 1, we are faced here with a different situation.

Whereas in complex (commutative unital) Banach algebras maximal ideals always have co-dimension 1, we are faced here with a different situation.

Theorem

Let $a \in \overline{\mathbb{D}}$. *Then*

1. M_a *has co-dimension 1 (in the real vector space* $A_{\mathbb{R}}(\mathbb{D})$*) if and only if* $a \in [-1, 1]$.
2. M_a *has co-dimension 2 if and only if* $a \in \overline{\mathbb{D}} \setminus [-1, 1]$.

Proof

For (1), let $a \in [-1, 1]$ and let $f \in A_{\mathbb{R}}(\mathbb{D})$. Then $f(a)$ is real and so

$$f = (f - f(a)) + f(a) \cdot 1 \in \text{vect}[M_a, 1].$$

Proof

For (1), let $a \in [-1,1]$ and let $f \in A_{\mathbb{R}}(\mathbb{D})$. Then $f(a)$ is real and so

$$f = (f - f(a)) + f(a) \cdot 1 \in \text{vect}[M_a, 1].$$

For (2), let $a \in \overline{\mathbb{D}} \setminus [-1,1]$. We claim that $A_{\mathbb{R}}(\mathbb{D}) = \text{vect}[M_a, 1, z]$.

Proof

For (1), let $a \in [-1, 1]$ and let $f \in A_{\mathbb{R}}(\mathbb{D})$. Then $f(a)$ is real and so

$$f = (f - f(a)) + f(a) \cdot 1 \in \text{vect}[M_a, 1].$$

For (2), let $a \in \overline{\mathbb{D}} \setminus [-1, 1]$. We claim that $A_{\mathbb{R}}(\mathbb{D}) = \text{vect}[M_a, 1, z]$.Let $f \in A_{\mathbb{R}}(\mathbb{D})$. Since $\{1, a\}$ is a Hamel base of the real vector space $\mathbb{C}$ over $\mathbb{R}$, every $f(a) \in \mathbb{C}$ writes as $f(a) = \sigma \cdot 1 + \beta \cdot a$ where $\sigma, \beta \in \mathbb{R}$.

Proof

For (1), let $a \in [-1, 1]$ and let $f \in A_{\mathbb{R}}(\mathbb{D})$. Then $f(a)$ is real and so

$$f = (f - f(a)) + f(a) \cdot 1 \in \text{vect}[M_a, 1].$$

For (2), let $a \in \overline{\mathbb{D}} \setminus [-1, 1]$. We claim that $A_{\mathbb{R}}(\mathbb{D}) = \text{vect}[M_a, 1, z]$.Let $f \in A_{\mathbb{R}}(\mathbb{D})$. Since $\{1, a\}$ is a Hamel base of the real vector space $\mathbb{C}$ over $\mathbb{R}$, every $f(a) \in \mathbb{C}$ writes as $f(a) = \sigma \cdot 1 + \beta \cdot a$ where $\sigma, \beta \in \mathbb{R}$.Hence $f - (\sigma + \beta z) \in M_a$ and so

$$f(z) = \big(f(z) - (\sigma + \beta z)\big) + \sigma + \beta z \in \text{vect}[M_a, 1, z].$$

Next we shall determine the (non-zero) multiplicative linear functionals ϕ on $A_{\mathbb{R}}(\mathbb{D})$; we have to distinguish two cases:

Next we shall determine the (non-zero) multiplicative linear functionals ϕ on $A_{\mathbb{R}}(\mathbb{D})$; we have to distinguish two cases:either the target space of ϕ is the field of reals or it is the real division algebra $\mathbb{C}$, regarded as a vector space over $\mathbb{R}$. We denote it by ${}^{\mathbb{R}}\mathbb{C}$.

Theorem

(1) *The only multiplicative* $\mathbb{R}$*-linear functionals* $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{R}$ *on* $A_{\mathbb{R}}(\mathbb{D})$ *are given by* $\phi(f) = f(a)$*, where* $a \in [-1, 1]$*.*

Theorem

(1) *The only multiplicative* $\mathbb{R}$*-linear functionals* $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{R}$ *on* $A_{\mathbb{R}}(\mathbb{D})$ *are given by* $\phi(f) = f(a)$*, where* $a \in [-1, 1]$*. Their kernels are the maximal ideals* M_a *that have co-dimension 1 in the real vector space* $A_{\mathbb{R}}(\mathbb{D})$*.*

Theorem

(1) *The only multiplicative* $\mathbb{R}$*-linear functionals* $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{R}$ *on* $A_{\mathbb{R}}(\mathbb{D})$ *are given by* $\phi(f) = f(a)$*, where* $a \in [-1, 1]$*. Their kernels are the maximal ideals* M_a *that have co-dimension 1 in the real vector space* $A_{\mathbb{R}}(\mathbb{D})$*.*

(2) *The remaining* $\mathbb{R}$*-linear multiplicative functionals have target space* ${}^{\mathbb{R}}\mathbb{C}$ *and are given by* $\phi(f) = f(a)$ *or* $\phi(f) = \overline{f(a)}$*, where* $a \in \overline{\mathbb{D}} \setminus [-1, 1]$*.*

Theorem

(1) *The only multiplicative* $\mathbb{R}$*-linear functionals* $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{R}$ *on* $A_{\mathbb{R}}(\mathbb{D})$ *are given by* $\phi(f) = f(a)$*, where* $a \in [-1, 1]$*. Their kernels are the maximal ideals* M_a *that have co-dimension 1 in the real vector space* $A_{\mathbb{R}}(\mathbb{D})$*.*

(2) *The remaining* $\mathbb{R}$*-linear multiplicative functionals have target space* ${}^{\mathbb{R}}\mathbb{C}$ *and are given by* $\phi(f) = f(a)$ *or* $\phi(f) = \overline{f(a)}$*, where* $a \in \overline{\mathbb{D}} \setminus [-1, 1]$*. Their kernels are the maximal ideals* M_a *that have co-dimension 2 in the real vector space* $A_{\mathbb{R}}(\mathbb{D})$*.*

Proof

First we note that the kernel of any multiplicative $\mathbb{R}$-linear functional $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{K}$ is a maximal ideal (here $\mathbb{K}$ is either $\mathbb{R}$ or ${}^{\mathbb{R}}\mathbb{C}$).

Proof

First we note that the kernel of any multiplicative $\mathbb{R}$-linear functional $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{K}$ is a maximal ideal (here $\mathbb{K}$ is either $\mathbb{R}$ or $^{\mathbb{R}}\mathbb{C}$).Indeed, let $I = \ker \phi$. Then I is easily seen to be an ideal.

Proof

First we note that the kernel of any multiplicative $\mathbb{R}$-linear functional $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{K}$ is a maximal ideal (here $\mathbb{K}$ is either $\mathbb{R}$ or $^{\mathbb{R}}\mathbb{C}$).Indeed, let $I = \ker \phi$. Then I is easily seen to be an ideal.Now let $f \in A_{\mathbb{R}}(\mathbb{D}) \setminus I$. Then $\phi(f) \neq 0$. Moreover, $\phi(1) = 1$.

Proof

First we note that the kernel of any multiplicative $\mathbb{R}$-linear functional $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{K}$ is a maximal ideal (here $\mathbb{K}$ is either $\mathbb{R}$ or ${}^{\mathbb{R}}\mathbb{C}$).Indeed, let $I = \ker\phi$. Then I is easily seen to be an ideal.Now let $f \in A_{\mathbb{R}}(\mathbb{D}) \setminus I$. Then $\phi(f) \neq 0$. Moreover, $\phi(1) = 1$.Using the identity $z^2 - (2\,\mathrm{Re}\, z)z + |z|^2 = 0$ for any $z \in \mathbb{C}$, we see that

$$\phi\Big((f - \phi(f))\,\big(f - \overline{\phi(f)}\big)\Big) = \phi(f^2) - (2\,\mathrm{Re}\,\phi(f))\phi(f) + |\phi(f)|^2 = 0.$$

Proof

First we note that the kernel of any multiplicative $\mathbb{R}$-linear functional $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{K}$ is a maximal ideal (here $\mathbb{K}$ is either $\mathbb{R}$ or ${}^{\mathbb{R}}\mathbb{C}$).Indeed, let $I = \ker \phi$. Then I is easily seen to be an ideal.Now let $f \in A_{\mathbb{R}}(\mathbb{D}) \setminus I$. Then $\phi(f) \neq 0$. Moreover, $\phi(1) = 1$.Using the identity $z^2 - (2\,\mathrm{Re}\,z)z + |z|^2 = 0$ for any $z \in \mathbb{C}$, we see that

$$\phi\big((f - \phi(f))\,(f - \overline{\phi(f)})\big) = \phi(f^2) - (2\,\mathrm{Re}\,\phi(f))\phi(f) + |\phi(f)|^2 = 0.$$

Thus we may conclude from

$$1 = \frac{(f - \phi(f))\ (f - \overline{\phi(f)})}{|\phi(f)|^2} - f\ \frac{f - (\phi(f) + \overline{\phi(f)})}{|\phi(f)|^2};$$

Proof

First we note that the kernel of any multiplicative $\mathbb{R}$-linear functional $\phi : A_{\mathbb{R}}(\mathbb{D}) \to \mathbb{K}$ is a maximal ideal (here $\mathbb{K}$ is either $\mathbb{R}$ or ${}^{\mathbb{R}}\mathbb{C}$).Indeed, let $I = \ker\phi$. Then I is easily seen to be an ideal.Now let $f \in A_{\mathbb{R}}(\mathbb{D}) \setminus I$. Then $\phi(f) \neq 0$. Moreover, $\phi(1) = 1$.Using the identity $z^2 - (2\,\mathrm{Re}\,z)z + |z|^2 = 0$ for any $z \in \mathbb{C}$, we see that

$$\phi\big((f - \phi(f))\,(f - \overline{\phi(f)})\big) = \phi(f^2) - (2\,\mathrm{Re}\,\phi(f))\phi(f) + |\phi(f)|^2 = 0.$$

Thus we may conclude from

$$1 = \frac{(f - \phi(f))\ (f - \overline{\phi(f)})}{|\phi(f)|^2} - f\ \frac{f - (\phi(f) + \overline{\phi(f)})}{|\phi(f)|^2};$$

that the ideal generated by I and f is the whole algebra. Hence I is maximal.

Thus, by Theorem 3, ker $\phi = M_a$ for some $a \in \overline{\mathbb{D}}$.

Thus, by Theorem 3, ker $\phi = M_a$ for some $a \in \overline{\mathbb{D}}$. To prove the assertions (1) and (2) we recall that $\phi(1) = 1$.

Thus, by Theorem 3, ker $\phi = M_a$ for some $a \in \overline{\mathbb{D}}$. To prove the assertions (1) and (2) we recall that $\phi(1) = 1$.

For (1), suppose that $a \in [-1, 1]$. Then, for any $f \in A_{\mathbb{R}}(\mathbb{D})$, the constant function $z \mapsto f(a)$ is in $A_{\mathbb{R}}(\mathbb{D})$ and so $f - f(a) \in M_a = \ker \phi$.

Thus, by Theorem 3, ker $\phi = M_a$ for some $a \in \overline{\mathbb{D}}$. To prove the assertions (1) and (2) we recall that $\phi(1) = 1$.

For (1), suppose that $a \in [-1, 1]$. Then, for any $f \in A_{\mathbb{R}}(\mathbb{D})$, the constant function $z \mapsto f(a)$ is in $A_{\mathbb{R}}(\mathbb{D})$ and so $f - f(a) \in M_a = \ker \phi$. Thus we obtain from $f = (f - f(a)) + f(a) \cdot 1$ that $\phi(f) = 0 + f(a)\phi(1) = f(a)$.

Thus, by Theorem 3, ker $\phi = M_a$ for some $a \in \overline{\mathbb{D}}$. To prove the assertions (1) and (2) we recall that $\phi(1) = 1$.

For (1), suppose that $a \in [-1, 1]$. Then, for any $f \in A_{\mathbb{R}}(\mathbb{D})$, the constant function $z \mapsto f(a)$ is in $A_{\mathbb{R}}(\mathbb{D})$ and so $f - f(a) \in M_a = \ker \phi$. Thus we obtain from $f = (f - f(a)) + f(a) \cdot 1$ that $\phi(f) = 0 + f(a)\phi(1) = f(a)$.

For (2), let $a \in \overline{\mathbb{D}} \setminus [-1, 1]$ and $f \in A_{\mathbb{R}}(\mathbb{D})$. Then, as in Proposition 4, we consider the affine function $\sigma + \beta z$, where $\sigma, \beta \in \mathbb{R}$ are chosen so that $\sigma + \beta a = f(a)$.

Thus, by Theorem 3, $\ker\phi = M_a$ for some $a \in \overline{\mathbb{D}}$. To prove the assertions (1) and (2) we recall that $\phi(1) = 1$.

For (1), suppose that $a \in [-1, 1]$. Then, for any $f \in A_{\mathbb{R}}(\mathbb{D})$, the constant function $z \mapsto f(a)$ is in $A_{\mathbb{R}}(\mathbb{D})$ and so $f - f(a) \in M_a = \ker\phi$. Thus we obtain from $f = (f - f(a)) + f(a) \cdot 1$ that $\phi(f) = 0 + f(a)\phi(1) = f(a)$.

For (2), let $a \in \overline{\mathbb{D}} \setminus [-1, 1]$ and $f \in A_{\mathbb{R}}(\mathbb{D})$. Then, as in Proposition 4, we consider the affine function $\sigma + \beta z$, where $\sigma, \beta \in \mathbb{R}$ are chosen so that $\sigma + \beta a = f(a)$. Then $f - (\sigma + \beta z) \in M_a = \ker\phi$; hence

Thus, by Theorem 3, $\ker \phi = M_a$ for some $a \in \overline{\mathbb{D}}$. To prove the assertions (1) and (2) we recall that $\phi(1) = 1$.

For (1), suppose that $a \in [-1, 1]$. Then, for any $f \in A_{\mathbb{R}}(\mathbb{D})$, the constant function $z \mapsto f(a)$ is in $A_{\mathbb{R}}(\mathbb{D})$ and so $f - f(a) \in M_a = \ker \phi$. Thus we obtain from $f = (f - f(a)) + f(a) \cdot 1$ that $\phi(f) = 0 + f(a)\phi(1) = f(a)$.

For (2), let $a \in \overline{\mathbb{D}} \setminus [-1, 1]$ and $f \in A_{\mathbb{R}}(\mathbb{D})$. Then, as in Proposition 4, we consider the affine function $\sigma + \beta z$, where $\sigma, \beta \in \mathbb{R}$ are chosen so that $\sigma + \beta a = f(a)$. Then $f - (\sigma + \beta z) \in M_a = \ker \phi$; hence

$$\phi(f) = \phi\left(\left(f - (\sigma + \beta z) + (\sigma + \beta z)\right)\right) = 0 + \sigma + \beta\phi(z). \qquad (3)$$

Now let $b := \phi(z)$. The function $H(z) = (z - a)(z - \overline{a}) \in A_{\mathbb{R}}(\mathbb{D})$ and so it belongs to $M_a = \ker \phi$.

Now let $b := \phi(z)$. The function $H(z) = (z-a)(z-\overline{a}) \in A_{\mathbb{R}}(\mathbb{D})$ and so it belongs to $M_a = \ker\phi$. Hence

$$0 = \phi(H) = \phi(z^2) - 2\mathrm{Re}\, a\, \phi(z) + |a|^2 = b^2 - 2b\,\mathrm{Re}\, a + |a|^2.$$

Now let $b := \phi(z)$. The function $H(z) = (z-a)(z-\overline{a}) \in A_{\mathbb{R}}(\mathbb{D})$ and so it belongs to $M_a = \ker\phi$. Hence

$$0 = \phi(H) = \phi(z^2) - 2\mathrm{Re}\, a\, \phi(z) + |a|^2 = b^2 - 2b\,\mathrm{Re}\, a + |a|^2.$$

Solving this quadratic, yields $b = \mathrm{Re}\, a \pm i\,\mathrm{Im}\, a$. Hence $b = a$ or $b = \overline{a}$.

Now let $b := \phi(z)$. The function $H(z) = (z-a)(z-\overline{a}) \in A_{\mathbb{R}}(\mathbb{D})$ and so it belongs to $M_a = \ker \phi$. Hence

$$0 = \phi(H) = \phi(z^2) - 2\mathrm{Re}\, a\, \phi(z) + |a|^2 = b^2 - 2b\,\mathrm{Re}\, a + |a|^2.$$

Solving this quadratic, yields $b = \mathrm{Re}\, a \pm i\, \mathrm{Im}\, a$. Hence $b = a$ or $b = \overline{a}$.By (3) we obtain $\phi(f) = \sigma + \beta a = f(a)$ or $\phi(f) = \sigma + \beta\overline{a} = \overline{f(a)}$.

$$\phi(f) = \phi\left(\left(f - (\sigma + \beta z) + (\sigma + \beta z)\right)\right) = 0 + \sigma + \beta\phi(z).$$

Bounded analytic functions

Let $H^\infty_{\mathbb{R}}(\mathbb{D}) = \{f \in H^\infty : f(z) = \overline{f(\overline{z})}\}$.
Corona Theorem for $H^\infty_{\mathbb{R}}(\mathbb{D})$

Theorem

$1 \in I(f_1, \dots, f_n) \Leftarrow: \inf \sum_{j=1}^n |f_j| > 0.$

Bounded analytic functions

Let $H^\infty_{\mathbb{R}}(\mathbb{D}) = \{f \in H^\infty : f(z) = \overline{f(\overline{z})}\}$.
Corona Theorem for $H^\infty_{\mathbb{R}}(\mathbb{D})$

Theorem

$1 \in I(f_1, \dots, f_n) \Leftarrow: \inf \sum_{j=1}^n |f_j| > 0.$

Proof

Carleson $\longrightarrow \exists g_j \in H^\infty : 1 = \sum_{j=1}^n g_j f_j$ Let
$h_j(z) := \frac{1}{2}(g_j(z) + \overline{g_j(\overline{z})})$. Then $h_j \in H^\infty_{\mathbb{R}}(\mathbb{D})$ and $\sum_{j=1}^n h_j f_j = 1$.

Theorem (Treil)

H^∞ has the Bass stable rank one; that is:
$|f| + |g| \geq \delta > 0 \longrightarrow \ \exists\, h \in H^\infty : f + hg$ *invertible.*

Theorem (Treil)

H^∞ has the Bass stable rank one; that is:
$|f| + |g| \geq \delta > 0 \longrightarrow \exists\, h \in H^\infty : f + hg$ invertible.

Does not work for $H^\infty_{\mathbb{R}}(\mathbb{D})$:

Theorem (Treil)

H^∞ has the Bass stable rank one; that is:
$|f| + |g| \geq \delta > 0 \longrightarrow \; \exists\, h \in H^\infty : f + hg$ *invertible.*

Does not work for $H^\infty_{\mathbb{R}}(\mathbb{D})$:
$f(z) = z, g(z) = 1 - z^2$; suppose
$u(z) := z + h(1 - z^2) \in H^\infty_{\mathbb{R}}(\mathbb{D}))^{-1}$.

Theorem (Treil)

H^∞ has the Bass stable rank one; that is:
$|f| + |g| \geq \delta > 0 \longrightarrow \; \exists\, h \in H^\infty : f + hg$ invertible.

Does not work for $H^\infty_{\mathbb{R}}(\mathbb{D})$:
$f(z) = z, g(z) = 1 - z^2$; suppose
$u(z) := z + h(1 - z^2) \in H^\infty_{\mathbb{R}}(\mathbb{D}))^{-1}$.Then on
$]-1, 1[$: $\lim_{x \to -1} u(x) = -1$ and $\lim_{x \to 1} u(x) = 1$. Hence
$\exists x_0 \in]-1, 1[$ such that $u(x_0) = 0$. ↯

Theorem (Treil)

H^∞ has the Bass stable rank one; that is:
$|f| + |g| \geq \delta > 0 \longrightarrow \exists\, h \in H^\infty : f + hg$ *invertible.*

Does not work for $H^\infty_{\mathbb{R}}(\mathbb{D})$:
$f(z) = z, g(z) = 1 - z^2$; suppose
$u(z) := z + h(1 - z^2) \in H^\infty_{\mathbb{R}}(\mathbb{D}))^{-1}$.Then on
$]-1, 1[$: $\lim_{x\to -1} u(x) = -1$ and $\lim_{x\to 1} u(x) = 1$. Hence
$\exists x_0 \in]-1, 1[$ such that $u(x_0) = 0$. ↯

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f| + |g| + |h| \geq \delta > 0 \longrightarrow \exists a, b \in H^\infty_{\mathbb{R}}(\mathbb{D})$ *such that*
$|f + ah| + |g + bh| \geq \delta > 0.$

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f|+|g|+|h|\geq\delta>0 \longrightarrow \exists a,b\in H^\infty_{\mathbb{R}}(\mathbb{D})$ *such that*
$|f+ah|+|g+bh|\geq\delta>0$.

Proof

Let $(x,y,t)\in(H^\infty_{\mathbb{R}})^3$ be such that $1=xf+yg+th$.

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f| + |g| + |h| \geq \delta > 0 \longrightarrow \exists a, b \in H^\infty_{\mathbb{R}}(\mathbb{D})$ *such that*
$|f + ah| + |g + bh| \geq \delta > 0$.

Proof

Let $(x, y, t) \in (H^\infty_{\mathbb{R}})^3$ be such that $1 = xf + yg + th$. The idea is to approximate (x, y) by $(\tilde{x}, \tilde{y})$, such that $(\tilde{x}, \tilde{y})$ is an invertible pair in $H^\infty_{\mathbb{R}}$.

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f| + |g| + |h| \geq \delta > 0 \longrightarrow \exists a, b \in H^\infty_{\mathbb{R}}(\mathbb{D})$ *such that*
$|f + ah| + |g + bh| \geq \delta > 0$.

Proof

Let $(x, y, t) \in (H^\infty_{\mathbb{R}})^3$ be such that $1 = xf + yg + th$. The idea is to approximate (x, y) by $(\tilde{x}, \tilde{y})$, such that $(\tilde{x}, \tilde{y})$ is an invertible pair in $H^\infty_{\mathbb{R}}$. Now

$$u := \tilde{x}f + \tilde{y}g + th = (xf + yg + th) + (\tilde{x} - x)f + (\tilde{y} - y)g = 1 + \varepsilon.$$

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f|+|g|+|h|\geq\delta>0 \longrightarrow \exists a,b\in H^\infty_{\mathbb{R}}(\mathbb{D})$ such that
$|f+ah|+|g+bh|\geq\delta>0$.

Proof

Let $(x,y,t)\in(H^\infty_{\mathbb{R}})^3$ be such that $1=xf+yg+th$. The idea is to approximate (x,y) by $(\tilde{x},\tilde{y})$, such that $(\tilde{x},\tilde{y})$ is an invertible pair in $H^\infty_{\mathbb{R}}$. Now
$u:=\tilde{x}f+\tilde{y}g+th=(xf+yg+th)+(\tilde{x}-x)f+(\tilde{y}-y)g=1+\varepsilon$.Thus, if ε has small norm, u is invertible.

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f|+|g|+|h|\geq\delta>0 \longrightarrow \exists a,b\in H^\infty_{\mathbb{R}}(\mathbb{D})$ *such that*
$|f+ah|+|g+bh|\geq\delta>0.$

Proof

Let $(x,y,t)\in(H^\infty_{\mathbb{R}})^3$ be such that $1=xf+yg+th$. The idea is to approximate (x,y) by $(\tilde{x},\tilde{y})$, such that $(\tilde{x},\tilde{y})$ is an invertible pair in $H^\infty_{\mathbb{R}}$. Now
$u:=\tilde{x}f+\tilde{y}g+th=(xf+yg+th)+(\tilde{x}-x)f+(\tilde{y}-y)g=1+\varepsilon$.Thus, if ε has small norm, u is invertible.
$(\tilde{x},\tilde{y})$ being invertible $\longrightarrow$ $t=a\tilde{x}+b\tilde{y}$; hence

Theorem (Mortini, Wick)

$H^\infty_{\mathbb{R}}(\mathbb{D})$ has the Bass stable rank 2; that is:
$|f|+|g|+|h|\geq\delta>0 \longrightarrow \exists a,b\in H^\infty_{\mathbb{R}}(\mathbb{D})$ such that
$|f+ah|+|g+bh|\geq\delta>0$.

Proof

Let $(x,y,t)\in(H^\infty_{\mathbb{R}})^3$ be such that $1=xf+yg+th$. The idea is to approximate (x,y) by $(\tilde{x},\tilde{y})$, such that $(\tilde{x},\tilde{y})$ is an invertible pair in $H^\infty_{\mathbb{R}}$. Now
$u:=\tilde{x}f+\tilde{y}g+th=(xf+yg+th)+(\tilde{x}-x)f+(\tilde{y}-y)g=1+\varepsilon$.Thus, if ε has small norm, u is invertible.
$(\tilde{x},\tilde{y})$ being invertible $\longrightarrow$ $t=a\tilde{x}+b\tilde{y}$; hence
$u=\tilde{x}f+\tilde{y}g+(a\tilde{x}+b\tilde{y})h=\tilde{x}(f+ah)+\tilde{y}(g+bh)$.