The generalized corona problem

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be the corresponding finitely generated ideal in H^∞ and let

$$J = J(f_1, \dots, f_n) = \{f \in H^\infty : \exists C = C(f) > 0, |f| \leq C \sum_{j=1}^{n} |f_j|\}$$

be its associated J-form, which we will call an ideal of finite type.

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iv) Is J finitely generated if and only if J contains a Carleson-Newman Blaschke product
(or equivalently if and only if $Z(J) \subseteq G$)?

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$$J(b_1^N, b_2^N) = \bigcap_{j=1}^{N} (I + b_j H^\infty),$$

where b_j is the inner factor of $b_1 + \varepsilon_j b_2$, $\varepsilon_j > 0$ small, $\varepsilon_j \neq \varepsilon_k$, $j = 3, \dots, N$,

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Theorem (Gorkin-Nicolau-Mortini, 1995)

Suppose that $f_1, f_2 \in H^\infty$ have no common factors. Let $I = I(f_1, f_2)$ and $J = J(f_1, f_2)$. Equivalent are:

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3. *I contains an interpolating Blaschke product;*
4. *J contains an interpolating Blaschke product;*
5. $|f_1(z)|^2 + (1 - |z|^2)|f_1'(z)| + |f_2(z)|^2 + (1 - |z|^2)|f_2'(z)| \geq \delta > 0$ *for every* $z \in \mathbb{D}$.

Theorem (Mortini 1997)

Let B and C be interpolating Blaschke products. Then

$$I(B^N, B^{N-1}C, B^{N-2}C^2, \dots, BC^{N-1}, C^N) =$$

$$= J(B^N, B^{N-1}C, B^{N-2}C^2, \dots, BC^{N-1}, C^N) = J(B^N, C^N).$$

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- (Mortini) [$>$ 1997]

Proposition

For $j = 1, 2$ let B_j, C_j be interpolating Blaschke products. Then

$$I(B_1B_2, B_1C_2, C_1C_2) = J(B_1B_2, B_1C_2, C_1C_2).$$

◂ symmetric case

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Then on $Z_{\mathbb{D}}(C_2)$ we have $|f| \leq |B_1 B_2|$. Thus, $f = xC_2 + yB_1 B_2$ for some functions $x, y \in H^\infty$ (here we have used that C_2 is an interpolating Blaschke product).

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$$\frac{f}{C_1C_2} = \frac{yB_1B_2}{C_1C_2} + \frac{x}{C_1} \tag{2}$$

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Hence, by (2), $\frac{x}{C_1}$ is bounded on $Z_{\mathbb{D}}(B_1)$. So $x \in I(C_1, B_1)$ (note that B_1 is an interpolating Blaschke product.)
Hence $f = xC_2 + yB_1B_2 \in I(C_1C_2, B_1C_2, B_1B_2)$. □

Theorem

Let $B_1, B_2, B_3, C_1, C_2, C_3$ be Blaschke products without common zeros and let
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(1) $I^* = (I^* + B_1H^\infty) \cap (I^* + B_2H^\infty) \cap (I^* + C_3H^\infty)$.

(2) $I = I^*$ *if and only if* $I(B_1, B_2) = I(B_1, C_1)$.

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If B_1, B_2 and C_3 are interpolating Blaschke products, then $I^* = J^*$, where J^* is the J-form of I^*.

Theorem

For $N \geq 3$ let I be the ideal

$$I\left(\prod_{j=1}^{N} B_j,\ (\prod_{j=1}^{N-1} B_j)C_N,\ (\prod_{j=1}^{N-2} B_j)C_{N-1}C_N, \dots, B_1(\prod_{j=2}^{N} C_j),\ \prod_{j=1}^{N} C_j\right),$$

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and J the associated J-ideal, where the B_j and C_k are interpolating Blaschke products without common zeros in $\mathbb{D}$.

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and J the associated J-ideal, where the B_j and C_k are interpolating Blaschke products without common zeros in $\mathbb{D}$. Suppose that

$\frac{C_j}{B_j}$ *is bounded on $Z(B_{k+1})$ for $j = 1, 2, \dots, k$ and $k = 1, 2, \dots, N-2$.*

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Then $I = J$. unsymmetric case

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In particular we have $J(BB^*, CC^*, BC) = I(BB^*, CC^*, BC).$

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$$I = I\left(\prod_{j=1}^N B_j,\ \ C_1 \prod_{j=2}^N B_j,\ \ C_2 \prod_{j=3}^N B_j,\ \ \cdots\cdots,\ \ C_{N-1} B_N,\ \ C_N\right).$$

The same result holds for J.

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If $\{f_1, \dots, f_m\}$ is another set of generators for I, then $m \geq N+1$; that is $N+1$ is the minimal number of generators for I.

Proposition

Let $f, g \in H^\infty$ have no common factor. Suppose that $Z^\infty(f) \cap Z^\infty(g) \neq \emptyset$. Then
$I(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) \neq J(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N)$ Binomi