# The stable rank of $A(K)$ and $A(K)_{\text{sym}}$

joint work with R. Rupp

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For $f \in C(K)$, let $Z(f) = \{z \in K : f(z) = 0\}$ be the zero set of $f$.

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2. *$\mathbb{C} \setminus K$ is connected, that is $K$ has no holes;*
3. *Every continuous function $f : K \to \mathbb{C} \setminus \{0\}$ has a continuous logarithm on $K$.*

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2. *If $a, b$ are in the same connected component of $\mathbb{C} \setminus K$, then there exists $h \in A(K)$ such that*

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$$\sum_{j=1}^n h_j f_j = \frac{1}{2}\Big[\sum_{j=1}^n g_j f_j + \sum_{j=1}^n g_j^* f_j^*\Big] = 1.$$

Answers to ii) and iii) later

An $n$-tuple $\mathbf{f} := (f_1, \ldots, f_n)$ in $A$ is said to be invertible (not.: $\mathbf{f} \in U_n(A)$), if there exists $(g_1, \ldots, g_n) \in A^n$ such that $\sum_{j=1}^n f_j g_j = 1$.
An element $(f_1, \ldots, f_n, g) \in U_{n+1}(A)$ is said to be *reducible*, if there exists $(x_1, \ldots, x_n) \in A^n$ so that

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The smallest integer $n$ for which every element in $U_{n+1}(A)$ is reducible is called the *Bass stable rank* of $A$ and is denoted by $\mathrm{bsr}(A)$. If no such integer exists, then $\mathrm{bsr}(A) = \infty$.

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### Theorem (Izzo)

*Let $K \subseteq \mathbb{C}$ be compact and suppose that $E$ is a compact subset of $K$ with $E^\circ = \emptyset$ such that each component of $\mathbb{C} \setminus E$ intersects $\mathbb{C} \setminus K$. Then $A(K)|_E$ is dense in $C(E)$.*

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### Lemma

*Let $K \subseteq \mathbb{C}$ be a real-symmetric compact set, $g \in A(K)_{\rm sym}$, and let $K' = K \setminus Z(g)^\circ$. Choose $(f_1, \dots, f_n, g) \in U_{n+1}(A(K)_{\rm sym})$. Suppose that the tuple $(f_1, \dots, f_n, g^2)$ is reducible in $A(K')_{\rm sym}$. Then the original tuple $(f_1, \dots, f_n, g)$ is reducible in $A(K)_{\rm sym}$. A similar result also holds for $A(K)$ on arbitrary compacta.*

**Proof**

Since $(f_1, \dots, f_n, g^2)$ is reducible in $A(K')_{\rm sym}$, there exist $h_j \in A(K')_{\rm sym}$ such that $(f_1 + h_1 g^2, \dots, f_n + h_n g^2)$ is an invertible $n$-tuple in $A(K')_{\rm sym}$.

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$$(f_1 + (h_1 g)g, \dots, f_n + (h_n g)g)$$

is an invertible $n$-tuple in $A(K)_{\rm sym}$ and so $(f_1, \dots, f_n, g)$ is reducible in $A(K)_{\rm sym}$.

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### Theorem

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**Theorem**

*Let $K \subseteq \mathbb{C}$ be compact and real-symmetric. Then*

1. $\mathrm{bsr}(C(K)_{\mathrm{sym}}) = 1$ *if and only if* $K^\circ = \emptyset$ *and* $K \cap \mathbb{R}$ *is totally disconnected or empty;*
2. $\mathrm{bsr}(C(K)_{\mathrm{sym}}) = 2$ *if and only if* $K^\circ \neq \emptyset$ *or* $K \cap \mathbb{R}$ *contains an interval.*

**Theorem**

*Let $K \subseteq \mathbb{C}$ be compact and real-symmetric. Then*

1. $\text{bsr}(C(K)_{\text{sym}}) = 1$ *if and only if* $K^\circ = \emptyset$ *and* $K \cap \mathbb{R}$ *is totally disconnected or empty;*
2. $\text{bsr}(C(K)_{\text{sym}}) = 2$ *if and only if* $K^\circ \neq \emptyset$ *or* $K \cap \mathbb{R}$ *contains an interval.*

**Theorem**

*Let $K \subseteq \mathbb{C}$ be compact and real-symmetric. Then*

1. $\text{bsr}(A(K)_{\text{sym}}) = 1$ *if and only if* $K \cap \mathbb{R}$ *is empty or totally disconnected.*
2. $\text{bsr}(A(K)_{\text{sym}}) = 2$ *if and only if* $K \cap \mathbb{R}$ *contains an interval.*

### Theorem

*Let $(f,g)$ be an invertible pair in $A(K)_{\rm sym}$. Then the following assertions are equivalent.*

1. *$(f,g)$ is reducible in $A(K)_{\rm sym}$;*
2. *$(\operatorname{sign} f)|_{Z(g)\cap\mathbb{R}}$ admits a continuous extension to a sign-function in the space $C(K\cap\mathbb{R})_{\rm sym}$;*
3. *$f$ has constant sign at each real zero of $g$ on fixed components of $K\cap\mathbb{R}$.*

## Theorem

*Let $(f, g)$ be an invertible pair in $A(K)_{\mathrm{sym}}$. Then the following assertions are equivalent.*

1. $(f, g)$ *is reducible in* $A(K)_{\mathrm{sym}}$;
2. $(\operatorname{sign} f)|_{Z(g)\cap\mathbb{R}}$ *admits a continuous extension to a sign-function in the space* $C(K \cap \mathbb{R})_{\mathrm{sym}}$;
3. $f$ *has constant sign at each real zero of* $g$ *on fixed components of* $K \cap \mathbb{R}$.

## Corollary

*Let $(f, g)$ be an invertible pair in $A(K)_{\mathrm{sym}}$. Suppose that $Z(g) \cap \mathbb{R} = \emptyset$. Then $(f, g)$ is reducible.*

**Definition**

Let $E \subseteq \mathbb{C}$ compact, $f \in C(E)$ zero free. Let $C$ be a bounded component (=hole) of $E \setminus C$. Then $C$ is called an essential hole for $f$ if the Brouwer degree $d(f|_{\partial C}, C, 0)$ of $f$ with respect to the component $C$ is not zero.

**Definition**

Let $E \subseteq \mathbb{C}$ compact, $f \in C(E)$ zero free. Let $C$ be a bounded component (=hole) of $E \setminus C$. Then $C$ is called an essential hole for $f$ if the Brouwer degree $d(f|_{\partial C}, C, 0)$ of $f$ with respect to the component $C$ is not zero.

**Theorem**

*Let $K \subseteq \mathbb{C}$ be compact and let $(f, g)$ be an invertible pair in $C(K)$. Then the following assertions are equivalent:*

1. $(f, g)$ *is reducible;*
2. *each essential hole for* $f|_{Z(g)}$ *contains a hole of* $K$.

**Theorem**

*Let $K \subseteq \mathbb{C}$ be real-symmetric and compact. Suppose that $(f, g)$ is an invertible pair in $C(K)_{\rm sym}$. Then the following assertions are equivalent:*

1. *$(f, g)$ is reducible;*
2. *each essential hole for $f|_{Z(g)}$ contains a hole of $K$ and $f$ has constant sign at each real zero of $g$ on fixed components of $K \cap \mathbb{R}$.*

Here is the answer to iii)

## Proposition

*The invertible pair* $(f, g)$ *in* $C(K)_{\rm sym}$ *is reducible if and only if* $f|_{Z(g)}$ *admits an extension to an invertible function in* $C(K)_{\rm sym}$.

Here is the answer to iii)

### Proposition

*The invertible pair $(f, g)$ in $C(K)_{\rm sym}$ is reducible if and only if $f|_{Z(g)}$ admits an extension to an invertible function in $C(K)_{\rm sym}$.*

### Proposition

*Let $(f, g)$ be an invertible pair in $A(K)_{\rm sym}$. Then the following assertions are equivalent.*

1. *$(f, g)$ is reducible in $A(K)_{\rm sym}$;*
2. *$f|_{Z(g)}$ admits an extension to an invertible function in $A(K)_{\rm sym}$ or equivalently in $C(K)_{\rm sym}$.*