A constructive proof of the Nullstellensatz for $A(K)$

joint work with R. Rupp

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For a compact planar set K, let $A(K)$ be the algebra of all continuous complex-valued functions on K that are holomorphic in the interior K° of K.
We present two elementary proofs of Arens' Nullstellensatz and the characterization of the maximal ideals of the algebra $A(K)$ without using Zorn's Lemma, the Hahn-Banach Theorem and the fundamental theorem of Gelfand theory. Our methods involve only elementary $\overline{\partial}$-calculus and Weierstrass' approximation theorem.

Theorem (Nullstellensatz)

Let $f_j \in A(K)$ *and suppose that* $\bigcap_{j=1}^n Z(f_j) = \emptyset$. *Then there are* $g_j \in A(K)$ *such that* $\sum_{j=1}^n g_j f_j = 1$.

Corollary

An ideal M *in the algebra* $A(K)$ *is maximal if and only if there is* $a \in K$ *such that*

$$M = M_a := \{f \in A(K) : f(a) = 0\}.$$

Standard proof: "$\Longrightarrow$"Let $M = \mathrm{Ker}\,\phi$ for some multiplicative linear functional ϕ on A. Put $a := \phi(z)$. Since $z - b$ is invertible in A for $b \notin K$ ($\frac{1}{z-b} \in A$), we see that $a \in K$. If $a \in K^\circ$, and $f \in A$, $\frac{f(z)-f(a)}{z-a} \in A$. If $a \in \partial K$, take a Tietze extension of f to $\mathbb{C}$, uniformly approximate f on $\mathbb{C}$ by $f_n \in C(\mathbb{C})$ with f_n holomorphic on K° and holomorphic on a neighborhood U_n of a. Then $\frac{f_n(z)-f_n(a)}{z-a} \in A$. Using that $\phi(1) = 1$,

$$\begin{aligned} \phi(f) - f(a) &= \lim[\phi(f_n) - f_n(a)] && (1) \\ &= \lim \phi(f_n - f_n(a)) && (2) \\ &= \lim \phi\left(\frac{f_n - f_n(a)}{z - a}\right)\phi(z - a) = 0 && (3) \end{aligned}$$

Hence $\phi(f) = f(a)$ and so $M = M_a$. The theorem now follows using Gelfand theory.

Idea for the new proof: replace this local approximation theorem with a global one:

Proposition (Izzo)

$$C(\partial K) = \overline{A(K)|\partial K + R(\partial K)},$$

where $R(\partial K)$ denotes the uniform closure in $C(\partial K)$ of the algebra of rational functions with poles outside ∂K.

Follows together with Weierstrass' approximation theorem from

Lemma

Let $K \subseteq \mathbb{C}$ be compact. For every function f, twice continuously differentiable in a neighborhood of K, there exists $g \in A(K)$ such that $f - g \in R(\partial K)$.

Proof Just let $g : K \to \mathbb{C}$ be defined as

$$g(z) = f(z) + \frac{1}{\pi} \int_{K^\circ} \frac{\overline{\partial} f(\xi)}{\xi - z} d\sigma(\xi),$$

and use the following two results.

Lemma

Let $K \subseteq \mathbb{C}$ be compact, $X \subseteq K$ Borel-measurable and $h \in C(K)$. The Cauchy transforms

$$\hat{\mu}_{h,X}(z) = -\frac{1}{\pi} \int_X \frac{h(\xi)}{\xi - z} d\sigma(\xi) \; (z \in \hat{\mathbb{C}})$$

have the following properties:

- *If $K_n \subseteq K^\circ$ is a monotone increasing sequence of compact sets with $K_n \nearrow K^\circ$, then $\hat{\mu}_{h,K_n}$ converges uniformly on $\hat{\mathbb{C}}$ to $\hat{\mu}_{h,K^\circ}$.*

Theorem

Let $K \subseteq \mathbb{C}$ be compact, $f \in C(K) \cap C^1(K^\circ)$ and for $z \in \mathbb{C}$ let

$$v(z) = -\frac{1}{\pi} \int_K \frac{f(\xi)}{\xi - z} d\sigma(\xi).$$

Then the following assertions hold:

(i) $v \in C(\hat{\mathbb{C}}), v$ *holomorphic in* $\mathbb{C} \setminus K$, *and* $v(\infty) = 0$;

(ii) $v \in C^1(K^\circ)$;

(iii) $\overline{\partial} v = f$ *in* K°.

Proof. $\bigcap Z(f_j) = \emptyset \implies \exists \delta > 0$ such that $\sum_{j=1}^n |f_j| \geq \delta > 0$ on K. Let

$$g_k = \frac{\overline{f}_k}{\sum_{j=1}^n |f_j|^2}.$$

Then $g_k \in C(\partial K)$ and $\sum_{k=1}^n g_k f_k = 1$ on ∂K. By Proposition 3, we can choose rational functions r_k without poles on ∂K and functions $h_k \in A(K)$ such that for every k

$$||g_k - (h_k + r_k)||_{\partial K} \leq \left(2 \sum_{j=1}^n ||f_j||_{\partial K}\right)^{-1}. \tag{4}$$

Let $r_k = p_k/q_k$, where p_k and q_k are polynomials such that q_k has no zeros on ∂K. Let $R_k = (q_k h_k + p_k)/q_k$.

Then, by (4),

$$\Big|\sum_{k=1}^n R_k f_k\Big| \geq \Big|\sum_{k=1}^n g_k f_k\Big| - \sum_{k=1}^n ||R_k - g_k||_{\partial K} |f_k| \tag{5}$$

$$\geq 1 - \frac{1}{2} = \frac{1}{2} \text{ on } \partial K. \tag{6}$$

If we define $q = \prod_{j=1}^n q_j$, then $qR_k \in A(K)$.
Hence $f := \sum_{k=1}^n (qR_k) f_k$ belongs to the ideal $I = I(f_1, \ldots, f_n)$ generated by the f_k.
Moreover, f has no zeros on ∂K.

Let $z_1, \ldots, z_p$ be the zeros of f in K°, including multiplicities. Since $\bigcap_{j=1}^n Z(f_j) = \emptyset$, there exists for every z_j a generator $F_j \in \{f_1, \ldots, f_n\}$ such that $F_j(z_j) \neq 0$. By using the stability of $A(K)$ (that is $f \in A(K), f(a) = 0$, $a \in K^\circ$, implies $(f - f(a)) \,/\, (z - a) \in A(K)$), and the formula

$$\frac{f(z)}{z - z_1} = \frac{1}{F_1(z_1)} \left[\frac{f(z)}{z - z_1} F_1(z) - \frac{F_1(z) - F_1(z_1)}{z - z_1} f(z) \right] \in I,$$

we obtain a factor $h_1 \in I$ of f that has one zero less than f.

Reapplying the same formula to the function h_1 and the zero z_2 (note that $z_1 = z_2$ is possible), yields that

$$h_2(z) := \frac{f(z)}{(z - z_1)(z - z_2)} \in I.$$

After p-steps, we achieve that $h_p := f/(\prod_{j=1}^{p}(z - z_j) \in I$. But h_p has no zeros at all. Since $A(K)$ is inversionally closed, we conclude that $I(f_1, \dots, f_n) = A(K)$.

Yet another constructive proof of the Nullstellensatz is based on the inhomogeneous Cauchy-Riemann equation $\overline{\partial} v = f$.
For a bounded open set Ω in the plane, let $C_{\overline{\partial}}(\overline{\Omega})$ be the set of all complex-valued functions f continuous on the closure $\overline{\Omega}$ of Ω, continuously differentiable in Ω with $\overline{\partial} f \in C^1(\Omega)$ and such that $\overline{\partial} f$ admits a continuous extension to $\overline{\Omega}$. $C_{\overline{\partial}}(\overline{\Omega})$ actually is an algebra.

Theorem

Suppose that the functions $f_1, \dots, f_n \in C_{\overline{\partial}}(\overline{\Omega})$ have no common zero on $\overline{\Omega}$. Then the Bézout equation $\sum_{j=1}^n x_j f_j = 1$ admits a solution $(x_1, \dots, x_n)$ in $C_{\overline{\partial}}(\overline{\Omega})$.

Proof. Let

$$q_j = \frac{\overline{f}_j}{\sum_{k=1}^n |f_k|^2}.$$

Then $q_j \in C(\overline{\Omega})$ and $\sum_{j=1}^n q_j f_j = 1$. By Weierstrass' approximation theorem choose a polynomial $p_j(z, \overline{z})$ such that on $\overline{\Omega}$

$$|p_j - q_j| \leq (2 \sum_{k=1}^n ||f_j||_\infty)^{-1}.$$

Then

$$|\sum_{j=1}^n p_j f_j| \geq \left|\sum_{j=1}^n q_j f_j\right| - \left|\sum_{j=1}^n (p_j - q_j) f_j\right| \geq \frac{1}{2}.$$

Note that $\sum_{j=1}^{n} p_j f_j \in C_{\overline{\partial}}(\overline{\Omega})$. Because $C_{\overline{\partial}}(\overline{\Omega})$ is inversionally closed, we get that

$$x_j = \frac{p_j}{\sum_{k=1}^{n} p_k f_k} \in C_{\overline{\partial}}(\overline{\Omega}).$$

Since $\sum_{j=1}^{n} x_j f_j = 1$, we see that $(x_1, \dots, x_n)$ is the desired solution to the Bézout equation in $C_{\overline{\partial}}(\overline{\Omega})$.

Third Proof of the Nullstellensatz for $A(K)$

Assume that $\sum_{j=1}^{n} |f_j| \geq \delta > 0$ on K. Applying Theorem 7 to $\Omega = K^\circ$, there is a solution $(x_1, \ldots, x_n) \in C_{\overline{\partial}}(\overline{\Omega})^n \subseteq C(K)$ to $\sum_{j=1}^{n} x_j f_j = 1$.

Consider $\boldsymbol{f} = (f_1, \ldots, f_n)$ as a row matrix; its transpose is denoted by $\boldsymbol{f}^t$. Let $|\boldsymbol{f}|^2 = \sum_{j=1}^{n} |f_j|^2$; that is $|f|^2 = \overline{\boldsymbol{f}}\boldsymbol{f}^t$.

The Bézout equation now reads as $\boldsymbol{x}\boldsymbol{f}^t = 1$. It is well-known that any other solution $\boldsymbol{u} \in C(K)$ to the Bézout equation $\boldsymbol{u}\boldsymbol{f}^t = 1$ is given by

$$\boldsymbol{u}^t = \boldsymbol{x}^t + H\boldsymbol{f}^t,$$

or equivalently

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{f}H,$$

where H is an antisymmetric matrix over $C(K)$; that is $H^t = -H$.

Let

$$F = \left(\left(\overline{\partial} \mathbf{x}^t \cdot \overline{\mathbf{f}} \right)^t - \overline{\partial} \mathbf{x}^t \cdot \overline{\mathbf{f}} \right) \frac{1}{|\mathbf{f}|^2}.$$

Since $\mathbf{x} \in C_{\overline{\partial}}(\overline{\Omega})^n$, we see that F is an antisymmetric matrix over $C(K) \cap C^1(K^\circ)$. Thus, by Theorem 6, the system $\overline{\partial} H = F$ admits a matrix solution H over $C(K) \cap C^1(K^\circ)$. Note that H is antisymmetric, too.

It is now easy to check that $\overline{\partial} \mathbf{u} = 0$

In fact

$$\overline{\partial}\boldsymbol{u} = \overline{\partial}\boldsymbol{x} - \boldsymbol{f}\cdot\overline{\partial}H = \overline{\partial}\boldsymbol{x} - \boldsymbol{f}\left(\overline{\boldsymbol{f}}^t\cdot\overline{\partial}\boldsymbol{x} - \overline{\partial}\boldsymbol{x}^t\cdot\overline{\boldsymbol{f}}\right)\frac{1}{|\boldsymbol{f}|^2} =$$

$$\frac{(\boldsymbol{f}\,\overline{\partial}\boldsymbol{x}^t)\cdot\overline{\boldsymbol{f}}}{|\boldsymbol{f}|^2} = \frac{\overline{\partial}(\boldsymbol{f}\boldsymbol{x}^t)\cdot\overline{\boldsymbol{f}}}{|\boldsymbol{f}|^2} = \frac{\overline{\partial}(\boldsymbol{x}\boldsymbol{f}^t)^t\cdot\overline{\boldsymbol{f}}}{|\boldsymbol{f}|^2} = 0$$

Thus $u = \boldsymbol{x} - \boldsymbol{f}H \in A(K)$. Hence $\boldsymbol{u}$ is the solution to our Bézout equation in $A(K)$.

Let us now consider the Aryabattha-Bézout equation

$$\sum_{j=1}^{N} X_j A_j = I_n,$$

where the A_j are given (n,n)-matrices over $A(K)$ and

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

is the identity matrix. If R is a unital ring, let $M_{(m,n)}(R)$ denote the set of all matrices with entries in R having m rows and n columns.

Theorem

For $j = 1, \ldots, N$, let $A_j \in M_{(n,n)}(A(K))$. The following assertions are equivalent:

(1) *There exist matrices $X_j \in M_{(n,n)}(C(K))$ with $I_n = \sum_{j=1}^N X_j A_j$;*

(2) *the matrix*

$$M := \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}$$

is left-invertible in $C(K)$;

(3) *there exists $\delta > 0$ such that for $M := \sum_{j=1}^N A_j^* A_j$ we have $M \geq \delta I_n$; that is*

$$\langle M(x)z, z\rangle \geq \delta \langle z, z\rangle \text{ for every } z \in \mathbb{K}^n \text{ and } x \in X.$$

Theorem

(4) *the determinants of the* (n,n)*-minors of* M *have no common zeros on* K*;*

(5) *there exist matrices* $B_j \in M_{(n,n)}(A(K))$ *with* $I_n = \sum_{j=1}^{N} B_j A_j$.